Exercise 1. Is there $A \subseteq B$ such that $A$ is isomorphic to $B$ but the inclusion map is not an elementary embedding?

Exercise 2. Define theories for vector spaces over $\mathbb{Q}$ and for metric spaces. (See Exercise 2 from Day 2.)

Exercise 3. (a) An $\mathcal{L}$-theory $T$ is called **absolutely categorical** if it is satisfiable and any two models of $T$ are isomorphic. Show that if $T$ is absolutely categorical, then every model of $T$ is finite.

(b) Give an example of a *finite* $\mathcal{L}$-theory $T$ all of whose models are infinite.

(c) Show that there is an $\mathcal{L}_{\text{group}}$-sentence $\phi$ such that $M \models \phi \iff M \cong \mathbb{Z}/2\mathbb{Z}.$

(d) More generally, let $\mathcal{L}$ be a finite language and $A$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence $\phi$ such that for any $\mathcal{L}$-structure $B$,

$$B \models \phi \iff B \cong A.$$ 

In particular,

$$B \equiv A \iff B \cong A.$$

Exercise 4. Let $\mathcal{C}_{\text{exp}} = (\mathbb{C}, 0, 1, +, \cdot, \exp)$, where $\exp : \mathbb{C} \to \mathbb{C}$ is the usual exponentiation $z \mapsto e^z$. Show that $\mathbb{Z}$ is definable in $\mathcal{C}_{\text{exp}}$. Conclude that $\mathbb{N}$ is too.

Exercise 5. (a) Let $A \subseteq B$ and assume that for any finite $S \subseteq A$ and any $b \in B$ there exists an automorphism $f$ of $B$ that fixes $S$ pointwise (i.e., $f(a) = a$ for every $a \in S$) and $f(b) \in A$. Show that $A \prec B$.

(b) Give an example to show that the converse of part (a) can fail. That is, find structures $A$ and $B$ such that $A \prec B$ but it isn’t the case that for every finite set $S$ and every $b \in B$ there’s an automorphism.

Exercise 6. Show that $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. (Hint: previous problem) Conclude that $(\mathbb{Q}, <) \equiv (\mathbb{R}, <)$ but $(\mathbb{Q}, <) \not\equiv (\mathbb{R}, <)$. (Also, see Problem 1(c) from Day 2.)

Exercise 7. Let $\mathcal{L}$ be a language whose only symbol is a unary function symbol $f$. For each $\mathcal{L}$-sentence $\sigma$, let $\text{Spec}(\sigma)$ be the finite spectrum of $\sigma$, i.e., the set of all cardinalities of finite models of $\sigma$. Let $a \geq 0$ and $b \geq 1$ be integers. Give an $\mathcal{L}$-sentence $\sigma$ such that

$$\text{Spec}(\sigma) = \{a + bn : n \in \mathbb{N}\}$$
Exercise 8.

(a) Let $\mathcal{L} = \{E\}$, where $E$ is a binary relation symbol, and let $\phi$ be the $\mathcal{L}$-sentence asserting that $E$ is an equivalence relation whose classes all have exactly 2 elements. Show that the finite spectrum of $\phi$ is all positive even integers.

(b) For each of the following subsets of $\mathbb{N}$, show that it is the finite spectrum of some sentence $\phi$ in some language $\mathcal{L}$:

(i) $\{2^n : n \in \mathbb{N}\}$
(ii) $\{2^n 3^m : n, m \in \mathbb{N}\}$
(iii) $\{n^2 : n \in \mathbb{N}\}$
(iv) $\{n \in \mathbb{N} : n \text{ is composite}\}$
(v) $\{p^n : p \text{ is prime and } n \in \mathbb{N}\}$
(vi) $\{p : p \text{ is prime}\}$.
(vii) $\{\binom{n}{3} : n \in \mathbb{N}\}$.

Open Question: Must the complement of a finite spectrum be a finite spectrum too?