Exercise 1.
(a) Find a sentence true in the structure \((\mathbb{N}, <)\) but not in the structure \((\mathbb{Z}, <)\).
(b) Find a sentence true in the structure \((\mathbb{Q}, <)\) but not in the structure \((\mathbb{Z}, <)\).
(c) Can you think of a sentence true in \((\mathbb{Q}, <)\) but false in \((\mathbb{R}, <)\)?

Exercise 2. Define appropriate languages for
(a) vector spaces over \(\mathbb{Q}\) (better yet, over any field);
(b) metric spaces.
Is there a language suitable to describe topological spaces? Explain your answer.

Exercise 3. Suppose that \(f : \mathbb{R} \to \mathbb{R}\) is a function. Prove that the set of points at which \(f\) is continuous is definable in the structure \((\mathbb{R}, <, f)\).

Exercise 4. A formula is called universal if it is of the form \(\forall \vec{x} \psi\), where \(\psi\) is quantifier-free. Existential formulas are defined similarly (with \(\exists\) instead of \(\forall\)). Let \(A \subseteq B\) be \(L\)-structures, \(\phi = \phi(v)\) be an \(L\)-formula, and \(a \in A^n\). Show that
(a) If \(\phi\) is quantifier-free, then \(A \models \phi(a) \iff B \models \phi(a)\).
(b) If \(\phi\) is universal, then \(B \models \phi(a) \implies A \models \phi(a)\).
(c) If \(\phi\) is existential, then \(A \models \phi(a) \implies B \models \phi(a)\).

Exercise 5. Let \(S\) be a finite set of vectors in \(\mathbb{Q}^d\). Show that \(S\) is linearly independent over \(\mathbb{Q}\) iff it is linearly independent over \(\mathbb{R}\). (Hint: previous exercise)

Exercise 6. Show that any definable set in \(\mathbb{N}\) is 0-definable.

Exercise 7. Let \((A, 0, +)\) be a non-trivial abelian group, and let \(R^A\) be the relation on \(A\) defined by
\[
R^A(x, y, z, w) \iff x + y = z + w
\]
Show that the addition map \((x, y) \mapsto x + y\) : \(A \times A \to A\) is definable in the structure \(A = (A, R^A)\) from the parameter 0, but is not definable without parameters.

Exercise 8. Determine whether the following are 0-definable.
(a) The set \(\mathbb{N}\) in \((\mathbb{Z}, +, \cdot)\). (Hint: You may need a nontrivial fact from elementary number theory.)
(b) The set of nonnegative numbers in \((\mathbb{Q}, +, \cdot)\).
(c) The set of nonnegative numbers in \((\mathbb{Q}, +)\).
(d) The set of positive numbers in \((\mathbb{R}, <)\).
(e) The function \(\max(x, y)\) in \((\mathbb{R}, <)\).
(f) The function \(\text{average}(x, y)\) in \((\mathbb{R}, +, \cdot)\).
(g) The function \(\text{average}(x, y)\) in \((\mathbb{R}, <)\).
(h) 2 in \((\mathbb{R}, +, \cdot)\).
(i) The relation \(d(x, y) \leq 2\) in a graph, where \(d\) denotes the distance function.
(j) The relation \(d(x, y) = 2\) in a graph (as above).

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Exercise 9. Suppose that \((X, <)\) and \((Y, <)\) are countably infinite linear orders that are dense, meaning that between any two distinct points there is a third point. (If \(x < y\) then there is \(z\) in the interval \((x, y)\).) Suppose also that there is no minimum or maximum in either of these orderings. Prove that \((X, <)\) and \((Y, <)\) are isomorphic.