Erdős–Rado

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3 June 2012

This is Jech's proof. Write $[X]^\lambda$ for the set of subsets of X of cardinality λ.

We use choice liberally, though apparently there is a sensible statement and proof of Erdős–Rado without choice (see `http://www.dpmms.cam.ac.uk/~tf/erdosrado.pdf`).

Theorem 1 (Erdős–Rado)**.** $(\beth_n)^+ \to (\aleph_1)^{n+1}_{\aleph_0}$.

Proof. The proof is by induction on n (notice the case $n = 0$ simply asserts that $\aleph_1$ is regular). Put $\kappa = (\beth_n)^+$ and let f be an $\aleph_0$-colouring of $[\kappa]^{n+1}$. Our task is to exhibit a subset of κ of cardinality $\aleph_1$ that is monochromatic for f.

For $x \in \kappa$ write f_x for the colouring of $[\kappa]^n$ given by

$$f_x\{y_1, \ldots, y_n\} = f\{x, y_1, \ldots, y_n\}.$$

If $X \subseteq \kappa$, say elements $y, z \in \kappa$ are **f-indistinguishable on** X if $y, z \notin X$ and

$$f_y\restriction_{[X]^n} = f_z\restriction_{[X]^n},$$

that is, when $f\{y, x_1, \ldots, x_n\} = f\{z, x_1, \ldots, x_n\}$ holds for all $x_1, \ldots, x_n \in X$. Say a set $A \subset \kappa$ **produces f-indistinguishables** if for every $a \in \kappa \smallsetminus A$ and every $C \subset A$ of cardinality $\leq \beth_{n-1}$, there is an element $y \in A \smallsetminus C$ such that y and a are f-indistinguishable on C.

Claim. There is an $A \subset \kappa$ of cardinality $\beth_n$ that produces f-indistinguishables.

Proof. We will construct A as the limit of an ascending sequence $\langle A_\alpha : \alpha < (\beth_{n-1})^+\rangle$. Pick your favourite set $A_0 \subset \kappa$ of cardinality $\beth_n$. Temporarily fix a $C \subset A_0$ of cardinality $\leq \beth_{n-1}$. The number of distinct functions of the form $f_x\restriction_{[C]^n}$ is at most

$$\left|{}^{\beth_{n-1}}\aleph_0\right| = {\aleph_0}^{\beth_{n-1}} = 2^{\beth_{n-1}} = \beth_n,$$

so we can choose $\leq \beth_n$ such x that account for all functions $f_x\restriction_{[C]^n}$ and add them to $A_{\alpha+1}$ (which should also contain all of A_α). There are only $\beth_n$ such C, so after we do this for every relevant $C \subset A_\alpha$, we still have $|A_{\alpha+1}| = \beth_n$. Setting $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$ for limit α, we conclude that $A = \bigcup_{\alpha<(\beth_n-1)^+}$ is as claimed. (The important point here is that every subset $C \subset A$ of size $\leq (\beth_{n-1})^+$ is contained in some A_α, so the αth stage of the construction ensures that A produces indistinguishables on C.) ∎

Now pick your favourite $a \in \kappa \smallsetminus A$ (the 'anchor') and some $x(0) \in A$. Since A produces f-indistinguishables, we can construct inductively a sequence $\langle x(\alpha) : \alpha < (\beth_{n-1})^+\rangle$ in A

such that (for every α) $x(\alpha)$ is f-indistinguishable from a on $\{x(\beta) : \beta < \alpha\}$. Set $X = \{x(\alpha) : \alpha < (\beth_{n-1})^+\}$.

By the inductive hypothesis (namely, that $(\beth_{n-1})^+ \to (\aleph_1)^n_{\aleph_0}$)[1] the function $f_a\restriction_{[X]^n}$ (an $\aleph_0$-colouring of the n-subsets of a set of size $(\beth_{n-1})^+$) has a monochromatic subset M of size $\aleph_1$. (That is, $M \subseteq X$ and f_a is constant on $[M]^n$.) Now it is clear that M is monochromatic for f. Indeed, for $x(\alpha_1), \ldots, x(\alpha_{n+1}) \in M$, $\alpha_1 < \cdots < \alpha_{n+1}$, we have

$$f\{x(\alpha_1), \ldots, x(\alpha_{n+1})\} = f\{x(\alpha_1), \ldots, x(\alpha_n), a\} = f_a\{x(\alpha_1), \ldots, x(\alpha_n)\},$$

which is, for example, the same as $f(Y \cup \{a\})$, where Y is the set of the first n elements of M. (So on $[M]^{n+1}$ the function f is constant and takes the value $f_a(Y)$.) The point is that M is a monochromatic set of size $\aleph_1$. ∎

[1]Notice that in the case $n = 1$, the inductive hypothesis is the assertion that $\aleph_1$ is a regular cardinal. Jech uses this fact at this point in his proof of the case $n = 1$.