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Problem 1. The picture below represents the tree of a strictly competitive game with no chance moves.

(a) Find the strategic form of the game;
(b) Find all Nash equilibria;
(c) Using Zermelo's algorithm, find all subgame-perfect equilibria.

Nash equilibria: 
(rlr, XY), where X, Y ∈ {R, L}

Subgame-perfect equilibria:
(rll, RX), where X ∈ {L, R}

Maximal score: 15.
Problem 2. The following game is being played by two players. First, player I rolls a dice. If the outcome is in the set \{1, 2, 3, 4\}, then the payoffs of player I and player II are 1 and 0 respectively. If the outcome is in the set \{5, 6\}, it is now player II’s turn to roll a dice. This time, if the outcome is in the set \{1, 2, 3, 4\}, the resulting payoffs are for player I and player II are 0 and 1 respectively. If the outcome is in the set \{5, 6\}, player I rolls the dice, and so on. Assume that both player I and player II are risk-neutral. How much should they be willing to pay to participate in this game?

The expected payoff for Player I is

\[
\frac{2}{3} \cdot 1 + \left( \frac{1}{3} \right)^2 \frac{2}{3} + \left( \frac{1}{3} \right)^4 \frac{2}{3} + \ldots = \\
= \frac{2}{3} \cdot \left( 1 + \frac{1}{9} + \left( \frac{1}{9} \right)^2 + \left( \frac{1}{9} \right)^3 + \ldots \right)
\]

Since \( S = 1 + a + a^2 + a^3 + \ldots \) is a geometric series, (with \( a = \frac{1}{9} \))

compute \( S \) as follows:

\[
S = 1 + a \cdot S \\
\Rightarrow S = \frac{1}{1-a} = \frac{1}{8/9} = \frac{9}{8}.
\]

Thus, the expected payoff for Player I is \( \frac{2}{3} \cdot \frac{9}{8} = \frac{3}{4} \). This is how much he should be willing to spend on the ticket for the game. (since he is risk-neutral).

Since the total payoff is always 1, the expected payoff for player II is \( 1 - \frac{3}{4} = \frac{1}{4} \). Since he is risk-neutral, this is how much he’ll spend.
Problem 3. (A version of Gale's Roulette). Player I begins by choosing a wheel and spinning it. While this wheel is still spinning, player II picks a wheel and also spins it. The player with the larger number wins. The numbers written on the wheels are as follows:

- wheel 1: 1 3 9
- wheel 2: 0 7 8
- wheel 3: 2 4 6

(Making a picture is probably useful).
(a) If Player I chooses wheel 1 and Player II chooses wheel 2, what are the chances that Player I will win the game?
(b) Draw a game tree for this game.
(c) Find a subgame-perfect equilibrium for this game.

In all the tables below W denotes win for the wheel whose number is listed vertically.

The probability that Player I wins is \[ \frac{5}{9}. \]

Subgame-perfect equilibrium:

Wheel 1 \begin{array}{c}
1 \\
3 \\
9
\end{array}

Wheel 2 \begin{array}{c}
0 \\
7 \\
W
\end{array}

Wheel 3 \begin{array}{c}
2 \\
4 \\
6
\end{array}
Problem 4. An open-air concert organizer is worried about the possibility of rain, which is predicted to occur with the probability $p = 0.2$ (i.e., 20%) on the day of the concert. If the day is sunny, the profits of the concert will be $100,000. However, if it rains, the profits will be only $40,000. (In case of rain, the loss of the profits is $60,000). The organizer considers buying an insurance. His utility for money function is $u(x) = \sqrt{x}$. The insurance company offers him an insurance for the full loss ($60,000) in the case of rain. Such an insurance costs $15,000. Will the organizer buy this insurance?

Solution.

We need to compare what happens if the organizer buys the insurance with what happens if he does not.

1. If he does not buy an insurance, he gets the lottery

$$L = \begin{bmatrix} 100 & 40 \\ 0.8 & 0.2 \end{bmatrix} \quad \text{with} \quad Eu(L) = 0.8\sqrt{100} + 0.2\sqrt{40} = 8 + 0.2\sqrt{40}$$

2. With insurance, he always gets $100,000 after the concert (either from the concert, if there is no rain, or from the concert + from insurance, if there is rain).

Thus, his gain in this case is

$$X = 100 - 15 = 85 \quad \text{(in thousands of dollars)}$$

With insurance, he gets the lottery

$$M = \begin{bmatrix} 85 & 85 \\ 0.2 & 0.8 \end{bmatrix} \quad \text{with} \quad Eu(M) = 0.8\sqrt{85} + 0.2\sqrt{85} = \sqrt{85}$$

We need to compare $\sqrt{85}$ and $8 + 0.2\sqrt{40}$.

Square: $85$ vs. $64 + 1.6 + 3.2\sqrt{40}$; $19.4$ vs. $3.2\sqrt{40}$; $\Rightarrow \quad 6.0625$ vs. $\sqrt{40}$. Square again $\Rightarrow 6.0625 < \sqrt{40} \Rightarrow Eu(M) < Eu(L) \Rightarrow$ will not buy insurance.
Problem 5. Two players play the children’s game of Paper-Rock-Scissors as follows. On the count 1-2-3, they have to show a hand sign representing either paper, rock, or scissors. Paper wins over Rock, but looses to Scissors. And Scissors loose to Rock. If they show the same sign, it’s a draw. Assume that in the case of a draw, each gets a 0, in all other cases, the winner gets 1 and his opponent gets −1.

(a) Draw the game tree. (Note: this is a simultaneous-move game).
(b) Find the strategic form.
(c) List all dominated strategies.

\[
\begin{array}{c|c|c|c}
 & P & R & S \\
\hline
 I & 0 & -1 & -1 \\
\hline
 P & 0 & -1 & -1 \\
 R & -1 & 0 & -1 \\
 S & -1 & -1 & 0 \\
\end{array}
\]

(c) There are no dominated strategies.