Problem 1. State the book’s definition of:
(a) A complete metric space
(b) $\limsup$ and $\lim\inf$
(c) Convergence of a series of real numbers
(d) Normed vector space; Banach space

Solution. See book.

Problem 2. Let $X$ be a metric space with a metric $\rho$. Let $x_n$ and $y_n$ be two Cauchy sequences in $X$. Show that $\lim_{n \to \infty} \rho(x_n, y_n)$ exists. Note: we do not assume that $X$ is complete.

Solution. Let $\varepsilon > 0$ be given. Choose $N$ so that for all $n, m > N$, $\rho(x_n, x_m) < \varepsilon/2$ and $\rho(y_n, y_m) < \varepsilon/2$. This is possible because the two sequences are Cauchy.

By the triangle inequality, we have that

$$\rho(x, y) + \rho(y, z) \leq \rho(x, z)$$

for all $x, y, z$; this means that

$$\rho(x, y) - \rho(x, z) \leq \rho(y, z).$$

Since $x, y, z$ are arbitrary, we can switch the roles of $y$ and $z$ and obtain that

$$|\rho(x, y) - \rho(x, z)| \leq \rho(y, z).$$

Thus:

$$|\rho(x_n, y_n) - \rho(x_m, y_m)| = |\rho(x_n, y_n) - \rho(x_n, y_m) + \rho(y_m, x_n) - \rho(x_m, y_m)|$$

$$\leq |\rho(x_n, y_n) - \rho(x_n, y_m)| + |\rho(x_n, y_m) - \rho(x_m, y_m)|$$

$$\leq \rho(y_n, y_m) + \rho(x_n, x_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

It follows that the sequence of numbers $\{\rho(x_n, y_n)\}$ is Cauchy, and so converges.

Problem 3. Let $\rho$ be the usual Euclidean metric on $\mathbb{R}$. We say that a subset $X \subset \mathbb{R}$ is closed if whenever $x_n \in X$ and $x_n \to x \in \mathbb{R}$, then $x \in X$. Show that a subset $X \subset \mathbb{R}$ is complete with respect to $\rho$ if and only if it is closed.

Assume that $X$ is closed. Let $x_n \in X$ be a Cauchy sequence. Then $x_n$ is Cauchy when regarded as a sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $x_n \to x$ in $\mathbb{R}$. Since $X$ is closed, $x \in X$ and $x_n \to x$ in $X$. Thus $X$ is complete.

Assume that $X$ is not closed. Thus there is a sequence $x_n \in X$ so that $x_n \to x$ in $\mathbb{R}$, but $x \notin X$. Since $x_n \to x$, it is Cauchy. Also, $x_n$ does not converge in $X$: if $x_n \to x'$ with $x' \in X$, it would follow by uniqueness of the limit that $x = x' \in X$, but $x \notin X$. Thus $x_n$ is a Cauchy sequence with no limit in $X$. Thus $X$ is not complete.

Problem 4. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (3 + 0.5x + 0.1y, 4 + 0.6x).$$
Show that there is a unique point \((x_0, y_0) \in \mathbb{R}^2\) with the property that \(f(x_0, y_0) = (x_0, y_0)\).

**Solution.** Let \(\rho\) be the metric on \(\mathbb{R}^2\) given by

\[
\rho((x, y), (x', y')) = \max(|x - x'|, |y - y'|).
\]

Then

\[
\rho(f(x, y), f(x', y')) = \max(0.5x + 0.1y - 0.5x' - 0.1y', 0.6x - 0.6x')
\]

\[
= \max(0.5(x - x') - 0.1(y - y'), 0.6|x - x'|)
\]

\[
\leq \max(0.5|x - x'| + 0.1|y - y'|, 0.6|x - x'|)
\]

\[
\leq \max(0.5 \max(|x - x'|, |y - y'|) + 0.1 \max(|y - y'|, |x - x'|), 0.6|x - x'|)
\]

\[
= \max(0.6 \max(|x - x'|, |y - y'|), 0.6|x - x'|)
\]

\[
= 0.6 \rho((x, y), (x', y')).
\]

It follows that \(f\) is a contraction. Since \(\mathbb{R}^2\) is complete, we can apply the Banach contraction principle to conclude that \(f\) has a unique fixed point \((x_0, y_0)\).

**Problem 5.** State and prove that Banach contraction principle.

**Solution.** See book.

**Problem 6.** Let \(\|f\|_\infty\) and \(\|f\|_1\) be norms on the space \(C[0, 1]\) of continuous functions on the interval \([0, 1]\), given by:

\[
\|f\|_\infty = \sup_{x \in [0, 1]} |f(x)|
\]

\[
\|f\|_1 = \int_0^1 |f(x)|\,dx.
\]

Show that the two norms are not equivalent.

**Solution.** If the two norms were equivalent, being a Cauchy sequence with respect to one of them would imply being a Cauchy sequence with respect to the other. We’ll show that this is not the case.

Let \(f_n\) be given as follows. For \(x \in [0, 0.5 - 1/n]\), \(f_n(x) = 0\). For \(x \in [0.5 + 1/n, 1]\), \(f_n(x) = 1\). For \(x \in (0.5 - 1/n, 0.5 + 1/n)\), \(f_n(x) = 0.5n(x - 0.5) + 0.5\). Then \(f_n \in C[0, 1]\).

Let \(f\) be given by: \(f(x) = 0\) if \(x \in [0, 0.5]\) and \(f(x) = 1\) if \(x \in (0.5, 1]\).

Then

\[
\|f - f_n\|_1 = \int_0^1 |f(x) - f_n(x)|\,dx.
\]

Since \(f(x) = f_n(x)\) outside of \((0.5 - 1/n, 0.5 + 1/n)\), we get that

\[
\|f - f_n\|_1 = \int_{0.5 - 1/n}^{0.5 + 1/n} |f_n(x) - f(x)|\,dx.
\]

Note that \(|f_n(x)| \leq 1\) and \(|f(x)| \leq 1\) for all \(x\). Thus \(|f_n(x) - f(x)| \leq 2\) for all \(x\). Hence

\[
\|f_n - f\|_1 \leq \int_{0.5 - 1/n}^{0.5 + 1/n} 2\,dx = \frac{4}{n} \to 0.
\]

It follows that \(f_n \to f\) in \(\|\cdot\|_1\). In particular, \(f_n\) are a Cauchy sequence for \(\|\cdot\|_\infty\), since this would
Problem 7. Let $A = \limsup a_n$ and $a = \liminf a_n$. Show that $A = a$ if and only if $a_n$ converges, and moreover that if this is the case, then $a_n \to a$.

_Solution._ Assume that $A = a$. The for any $\varepsilon > 0$, there is an $N$ so that for all $n > N$, one has
\[
\liminf a_n - \varepsilon < a_n < \limsup a_n + \varepsilon
\]
Since $\liminf a_n = \limsup a_n = a$ in this case, we have that for all $n > N$,
\[
a - \varepsilon < a_n < a + \varepsilon.
\]
But then
\[
|a - a_n| < \varepsilon.
\]
Thus by the definition of limit, $a_n \to a$.

Conversely, suppose that $a_n \to a$. Then for any $\varepsilon > 0$, there is an $N$ so that for all $n > N, |a - a_n| < \varepsilon$. Hence for $n > N$, we have
\[
a - \varepsilon < a_n < a + \varepsilon.
\]
It follows that for any $M > N$,
\[
a - \varepsilon \leq \inf \{a_n : n > M\} \leq \sup \{a_n : n > M\} \leq a + \varepsilon.
\]
Thus by the definition of $\liminf$ and $\limsup$, we have
\[
a - \varepsilon \leq \liminf a_n \leq \limsup a_n \leq a + \varepsilon.
\]
Since $\varepsilon > 0$ was arbitrary, it follows that $\liminf a_n = \limsup a_n = a$.

Problem 8. State and prove the comparison test.

_Solution._ See book.