Consider the linear system of equations

\[ A \vec{x} = \vec{b} \]  

where \( A \) is an \( N \times N \) matrix and \( \vec{x} \) and \( \vec{b} \) are vectors in \( \mathbb{R}^n \).

Assume that \( A \) has \( N \) orthogonal eigenvectors forming a basis for \( \mathbb{R}^n \). Specifically, assume there is a set of \( N \) vectors \( \{ \vec{v}_k \} \), and corresponding set of real or complex values \( \{ \lambda_k \} \), such that

\[ A \vec{v}_k = \lambda_k \vec{v}_k \quad k = 1 \ldots N \]

and

\[ \langle \vec{v}_i, \vec{v}_j \rangle = 0 \text{ if } i \neq j \]

where \( \langle \ast, \ast \rangle \) is the standard vector inner (dot) product; \( \langle \vec{x}, \vec{y} \rangle = \sum_{k=1}^{N} x_k y_k \).

Since the set of vectors \( \{ \vec{v}_k \} \) form a basis, one can express \( \vec{b} \) as a linear combination of those vectors,

\[ \vec{b} = \sum_{k=1}^{N} \beta_k \vec{v}_k. \]  

As the set of vectors is orthogonal, one can determine equations that define the coefficients \( \beta_k \) by forming inner products of (2) with each of the elements of the set \( \{ \vec{v}_k \} \). For example, to obtain an expression for \( \beta_j \) we form the inner product of (2) with \( \vec{v}_j \) and then simplify using properties of the inner product and the orthogonality property of the vectors \( \{ \vec{v}_k \} \);

\[
\langle \vec{b}, \vec{v}_j \rangle = \left\langle \sum_{k=1}^{N} \beta_k \vec{v}_k, \vec{v}_j \right\rangle \\
= \sum_{k=1}^{N} \langle \beta_k \vec{v}_k, \vec{v}_j \rangle \\
= \sum_{k=1}^{N} \beta_k \langle \vec{v}_k, \vec{v}_j \rangle \\
= \beta_j \langle \vec{v}_j, \vec{v}_j \rangle \\
\therefore \beta_j = \frac{\langle \vec{b}, \vec{v}_j \rangle}{\langle \vec{v}_j, \vec{v}_j \rangle}
\]
We can also consider expressing the solution, \( \tilde{x} \), of (1) as a linear combination of the vectors \( \{ \tilde{v}_k \} \),

\[
\tilde{x} = \sum_{k=1}^{N} c_k \tilde{v}_k. \tag{3}
\]

If \( \tilde{x} \) is to be a solution of the equation \( A \tilde{x} = \tilde{b} \) then we must have

\[
A \left( \sum_{k=1}^{N} c_k \tilde{v}_k \right) = \tilde{b} \\
\equiv A \left( \sum_{k=1}^{N} c_k \tilde{v}_k \right) = \sum_{k=1}^{N} \beta_k \tilde{v}_k \\
\equiv \sum_{k=1}^{N} A(c_k \tilde{v}_k) = \sum_{k=1}^{N} \beta_k \tilde{v}_k \\
\equiv \sum_{k=1}^{N} c_k A \tilde{v}_k = \sum_{k=1}^{N} \beta_k \tilde{v}_k \\
\equiv \sum_{k=1}^{N} c_k \lambda_k \tilde{v}_k = \sum_{k=1}^{N} \beta_k \tilde{v}_k \tag{4}
\]

The last simplification uses the fact that the vectors \( \{ \tilde{v}_k \} \) are eigenvectors of \( A \).

One can determine equations defining the coefficients \( c_k \) by forming inner products of (4) with each of the elements of the set \( \{ \tilde{v}_k \} \). Specifically, to obtain the \( j \)th coefficient, \( c_j \), of the solution we form the inner product of (4) with \( \tilde{v}_j \) and then simplify:

\[
\sum_{k=1}^{N} c_k \lambda_k \tilde{v}_k = \sum_{k=1}^{N} \beta_k \tilde{v}_k \\
\implies \left\langle \sum_{k=1}^{N} c_k \lambda_k \tilde{v}_k, \tilde{v}_j \right\rangle = \left\langle \sum_{k=1}^{N} \beta_k \tilde{v}_k, \tilde{v}_j \right\rangle \\
\implies \sum_{k=1}^{N} c_k \lambda_k \left\langle \tilde{v}_k, \tilde{v}_j \right\rangle = \sum_{k=1}^{N} \beta_k \left\langle \tilde{v}_k, \tilde{v}_j \right\rangle \\
\implies c_j \lambda_j \left\langle \tilde{v}_j, \tilde{v}_j \right\rangle = \beta_j \left\langle \tilde{v}_j, \tilde{v}_j \right\rangle \\
\implies c_j = \frac{\beta_j}{\lambda_j}
\]
Summary: If the matrix $A$ possesses a set of $N$ orthogonal eigenvectors then the solution to

$$A \vec{x} = \vec{b}$$

is given by an eigenvector expansion

$$\vec{x} = \sum_{k=1}^{N} \frac{\beta_k}{\lambda_k} \vec{v}_k$$

where $\lambda_k$ is the eigenvalue associated with $\vec{v}_k$ and the $\beta_k$’s are determined by the eigenvector expansion of $\vec{b}$,

$$\vec{b} = \sum_{k=1}^{N} \beta_k \vec{v}_k$$

Questions:
(Q1) Are there matrices for which we know there will always be a set of $N$ orthogonal eigenvectors?
(A1) Yes. If $A$ is symmetric, the Principal Axis Theorem guarantees that there is a set of $N$ orthogonal eigenvectors.

(Q2) If $A$ is not symmetric, can you still apply this solution procedure?
(A2) Yes, if there is a basis of orthogonal eigenvectors. If there isn’t a basis of eigenvectors, a generalization of this approach leads to procedures that use the singular value decomposition (SVD) of the matrix to create solutions of the system.