Solution:

1. Find the cosine of the angle between the diagonal of a cube and the diagonal of one of the cube’s faces, when the two diagonals have the same initial point.

Solution: The angle will not change if we replace \((x, y, z)\) by \((ax, ay, az)\) for any \(a > 0\). Hence we may assume the cube has edges of unit length. We can also assume \((0, 0, 0)\) is one vertex of the cube and we can assume the edges are parallel to the coordinate axes.

Then the diagonal is the vector \(\vec{v} = \langle 1, 1, 1 \rangle\) and the diagonal of one face is \(\vec{w} = \langle 1, 1, 0 \rangle\). These vectors have lengths \(\sqrt{3}\) and \(\sqrt{2}\) and dot product \(\vec{v} \cdot \vec{w} = 2\). If \(\theta\) is the angle, then

\[
\cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|} = \frac{2}{\sqrt{6}}.
\]

The answer is the same for the diagonals of the other faces.