**Exercise 1.** For the purposes of this problem, you are allowed to assume Gödel’s Second Incompleteness theorem, i.e. $\text{PA} \not\vdash \text{Con}(\text{PA})$, where $\text{Con}(\text{PA})$ is a sentence in the language of arithmetic encoding “$\text{PA}$ is consistent” in the natural way. ($\text{Con}(\text{PA}) \equiv \lnot \text{Provable}_{\text{PA}}(\ulcorner \bot \urcorner)$.)

(a) Prove that there is a nonstandard model $\mathcal{M}$ of $\text{PA}$ and an a nonstandard element $a$ of the underlying set of $\mathcal{M}$ which is definable.

(b) Prove that there is a nonstandard model $\mathcal{M} \models \text{PA}$ without a proper elementary substructure.

**Exercise 2.** Prove that there is a partial recursive function which has no total recursive extension, i.e., there is $e \in \mathbb{N}$ such that there is no total recursive $f \supseteq \varphi_e$.

**Exercise 3.** Recall that a set $A \subseteq \mathbb{N}$ is $\Pi^0_2$ iff we may write $A(x) \iff \forall y \exists z R(x, y, z)$ for some recursive $R \subseteq \mathbb{N}^3$. A set $B \subseteq \mathbb{N}$ is called $\Pi^0_2$-complete iff $B$ is $\Pi^0_2$ and for all $\Pi^0_2$ sets $A$, $A \leq_1 B$, i.e. there exists a 1-1 total recursive function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $x \in A \iff f(x) \in B$. Prove that $\text{Tot} = \{e \in \mathbb{N} : \varphi_e \text{ is total}\}$ is $\Pi^0_2$-complete.

The following two problems make use of Kleene’s Second Recursion Theorem, which states that if $f(e, \bar{x})$ is a partial recursive function then there exists $e_0 \in \mathbb{N}$ such that $\varphi_{e_0}(\bar{x}) = f(e_0, \bar{x})$ for all $\bar{x}$.

**Exercise 4.** Suppose that $f$ is a total recursive function. Prove or give a counter-example to each of the following:

(a) There is an $e$ such that $W_{f(e)} = \{e\}$.

(b) There is an $e$ such that $W_e = \{f(e)\}$.

**Exercise 5.** Let $f(e)$ be a partial recursive function such that for all $e$,

$$W_e = \emptyset \implies f(e) \downarrow$$

Prove that there is some $m$ such that $W_m = \{m\}$ and $f(m) \downarrow$.

**Exercise 6.** For $A \subseteq \mathbb{N}^2$ let $A_a = \{b \in \mathbb{N} : (a, b) \in A\}$.

(a) Let $A$ be recursively enumerable and suppose $n \in \mathbb{N}$ is such that $|A_a| = n$ for all $a \in \mathbb{N}$. Show that $A$ is recursive.

(b) For every pair $n > m$ of natural numbers, give an example of an r.e. set $A$ such that for all $a A_a$ has size $n$ or size $m$, but $A$ is not recursive.

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Exercise 7. Instead of finding monochromatic sets, you might try looking for polychromatic ones. Suppose that \([\mathbb{N}]^2\) is colored (using infinitely many colors) in a way that is \(k\)-bounded, meaning that each color is used at most \(k\) times. Prove that there is an infinite fully polychromatic set \(X\), i.e., a set \(X\) such that on pairs of elements of \(X\) each color is used at most once.

Exercise 8. This exercise gives a different (possibly more tangible) way to think about the non-axiomatizability of wellfoundedness. (See #5 from Tuesday’s set.) For \(f, g: \mathbb{N} \to \mathbb{N}\) define \(f \leq^* g\) if \(\{n \in \mathbb{N} : f(n) > g(n)\}\) is finite. That is, \(f \leq^* g\) means that \(f(n) \leq g(n)\) for cofinitely many \(n\). Say \(f <^* g\) if \(f(n) < g(n)\) for cofinitely many \(n\). (Note that \(f <^* g\) doesn’t simply mean that \(f \leq^* g\) and \(f \neq g\).)

(a) Find a sequence \(f_0, f_1, \ldots\) of functions \(\mathbb{N} \to \mathbb{N}\) that is \(<^*\)-decreasing: \(f_0 >^* f_1 >^* f_2 >^* \cdots\).

(b) Let \(A = (\mathbb{N}, <)\) be the ordered structure of natural numbers. Let \(U\) be a nonprincipal ultrafilter on \(\omega\) and form the ultrapower \(A^{\omega}/U\). (Recall that this means the ultraproduct of structures \(M_i\) where \(M_i = A\) for every \(i\).) Use your answer to part (a) to give an explicit strictly decreasing sequence in the ultrapower \((A^{\omega}/U, <^{A^{\omega}/U})\). Of course, by Loš’s theorem this ultrapower is elementarily equivalent to \(A\), so deduce again that the class of wellorders is not axiomatizable.