Exercise 1. Prove that the adjective “total” in the definition of “r.e.” is unnecessary. That is, prove that if $A \subseteq \mathbb{N}$ is the range of a partial recursive function, then $A$ is r.e., meaning that $A$ is empty or the range of a total recursive function.

Exercise 2.
(a) Check that the set $\mathbb{N}[X]$ of polynomials is the underlying set (in a natural way) of a model of Robinson’s $Q$. Deduce that $Q$ doesn’t prove that every element is even or odd.
(b) Find a model of $Q$ in which there exists an $x$ with $S(x) = x$.
(c) Find a model of $Q$ in which addition is noncommutative.
(d) Find a model of $Q$ in which multiplication is noncommutative.

Exercise 3. Let $A \subseteq \mathbb{N}$. Prove that the following are equivalent:
(i) $A$ is r.e.
(ii) $A = W_e$ for some code $e$.
(iii) $A$ is the range of an injective total recursive function.

Exercise 4. Prove that $A \subseteq \mathbb{N}$ is recursive iff $A$ is finite or the range of an increasing total recursive function.

Exercise 5. Prove that every infinite r.e. set has an infinite recursive subset.

Exercise 6. Find a set that is Turing-reducible to $K$ (i.e., $\leq_T K$) but isn’t r.e.

Exercise 7.
(a) Use a previous exercise or argue directly that none of the following sets is recursive:

   (i) $\{ e \in \mathbb{N} : W_e = \emptyset \}$
   (ii) $\{ e \in \mathbb{N} : W_e \neq \emptyset \}$
   (iii) $\{ e \in \mathbb{N} : W_e = \mathbb{N} \}$
   (iv) $\{ e \in \mathbb{N} : W_e \neq \mathbb{N} \}$
   (v) $\{ e \in \mathbb{N} : |W_e| = 1 \}$
   (vi) $\{ e \in \mathbb{N} : |W_e| \leq 4 \}$
   (vii) $\{ e \in \mathbb{N} : |W_e| \geq 4 \}$
   (viii) $\{ e \in \mathbb{N} : W_e \text{ is finite} \}$

(b) Which of the sets listed in part (a) are r.e.? Prove your answers.

Exercise 8. Prove that the set $\{ e \in \mathbb{N} : \forall x \{e\}(x)\downarrow \}$ of codes of total recursive functions isn’t even r.e.

Date: 6 July 2015.
Exercise 9. We say that two disjoint sets $A, B \subseteq \mathbb{N}$ are **recursively inseparable** if there is no recursive set $R$ such that $A \subseteq R$ and $B \subseteq R^c$. Show that the sets

$$A = \{ e : \{e\}(e) \downarrow \text{ and } \{e\}(e) = 0 \}$$

and

$$B = \{ e : \{e\}(e) \downarrow \text{ and } \{e\}(e) = 1 \}$$

are recursively inseparable.

Exercise 10. Show that the recursive version of König’s lemma fails: there is an infinite recursive binary tree with no infinite recursive branch.