FORCING EXERCISES – DAY 7

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. Suppose that $M$ is a transitive set and $\alpha \in M$. Then

\[ \langle M, \in \rangle \models \text{“} \alpha \text{ is a cardinal} \text{”,} \]

if $\alpha$ is truly a cardinal. (NB. The converse is false.)

Exercise 2. Let $X$ be a countable elementary substructure of some $H_\theta$, where $\theta$ is regular and uncountable. (Note that $X$ is not transitive!)

(a) Suppose that $A \subseteq X$ and $H_\theta \models \text{“} A \text{ is countable} \text{”}$. Show that then $X \models \text{“} A \text{ is countable} \text{”}$ and $A \subseteq X$.

(b) Show that $X \cap \omega_1 \in \omega_1$, i.e., $X \cap \omega_1$ is a countable ordinal.

(c) Define a countable set of ordinals that is not a member of $X$.

(d) Show that $\omega_1 \in X$ if $\theta > \omega_1$.

(e) Describe a subset of $\omega$ that is not a member of $X$.

Exercise 3. A cardinal $\kappa$ is called inaccessible if it is a regular limit cardinal; it’s called strongly inaccessible if in addition $2^\lambda < \kappa$ for every $\lambda < \kappa$.

(a) If $\kappa$ is strongly inaccessible, then $V_\kappa \models \text{ZFC}$. Sketch a proof.

(b) If $V_\kappa \models \text{ZFC}$, does it follow that $\kappa$ is an inaccessible cardinal?

Exercise 4. A theory $T$ is finitely axiomatizable if there is a finite set $A \subseteq T$ such that $A \models T$; that is, if $A \models \sigma$ for every sentence $\sigma \in T$. Show that ZFC is not finitely axiomatizable.

Exercise 5. Suppose $a < 2^{\aleph_0}$. Prove that there is an open dense set $D \subseteq [\omega]^\omega$ that doesn’t include a mad family of size $a$ (though it does include a mad family).

Exercise 6. Instead of finding monochromatic sets, you might try looking for polychromatic ones. Suppose that $[\omega]^2$ is colored (using infinitely many colors) in a way that is $k$-restricted, meaning that each color is used at most $k$ times. Prove that there is an infinite fully polychromatic set $X$, i.e., a set $X$ such that on pairs of elements of $X$ each color is used at most once.

Date: 19 July 2016.