FORCING EXERCISES – DAY 3

Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. Prove the following:
(a) If $A$ is a mad family, then its downward closure
    $\{ x \in [\omega]^{\omega} : (\exists y \in A) x \subseteq^* y \}$
    is a dense open family.
(b) Every dense open family includes a mad family.

(For the definition of the tower number $t$, see yesterday’s problem set.)

Exercise 2. Suppose that $\kappa$ is a cardinal and that $\aleph_0 \leq \kappa < t$. Prove
    that $2^\kappa = 2^{\aleph_0}$.
    Deduce that $t \leq cf(2^{\aleph_0})$.

Exercise 3. Prove that MA implies $t = 2^{\aleph_0}$. Deduce (again) that MA
    implies that $2^{\aleph_0}$ is regular.

Definition. The distributivity number or scattering number $h$
    is the smallest number of dense open families $\subseteq [\omega]^{\omega}$ with empty intersection.

In prep exercise #19, you proved that $h$ is uncountable. (Make sure you know how to do that one!)

Exercise 4.
(a) Prove that $h$ is a regular cardinal.
(b) Prove that $t \leq h$.

Exercise 5. Let $3$ be the least size of a family $F \subseteq [\omega]^{\omega}$ such that every dense open family $D \subseteq [\omega]^{\omega}$ meets $F$.
(a) Prove that $3 = 2^{\aleph_0}$.
(b) Prove that $h \leq cf(2^{\aleph_0})$. (Hint: I put this here for a reason.)

Date: 13 July 2016.