Exercise 1. Express the axioms for a dense linear order without endpoints as sentences in the language of a single binary relation <.

Exercise 2. The language for set theory consists of a single binary relation symbol, \(\in\). Consider the following as \(\{\in\}\)-structures. Which axioms of ZFC does each satisfy?
(a) \((\mathbb{Z}, <)\)
(b) \((\omega, \in)\)
(c) \((\omega_1, \in)\)
(d) \((V_\omega, \in)\)

Exercise 3. Prove that if ZFC is consistent, then there is an illfounded model of ZFC, i.e., a model \((M, E) \models \text{ZFC}\) such that \(E = \in^M\) is an illfounded binary relation on \(M\).

Exercise 4. Prove that \(2^\omega\) is a compact metric space. Deduce that \(2^{[\omega]^2}\) is compact. Use this fact to derive the following finitary version of Ramsey’s theorem from the infinitary one:

\textbf{Theorem.} For every \(k < \omega\) there is an \(n < \omega\) such that every coloring \([n]^2 \to 2\) has a monochromatic set of size \(k\).\(^1\)

Exercise 5. Recall that a filter \(F\) on \(\omega\) is a Ramsey filter if every 2-coloring of \([\omega]^2\) has a monochromatic set in \(F\). Prove that a Ramsey filter must be a nonprincipal ultrafilter.

Exercise 6. If \(A \subseteq [\omega]^\omega\) is a countably infinite almost-disjoint family and there is a mad family of size \(\kappa\), then \(A\) extends to a mad family of size \(\kappa\). Prove this.

\textbf{Definition.} A family \(I \subseteq [\omega]^\omega\) is called \textbf{independent} if the intersection of any finitely many members of \(I\) with the complements of any finitely many other members of \(I\) is infinite.

Exercise 7. Prove that there is an independent family of size \(2^{\aleph_0}\). Prove that there are \(2^{2^{\aleph_0}}\) ultrafilters on \(\omega\) (using the independent family or otherwise).

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\(^1\)NB. The least \(n\) suitable for \(k = 5\) is not known! It’s known to be between 43 and 49, though.