The Gamma function

As you know, factorials $n!$ are defined for integer values of $n$ as $n! = n(n-1) \cdots 1$, so that the graph of $y = n!$ consists of just a point above each integer $n$. However, there is a well known function, the **Gamma function** $\Gamma(x)$ that makes it possible to draw a smooth graph through the factorials\(^1\). The Gamma function is shifted by 1, though, compared to factorials, so we have $\Gamma(n+1) = n!$ when $n$ is an integer. See Figure 1\(^2\).

The Gamma function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$ 

Notice that $x$ is a constant as far as the integral is concerned. For $x > 1$, as $t$ gets larger the power $t^{x-1}$ is getting bigger and is fighting $e^{-t}$, which is getting smaller. (Which wins?)

Some people like to turn things around and use the Gamma function to define factorials of non-integer values: Define $x! = \Gamma(x + 1)$.

For example, in the Mathematica computer algebra package, if you type 1.5!, it won’t complain; it will simply give you the value of $\Gamma(2.5)$.

\(^1\)The letter $\Gamma$ is Greek upper-case gamma.

\(^2\)Notice that the two axes in the figure are not drawn to the same scale.
Comment:

The procedure just described is analogous to the way we develop a growing understanding of the concept of powers of a positive real number $a$. We start at the beginning of algebra with the idea of $a^n$ as the product of $n$ copies of $a$, which makes sense only when $n$ is a positive integer. We later learn that $a^x$ also is defined when $x$ is any real number, so that $y = a^x$ has a smooth graph.

But what does $a^x$ mean, really? There must be some solid definition. The answer starts from a trick that you often use in calculus:

$$a^x = (e^{\log a})^x = e^{x \log a} = \exp(x \log a).$$

(Often $e^x$ is written as $\exp(x)$.)

So if we can find definitions of $\log(x)$ [meaning base $e$] and $\exp(x)$ that do not depend on knowing about powers, we can use them to define $a^x$ by saying $a^x$ means $\exp(x \log a)$.

For a good definition for logs, again start with a result you know:

$$\int_1^a \frac{1}{x} \, dx = [\log x]_1^a = \log a - \log 1 = \log a.$$  

Now turn it into a definition: Define $\log a$ to be $\int_1^a \frac{1}{x} \, dx$. This is a solid idea since this integral has an existence that doesn’t depend on logs.

The exponential function $\exp$ can be defined either as the inverse function of $\log$ (in other words, $y = \exp(x)$ means $x = \log(y)$) or using a power series. Either way there are some details to check, but everything works fine.

It’s interesting that for powers of $a$ there is an integral behind the scenes, just as there is for factorials.

Problem D-1. 3 Check by doing an integral that $\Gamma(1) = 1$, so $\Gamma(1) = 0!$. (Since it’s an integral to infinity, you need to integrate from 0 to an unspecified number $A$ and then let $A$ go to infinity.)

Problem D-2. 4 (a) Show that $\Gamma(x + 1) = x \Gamma(x)$ for $x > 0$.

You will need to use integration by parts; if you’re rusty on that, look it up. Also, for values of $x$ that are less than 1, the value of the “integrand” is undefined for $t = 0$ (so the integral is improper at the low end as well as at the high end). Therefore you need to integrate from $h$ to $A$ and then let $h \to 0$ and $A \to \infty$.

(b) Prove by induction that $\Gamma(n) = (n - 1)!$ for $n = 1, 2, \ldots$

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3Same as C-2, already assigned.

4Not to be done until assigned. We’ll discuss induction.