A Primal-Dual Projected Gradient Algorithm for Efficient Beltrami Regularization

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Introduction

Inverse problems in imaging and computer vision are typically addressed as data-fidelity optimization problems, where data regularizers are included to render the problem well-posed. While H1 regularization is known to produce overly smooth reconstructions, the TV model is feature-preserving but introduces staircasing artifacts. The geometrically derived Beltrami framework [SKM98] offers an ideal compromise. One of the limiting factors of the Beltrami regularizers is the lack of really efficient optimization schemes. Here, we start from one of the most efficient TV-optimization methods, primal-dual projected gradients [ZWLC10], and apply it to the Beltrami functional.

Beltrami framework

The Beltrami embedding defines a diffeomorphic map as follows:

\[ X : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R} \times \mathbb{R} \]

\( (x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_n, l_1(x_1, \ldots, x_n)) \)

This amounts to seeing the function \( f \) as a non-flat surface, \( M \) embedded in a higher dimensional space \( \mathbb{R} \times \mathbb{R} \), much like a topographic map corresponds to a three-dimensional surface in the real world. The relative scaling between these components is arbitrary and we choose a metric tensor \( h_0 \) that incorporates tuning of the aspect ratio [SKM98], by a factor \( \beta \):

\[ h_0 = \text{diag}(1, \ldots, 1, \beta^2) \]

Now we pullback the metric tensor to get \( g_{\mathbb{R}^2} \) on the original image domain manifold \( \Omega \):

\[ g_{\mathbb{R}^2} = \beta^2 g_{\mathbb{R}^n} \]

The determinant of this metric tensor is

\[ g = \det g_{\mathbb{R}^2} = 1 + \beta^2|\nabla f|^2 \]

and the hypersurface energy functional corresponding to the image reads

\[ \mathcal{E}_{\mathbb{R}^2}(f, \beta^2) := \int_0^1 \sqrt{1 + \beta^2|\nabla f|^2} \, dx. \]

Legendre-Fenchel Transform

The convex conjugate of a function \( f \) is the function \( f^* \) defined by:

\[ f^*(s) = \sup_{x \in \mathbb{R}^n} \{ sx - f(x) \} \quad \forall s \in \mathbb{R}^n \]

Let \( F : W \rightarrow \mathbb{R} \) be a closed and convex functional on the set \( W \), \( G \) a closed and convex functional on the set \( V \) and let \( K : V \rightarrow W \) be a continuous linear operator. Then we have the following equivalence:

\[ \min_{v \in V} \left\{ F(Kv) + G(v) \right\} \quad \text{and} \quad \max_{u \in W} \left\{ f(u) - \langle u, K^* \rangle \right\} \]

Beltrami Denoising Algorithm

Initialize \( f = b, \varphi = 0 \).

Repeat:

\[ \frac{\partial^{k+1}}{\partial t} \left( f - \varphi \right) = \left( 1 - \lambda \right) f + \rho \partial_t \nabla f \sqrt{\beta^2 + |\nabla f|^2} \]

until convergence

Performance Comparison

Typically, Beltrami results are

- 0.1 dB better (SNR)
- 0.005 better (SSIM)

Most algorithms take tens or hundreds of iterations to converge and only a 40% less CPU time compared to similarly structured ROF algorithm.

Conclusions and Outlook

We achieve better performance than ROF denoising for the basic grayscale problem, then extend the method to more involved problems such as inpainting, deconvolution, and the color case. With the proposed primal-dual projected gradients optimization algorithm, the benefits of the geometric Beltrami regularizer become available at almost no extra computational cost. Future work will focus on convex relaxations of the true multichannel Beltrami model.

References


A MATLAB implementation of the proposed algorithms will be made available at http://www.math.ucla.edu/~zosso/code.html.