Exercise 0. Make sure you know how to do the problems from yesterday.

Exercise 1. A poset $P$ is kinda homogeneous if for any $p, q \in P$ there is an automorphism $i : P \to P$ such that $i(p)$ and $q$ are compatible.

(a) Show that if $I$ is an infinite set and $J$ is any nonempty set then the poset $\text{Fn}(I, J) = \{p : p \text{ is a function and } \text{dom}(p) \subseteq I \text{ is finite and } \text{ran}(p) \subseteq J\}$, ordered by reverse inclusion, is kinda homogeneous.

(b) Suppose that $P$ is kinda homogeneous and $G$ is $P$-generic. Let $x \in M$. Show that if $M[G] \models \phi[x]$, then in fact $1_P \models \phi(x)$.

(c) Conclude that if $P$ is kinda homogeneous then for any $P$-generic filters $G$ and $H$, the generic extensions $M[G]$ and $M[H]$ have the same first-order theory (that is, they’re elementarily equivalent).

Definition. If $M \subseteq N$ are countable transitive models of $\text{ZFC}$, then we say $f \in \omega^\omega \cap N$ is a dominating real over $N$ if $f$ dominates every function $\omega \to \omega$ that belongs to $M$.

Exercise 2.

(a) Prove that the Cohen-real forcing (from Weekend 2 #4 — conditions are finite partial functions $\omega \to \omega$) doesn’t add any dominating reals over the ground model. (Caution: there’s more to this than proving that the generic real is not a dominating real, but that might be a good place to start.)

(b) Prove that if $A \in M$ is a mad family (in $M$) and $d \in N$ is a dominating real, then $A$ is not mad (i.e., not maximal) in $N$. (Hint: look at the proof of $b \leq a$.)

Exercise 3 (Weekend 2, #7). Suppose that $P$ is a separative poset in $M$. Show that the set

$\{\tau \in M^P : \tau[G] = \emptyset \text{ for every } M\text{-generic filter } G\}$

is an element of $M$, but the set

$\{\tau \in M^P : \tau[G] = \emptyset \text{ for some } M\text{-generic filter } G\}$

is not. (Hint: Think about $P$-rank.)

Exercise 4. ($*$) Assume $\text{CH}$. Prove that there is an infinite mad family $A \subseteq [\omega]^\omega$ that remains mad in $M[G]$ for any $G$ that is Cohen-generic over $M$. (The poset here is the poset of finite partial functions $\omega \to \omega$ again.)

Email Zach if you have questions or need a hint.

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