PRACTICE PROBLEMS FOR MIDTERM 1 (MATH 115AH)

**Problem 1.** Prove that if $m$ is not a prime number, then $\mathbb{Z}_m$ is not a field.

**Problem 2.** Let $p$ be a prime number. Compute the multiplicative inverse of $(p + 1)/2$ in the field $\mathbb{Z}_p$.

**Problem 3.** Let $p$ be a prime. What is the number of elements in the field $\mathbb{Z}_p$? What is the number of elements in the vector space $(\mathbb{Z}_p)^n$? What is the dimension of $\mathbb{Z}_p^n$ over $\mathbb{Z}_p$?

**Problem 4.** Let $P_3(\mathbb{R})$ be the vector space of polynomials of degree at most 3 with real coefficients. Let 
\[ W = \{ f \in P_3(\mathbb{R}) : f(0) = 2f'(0) \} . \]

(a) Show that $W$ is a subspace of $P_3(\mathbb{R})$.
(b) Find a basis of $W$.
(c) Find the dimension of $W$.

**Problem 5.** Let $\{v_1, \ldots, v_n\}$ be a linearly independent set of vectors in $V$. Let $\{u_1, \ldots, u_m\}$ be another linearly independent set of vectors in $V$. Suppose that $n < m$. Show that the vectors $\{v_1, \ldots, v_n\}$ can not form a basis of $V$.

**Problem 6.** Let $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ be given by 
\[ T(f(x)) = (f(0), f'(0)) \]
and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be given by 
\[ U(a,b) = (a + b, a - b) \]
Let $\alpha = \{1, x, x^2\}$ be a basis of $P_2(\mathbb{R})$ and $\beta = \{(1,0), (0,1)\}$ be a basis of $\mathbb{R}^2$. Compute the matrix $[U \circ T]_\alpha^\beta$ of the composition of $T$ and $U$.

**Problem 7.** True or False. For each of the following statements, indicate if it is true or false. This problem will be graded as follows: you will receive $n$ points for a correct answer, 0 points if there is no answer, and $-n$ points if the answer is wrong.

1. The set of polynomials of degree exactly 3 is not a vector space.
2. The set \( W = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + a_2 + a_3 = 1\} \) is a subspace of \( \mathbb{R}^3 \).

3. A subset of a linearly dependent set is linearly dependent.

4. If \( \dim(V) = n \), any generating set of \( V \) contains at least \( n \) vectors.

5. If a set of vectors \( S \) generates vectors space \( V \), any vector in \( V \) can be written as a linear combination of vectors in \( S \) in a unique way.

6. A linear transformation \( T : V \to V \) carries linearly independent subsets of \( V \) into linearly independent subsets of \( V \).

7. In a vector space \( V \) the equality \( av = aw \) for \( a \in F, v, w \in V \) implies that \( v = w \).

8. If \( W_1 \) and \( W_2 \) are subspaces of a vector space \( V \), then the intersection \( W_1 \cap W_2 \) is a subspace iff \( W_1 \subseteq W_2 \) or \( W_2 \subseteq W_1 \).

9. If \( S_1 \subseteq S_2 \) are subsets of a vector space \( V \) and \( S_1 \) is linearly independent, then \( S_2 \) is also linearly independent.

10. For any \( a \in \mathbb{R} \), the set of real-values functions \( W = \{f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) : f(a) = 0\} \) is a subspace of the vector space \( \mathcal{F}(\mathbb{R}, \mathbb{R}) \) of all real-values functions on the line.

11. If \( S \) is a subset of a vector space \( V \), then \( \text{span}(S) \) is the intersection of all subspaces of \( V \) that contain \( S \).

12. If a vector space \( V \) is generated by a finite set \( S \), then some subset of \( S \) is a basis of \( V \).

13. The dimension of the space \( M_{2 \times 3}(F) \) over \( F \) is 5.

**Problem 8.** Let \( V \) be the set of all pairs \((x, y)\), where \( x \) is a real number and \( y \) is a positive real number. Define addition on \( V \) by

\[
(x, y) + (x', y') = (x + x', y \cdot y')
\]

and scalar multiplication by

\[
c(x, y) = (cx, y^c)
\]

for \( c \in \mathbb{R} \).

Let \( \overrightarrow{0} = (0, 1) \).

1. Show that \( V \) is a vector space with these operations.
2. Find the dimension of \( V \).
3. Let \( n \) be the dimension of \( V \) which you found in part 2 of this problem. Construct an explicit isomorphism from \( V \) to \( \mathbb{R}^n \).
Problem 9. Let $W_1$ and $W_2$ be subspaces of a vector space $V$. Prove that the following conditions are equivalent:

1. Each vector $x$ in $V$ can be uniquely written in the form $x = x_1 + x_2$, where $x_1 \in W_1$ and $x_2 \in W_2$;
2. $W_1 \cap W_2 = \{0\}$ and $V = W_1 + W_2$, where $W_1 + W_2 = \{w_1 + w_2, w_1 \in W_1, w_2 \in W_2\}$.

(If either of these conditions is satisfied, $V = W_1 \oplus W_2$).

Problem 10. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$. Prove that $T$ is an isomorphism and find $T^{-1}$.

Problem 11. Let $T : M_{2 \times 2}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ be a linear transformation given by $T(A) = A^t$, the transpose of $A$. Let $U : M_{2 \times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation given by $U \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + 2bx + 3cx^2$.

Let $\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ be a basis of $M_{2 \times 2}(\mathbb{R})$ and $\beta = \{1, x, x^2\}$ be a basis of $P_2(\mathbb{R})$. Find the matrix $[U \circ T]_\alpha^\beta$ of the composition of linear transformations $T$ and $U$.

Problem 12. Prove that vectors $(a, b)$ and $(c, d)$ in $\mathbb{C}^2$ are linearly dependent iff $ad = bc$.

Problem 13. Let $V$ be a vector space, $\dim(V) = 4$. Show that if $W_1, W_2$ are both subspaces of dimension 3, then there is a non-trivial intersection of $W_1$ and $W_2$.

Problem 14. Find a linear functional $f$ on the vector space $P_3(\mathbb{R}) = \{p(t) = \sum_{i=0}^{3} a_i \cdot t^i | a_i \in \mathbb{R}\}$ such that

- $f(1) = 1$
- $f(x^3 + 2x) = 1$
- $f(x^3 + 3x^2) = 2$
- $f(x^2 + 5x) = 6$

How many linearly independent linear functionals with this property can you find?

Problem 15. Let $v$ and $u$ be vectors in $V$ such that $\{v\}^0 = \{u\}^0 \subseteq V'$, where $S^0$ denotes the annihilator of a set $S \subseteq V$. Prove that $v = ku$ for some $k \in F$.

Problem 16. Prove that for any $v \in V$, $\dim(V) = n$, there exists a linear functional $f$ such that $f(v) = \alpha$ for a given $\alpha \in F$. How many linearly independent linear functionals with this property can you find?
Problem 17. Is there a finite-dimensional vector space $V$ with a subspace $W$ such that $W$ has a unique complement in $V$? If yes, give an example. If not, explain why it cannot exist.

Problem 18. a) For a finite-dimensional vector space $V$, is it true that a linear transformation is onto if and only if it is one-to-one? Prove that it is true, or give a counterexample.

b) For a vector space of sequences $\mathbb{R}^\infty$ with entries in $\mathbb{R}$, give an example of a linear transformation on $\mathbb{R}^\infty$ which is onto, but not one-to-one, and another linear transformation which is one-to-one but not onto.

In addition, please review also the problems related to invariant subspaces and projections, rank and null space, similar to the last homework assignment.