This is Jech’s proof. Write \([X]^\lambda\) for the set of subsets of \(X\) of cardinality \(\lambda\).

We use choice liberally, though apparently there is a sensible statement and proof of Erdős–Rado without choice (see http://www.dpmms.cam.ac.uk/~tf/erdosrado.pdf).

**Theorem 1** (Erdős–Rado). \((\beth_n)^+ \rightarrow (\aleph_1)^0_n+1\).

**Proof.** The proof is by induction on \(n\) (notice the case \(n = 0\) simply asserts that \(\aleph_1\) is regular). Put \(\kappa = (\beth_n)^+\) and let \(f\) be an \(\aleph_0\)-colouring of \([\kappa]^n\). Our task is to exhibit a subset of \(\kappa\) of cardinality \(\aleph_1\) that is monochromatic for \(f\).

For \(x \in \kappa\) write \(f_x\) for the colouring of \([\kappa]^n\) given by

\[f_x\{y_1, \ldots, y_n\} = f\{x, y_1, \ldots, y_n\} = \text{colour of } \{x, y_1, \ldots, y_n\}\]

that is, when \(f\{x, y_1, \ldots, x_n\} = f\{z, y_1, \ldots, z_n\}\) holds for all \(x_1, \ldots, x_n \in X\). Say a set \(A \subset \kappa\) produces \(f\)-indistinguishables if for every \(a \in \kappa \setminus A\) and every \(C \subset A\) of cardinality \(\leq (\beth_{n-1})^+\), there is an element \(y \in A \setminus C\) such that \(y\) and \(a\) are \(f\)-indistinguishable on \(C\).

**Claim.** There is an \(A \subset \kappa\) of cardinality \(\beth_n\) that produces \(f\)-indistinguishables.

**Proof.** We will construct \(A\) as the limit of an ascending sequence \(\langle A_\alpha : \alpha < (\beth_{n-1})^+ \rangle\).

Pick your favourite set \(A_0 \subset \kappa\) of cardinality \(\beth_n\). Temporarily fix a \(C \subset A_0\) of cardinality \(\leq (\beth_{n-1})^+\). The number of distinct functions of the form \(f_x|_{[C]^n}\) is at most

\[|\beth_{n-1}\aleph_0| = \aleph_0\beth_{n-1} = 2^{\beth_{n-1}} = \beth_n,\]

so we can choose \(\leq \beth_n\) such \(x\) that account for all functions \(f_x|_{[C]^n}\) and add them to \(A_{\alpha+1}\) (which should also contain all of \(A_\alpha\)). There are only \(\beth_n\) such \(C\), so after we do this for every relevant \(C \subset A_\alpha\), we still have \(|A_{\alpha+1}| = \beth_n\). Setting \(A_\alpha = \bigcup_{\beta < \alpha} A_\beta\) for limit \(\alpha\), we conclude that \(A = \bigcup_{\alpha < (\beth_{n-1})^+} A_\alpha\) is as claimed. (The important point here is that every subset \(C \subset A\) of size \(\leq (\beth_{n-1})^+\) is contained in some \(A_\alpha\), so the \(\alpha\)th stage of the construction ensures that \(A\) produces indistinguishables on \(C\).)

Now pick your favourite \(a \in \kappa \setminus A\) (the ‘anchor’) and some \(x(0) \in A\). Since \(A\) produces \(f\)-indistinguishables, we can construct inductively a sequence \(\langle x(\alpha) : \alpha < (\beth_{n-1})^+ \rangle\) in \(A\)
such that (for every $\alpha$) $x(\alpha)$ is $f$-indistinguishable from $a$ on $\{x(\beta) : \beta < \alpha\}$. Set $X = \{x(\alpha) : \alpha < (\beth_{n-1})^+\}$.

By the inductive hypothesis (namely, that $(\beth_{n-1})^+ \rightarrow (\aleph_1)^n_{\aleph_0}$) the function $f_a |_{[X]^n}$ (an $\aleph_0$-colouring of the $n$-subsets of a set of size $(\beth_{n-1})^+$) has a monochromatic subset $M$ of size $\aleph_1$. (That is, $M \subseteq X$ and $f_a$ is constant on $[M]^n$.) Now it is clear that $M$ is monochromatic for $f$. Indeed, for $x(\alpha_1), \ldots, x(\alpha_{n+1}) \in M$, $\alpha_1 < \cdots < \alpha_{n+1}$, we have

$$f\{x(\alpha_1), \ldots, x(\alpha_{n+1})\} = f\{x(\alpha_1), \ldots, x(\alpha_n), a\} = f_a\{x(\alpha_1), \ldots, x(\alpha_n)\},$$

which is, for example, the same as $f(Y \cup \{a\})$, where $Y$ is the set of the first $n$ elements of $M$. (So on $[M]^{n+1}$ the function $f$ is constant and takes the value $f_a(Y)$.) The point is that $M$ is a monochromatic set of size $\aleph_1$.

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\(^1\)Notice that in the case $n = 1$, the inductive hypothesis is the assertion that $\aleph_1$ is a regular cardinal. Jech uses this fact at this point in his proof of the case $n = 1$. 

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