NOTES ON JECH, CHAPTER 8

Z NORWOOD

Health Warning:[1] I wrote these notes for myself. They come with no guarantee that they'll be helpful to you. In fact, it's likely that some of the commentary in these notes will make a nonpositive contribution to your understanding of the chapter's material. That aside, if you find anything in these notes that's definitely false, please let me know.

1. Definitions, etc.

Fix an uncountable regular cardinal κ. Two types of 'large', 'thick' (whatever) subsets of κ are the central objects of study in this chapter:

Definition. A subset C of κ is *closed unbounded* if it is, well, closed and unbounded:

(1) Closed: if $\langle \alpha_\eta : \eta < \lambda \rangle$ is an increasing sequence (of length $\lambda < \kappa$) of members of C, then its supremum (= limit) $\lim \alpha_\eta$ is also a member of C. (Not surprisingly, C is closed (in this sense) in κ iff C is a closed subset of κ endowed with the order topology.)

(2) Unbounded: C is unbounded in κ; i.e., $\sup C = \kappa$. That is, for every $\alpha < \kappa$, there is a member of C greater than α, a $\beta \in C$ such that $\alpha < \beta$.

We often abbreviate 'closed unbounded set' by 'club'.

A subset S of κ is *stationary*[2] if it meets every club in C: $C \cap S \neq \varnothing$ for every club $C \subseteq \kappa$.

Because the clubs in κ are closed under finite intersection (Lemma 8.2), they generate a filter, called the *closed unbounded filter* on κ. (That is, the set of supersets of clubs is a filter.) It turns out the notion of club is somehow not inclusive enough for us, so stationary sets are actually the fundamental objects of study here. I find it useful to record some trivialities to help cement my intuition for these sets:

Facts (mostly trivial).

(1) The intersection of two (hence finitely many) clubs is a club (Lemma 8.2).

(2) Every club is stationary.

(3) Every terminal segment $\langle \alpha : \alpha > \beta \rangle$ is a club (hence stationary). As a corollary, every bounded set is nonstationary.

(4) Every stationary set is unbounded, but not necessarily closed.

(5) Every superset of a stationary set is stationary.

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[1] This is TF's phrase, of course. He must've borrowed it from the government or something.

[2] not *stationery*, despite what Google might think

(6) The stationary sets *do not* form (or even generate) a filter. (In fact, there are many pairs of disjoint stationary sets, as we'll show.)

(7) The set of clubs generates a filter; that is, the set of supersets of clubs is a filter (dual to the filter of nonstationary sets).

(8) Every club and every stationary set must have cardinality κ.

(9) To partition a set X into stationary sets, it suffices to partition a subset $Y \subseteq X$ (by (5)).

(10) The intersection of a club and a stationary set is a stationary set.

(11) If S is stationary and $S = X_1 \cup X_2$, then X_1 or X_2 is stationary. (Assume not and use (1).)

Theorem (8.3)**.** The intersection of fewer than κ clubs is a club.

Proof sketch. By induction on the number $\lambda < \kappa$ of clubs. ∎

Definition. Let $\langle X_\alpha : \alpha < \kappa \rangle$ be κ subsets of κ. The *diagonal intersection* of the X_α is defined as

$$\mathop{\triangle}_{\alpha<\kappa} X_\alpha = \left\{ \xi < \kappa : \xi \in \bigcap_{\alpha<\xi} X_\alpha \right\}.$$

I try to keep the following slogan in mind to remember what the diagonal intersection does: 'To make it into the diagonal intersection you must be in every set indexed by one of your predecessors.' Notice that changing the order in which the X_α are indexed can change the diagonal intersection.

Lemma (8.4)**.** The diagonal intersection of a κ-sequence of clubs is a club.

Proof sketch. I think the only real idea in this proof is to use Theorem 8.3 to reduce to the case where the sequence is nonincreasing: $C_0 \supseteq C_1 \supseteq \cdots$. ∎

I have to admit, the first time I read the statement of Lemma 8.4, I thought 'Surely that's false.' I'll give a taste of the poor intuition that led me to that (false) conclusion. What I figured was that I could take a sequence of terminal segments of κ, carefully chosen so that, when stacked on top of κ, they always remained to the right of the diagonal. That is, I wanted a function $f\colon \kappa \to \kappa$ (not necessarily normal or anything fancy) with the property that, for every α, there is a $\gamma < \alpha$ such that $f(\gamma) > \alpha$. Then I'd use the terminal segments $X_{f(\alpha)}$ to break the lemma. (In fact, if this had worked, the diagonal intersection would have not only failed to be a club; it would be empty!) (Un?)fortunately, Exercise 8.2 asks you to prove that no such function exists. Note that you *can* build such a function f (without much difficulty) for e.g. ω or $\aleph_\omega$. Anyway, it turns out the lemma is true. Somehow a function $f\colon \kappa \to \kappa$ on a regular uncountable κ just can't hope to grow quickly enough to break the lemma. It gets caught up in more limit points than it can handle and just gets stuck.

In the discussion on page 93 Jech describes one connection between stationary sets and the diagonal intersection: Let I_{NS} be the ideal of nonstationary sets on κ. Then in the quotient algebra $B := \mathscr{P}(\kappa)/I_{NS}$, infimums and supremums are given by (respectively) diagonal intersections and diagonal unions. I proved for myself that the diagonal intersection is the supremum of a family in B, so I'll share that with you here (it's not particularly exciting):

Fact. In $B = \mathscr{P}(\kappa)/I_{NS}$, $[\triangle_{\alpha<\kappa} X_\alpha]$ is the infimum of the family $[X_\alpha]_{\alpha<\kappa}$.

Proof. To show this, we must prove (1) that for every $\alpha < \kappa$ the inequality $[\triangle_{\alpha<\kappa} X_\alpha] \leq [X_\alpha]$ holds in B; and (2) that if $[Y] \leq [X_\alpha]$ for every $\alpha < \kappa$ then $[Y] \leq [\triangle_{\alpha<\kappa} X_\alpha]$. Passing back from the quotient algebra to $\mathscr{P}(\kappa)$, we see that we need to prove

(1) $(\triangle X_\alpha) \smallsetminus X_\beta$ is nonstationary for every β; and

(2) if Y is such that $Y \smallsetminus X_\alpha$ is nonstationary for every α, then $Y \smallsetminus \triangle X_\alpha$ is nonstationary.

Luckily neither of these is difficult to prove. To see that $(\triangle X_\alpha) \smallsetminus X_\beta$ is nonstationary, notice that $(\triangle X_\alpha) \smallsetminus X_\beta$ is a subset of the (nonstationary, since it's bounded) set β. (This just follows from the definition of the diagonal intersection: every element $\gamma > \beta$ of the diagonal intersection $\triangle X_\alpha$ must have been a member of X_β, and is therefore excluded from the relative complement $(\triangle X_\alpha) \smallsetminus X_\beta$.)

Suppose that Y is such that the relative complement $Y \smallsetminus X_\alpha$ is nonstationary for every $\alpha < \kappa$. For every α we have a club C_α disjoint from $Y \smallsetminus X_\alpha$. Set $C := \triangle_\alpha C_\alpha$, and notice that C (the diagonal intersection of clubs) is a club. If $\beta \in C$, then $\beta \in C_\gamma \subseteq \kappa \smallsetminus (Y \smallsetminus X_\alpha)$ for every $\gamma < \beta$. (You see where this is going.) If you stare at this for a moment, you'll agree that $\beta \in \kappa \smallsetminus (Y \smallsetminus \triangle X_\alpha)$. But C is a club disjoint from $Y \smallsetminus \triangle X_\alpha$, so $Y \smallsetminus \triangle X_\alpha$ is nonstationary. ∎

2. Fodor's theorem

The main cool thing about Fodor's theorem is (apparently) its ability to provide a stationary set with a certain property.

A(n ordinal-valued) function f is *regressive* if $f(\alpha) < \alpha$ for every $\alpha > 0$.

Theorem (Fodor, 8.7)**.** If f is a regressive function on a stationary set S, then there is a stationary subset of S on which f is constant.

To prove Fodor's theorem, you remember that the diagonal intersection of clubs is club (8.4), and then you do the obvious thing: suppose each fibre $f_{-1}\{\alpha\} \subseteq S$ is nonstationary, get a club disjoint from each, take the diagonal intersection... I've already said too much.

3. Solovay's theorem

For a regular cardinal $\lambda < \kappa$ define

$$E^\kappa_\lambda := \{\alpha < \kappa : \operatorname{cf} \alpha = \lambda\}\,.$$

(Why do we define E^κ_λ only for regular λ? Because a cofinality is always regular, so E^κ_λ would be empty if λ were singular.)

Let's check carefully that each E^κ_λ is (a) stationary (subset of κ): Suppose $C \subseteq \kappa$ is a club. We need to locate a member of C with cofinality λ. Our only real option is to find a member of C that's the limit of a λ-sequence; such an element would have to have cofinality

λ (since λ is regular!). Well, if we start picking elements $\alpha_0, \alpha_1, \dots$ of C, we can pick λ of them since $\lambda < \kappa$ and C is unbounded. Then the limit of the sequence $\langle \alpha_\eta : \eta < \lambda \rangle$ belongs to C, because C is closed. Done.

The main result of this section is:

Theorem (8.10, Solovay)**.** As always, κ is an uncountable regular cardinal. Every stationary subset of κ can be partitioned into κ (disjoint) stationary sets.

Proof skeleton. Our strategy for proving Solovay's theorem: We're given a stationary subset A of κ and we hope to partition it into κ disjoint stationary sets. Since it suffices to partition the set of limit ordinals of A in this way, let's assume that every member of A is a limit, so that it makes sense to say 'every member of A is either regular or singular'. Write SNG for the set of singular (limit) ordinals below κ and REG for the set of regular (limit) ordinals (cardinals)[3] below κ. Now either $A \cap$ SNG or $A \cap$ REG must be stationary (don't forget Trivial Fact (11)!). We consider these cases separately. The purpose of Lemma 8.8 (and the discussion following it) is to deal with the case when $A \cap$ SNG is stationary.

Case 1: $A \cap$ SNG is stationary. First we show that every stationary subset W of E^κ_λ can be partitioned into κ stationary sets (L8.8). Here's a sketch of how to do this:

(1) For every $\alpha \in W$ pick an increasing cofinal λ-sequence $\langle a^\alpha_\gamma : \gamma < \lambda \rangle$ that converges to α.

(2) Prove the following claim: There is an index δ such that for every $\eta < \kappa$ the set $\{\alpha \in W : a^\alpha_\delta \geq \eta\}$ is stationary.[4]

(3) Now let f be the function that sends each $\alpha \in W$ to its 'δth approximation' a^α_δ (where δ is the special δ from part (2)). This function is regressive, so Fodor says there is, for each $\eta < \kappa$, a stationary set $S_\eta \subseteq \{\alpha \in W : a^\alpha_\delta \geq \eta\}$ on which f is constant. Write γ_η for the unique value f takes on S_η.

(4) Note that $\gamma_\eta \neq \gamma_{\eta'}$ implies $S_\eta \cap S_{\eta'} = \varnothing$. Prove $|\{S_\eta : \eta < \kappa\}| = \kappa$, and don't think it's completely trivial. (You have to use the fact that κ is regular: the γ_ηs form an unbounded sequence in κ, so the set $\{\gamma_\eta : \eta < \kappa\}$ has cardinality κ.)

That finishes the proof of L8.8. Now return to the big picture: $A \cap$ SNG is stationary. The map $\alpha \mapsto \operatorname{cf} \alpha$ is regressive on SNG, so by Fodor it is constant on a stationary subset of $A \cap$ SNG; such a subset looks like $A \cap E^\kappa_\lambda$. So we have a stationary subset $A \cap E^\kappa_\lambda$ of E^κ_λ; by what we proved in the first part of this case, $A \cap E^\kappa_\lambda$ can be partitioned into κ disjoint stationary sets. Done, since it suffices to partition the subset $A \cap E^\kappa_\lambda$ of A.

Case 2: $A \cap$ REG is stationary. Put $S = A \cap$ REG. Wlog $\omega \notin S$.[5] First prove that (L8.9) the set

$$T = \{\alpha \in S : S \cap \alpha \text{ is a nonstationary subset of } \alpha\}$$

is stationary (in κ).[6] To prove this, you want to show that every club C intersects T. Show that the least limit point of C in S belongs to T.

[3]Every regular ordinal is definitely a cardinal.

[4]This is something like 'there is an index δ such that for every notion of *far enough* (ie η) stationarily many $\alpha \in W$ have δth approximations (ie a^α_δ) that are *far enough*'. Ignore this if it means nothing to you.

[5]In the proof of L8.9, it's essential that every member of S be uncountable.

[6]'Stationarily many members of S have few members of S below them.'

Now it suffices to prove that T can be partitioned into κ disjoint stationary sets. The proof of this is similar to the proof of L8.8. As in the beginning of the proof of L8.8, we want to pick, for every $\alpha \in T$, a sequence that approximates α from below. Because $T \cap \alpha$ is nonstationary in α, there is a club in α that is disjoint from $T \cap \alpha$. There is a normal sequence a^α_ξ that enumerates this club,[7] so $\left\langle a^\alpha_\xi : \xi < \alpha \right\rangle$ is an increasing continuous sequence whose limit is α, such that $a^\alpha_\xi \notin T$ for every ξ.

As in L8.8 we want to show (∗) that there is ξ such that for every $\eta < \kappa$, the set

$$\left\{\alpha \in T : a^\alpha_\xi \geq \eta\right\}$$

is stationary. (The proof of this fact is the main difference between this proof and the proof of L8.8.) If there is no such ξ, then for each ξ there is some $\eta(\xi)$ and a club C_ξ such that $a^\alpha_\xi < \eta(\xi)$ for all $\alpha \in C_\xi \cap T$ if a^α_ξ is defined. Let C be the diagonal intersection of the C_ξ. By definition of the diagonal intersection, if $\alpha \in C \cap T$, then $a^\alpha_\xi < \eta(\xi)$ for every $\xi < \alpha$.

Let D be the set $\{\gamma \in C : \xi < \gamma \text{ implies } \eta(\xi) < \gamma\}$. Note that D is a club: Put $D_\xi = \{\gamma \in C : \eta(\xi) < \gamma\}$. Then D_ξ is a club, as it's the intersection of C with a terminal segment. In addition, $D = \triangle D_\xi$, so D is a club.[8] Now we see that $D \cap T$ (the intersection of a club and a stationary set) is stationary. Pick two ordinals $\gamma < \alpha$ in $D \cap T$. If $\xi < \gamma$, then $a^\alpha_\xi < \eta(\xi) < \gamma$. Since the sequence $\left\langle a^\alpha_\xi : \xi < \gamma \right\rangle$ is a continuous increasing sequence of length γ, it can't very well have a supremum that's less than γ. Each term in the sequence is less than γ, so $a^\alpha_\gamma = \lim_{\xi<\gamma} a^\alpha_\xi = \gamma$. But $\gamma \in T$ (by assumption) and $a^\alpha_\gamma \notin T$ (by the construction of the a^α_ξs); this contradiction establishes (∗).

The remainder of the proof is virtually identical to the last part of the proof of L8.8. ∎

4. Mahlo cardinals

Inaccessible cardinals (= uncountable regular limit cardinals) are pretty big; in fact, an inaccessible cardinal κ must be a fixed point of the $\aleph$ function: $\kappa = \aleph_\kappa$. A Mahlo cardinal κ is so big that it must be the κth inaccessible cardinal (that is, it must be a fixed point of the function $\alpha \mapsto$ the αth inaccessible cardinal), and (Exercise 8.6) it can't even be the least such cardinal!

Why must a Mahlo cardinal κ be the κth inaccessible cardinal? Jech says this, but (big surprise) doesn't provide all the details. The first paragraph of the section on Mahlo cardinals reads:

> The set of all cardinals below κ is a closed unbounded subset of κ, and so is the set of its limit points, the set of all limit cardinals. In fact, the set of all strong limit cardinals below κ is closed unbounded.

It shouldn't be too hard to see that the set of cardinals below an inaccessible κ is a club, and the set of limit points of a club is a club (this is mentioned at the top of page 92), as you can convince yourself. The set of strong limit cardinals below κ is

[7] See the top of page 92.
[8] Thanks for Philipp for this argument.

(1) **unbounded** because the limit of the sequence $\alpha, 2^\alpha, 2^{2^\alpha}, \dots$ is always a strong limit of countable cofinality (and the limit is $< \kappa$ because κ has uncountable cofinality);

(2) **closed** because (as it's hopefully not difficult to see) the limit of strong limits is a strong limit.

So if κ is a Mahlo cardinal, the set R of regular cardinals below κ is stationary and the set L of (strong) limit cardinals below κ is a club. The inaccessibles below κ are exactly the members of the intersection $R \cap L$, which (intersection) is a stationary set. In particular, the set $R \cap L$ of inaccessibles below κ must have cardinality κ (triviality (8)!), so κ is the κth inaccessible.

5. Normal filters

I don't think there's anything exciting to say here. For your convenience, here's the only result:

Lemma (8.11)**.** If κ is regular and uncountable and if F is a normal filter on κ that contains all final segments $\{\alpha : \alpha_0 < \alpha < \kappa\}$, then F contains all closed unbounded sets.

6. Silver's theorem

Could this section in the book be **any more** annoying?! No. Definitely not. Here's the digraph of implications in this section:

$$\text{Lemma 16} \Rightarrow \text{Lemma 15} \Rightarrow \text{Lemma 14} \Rightarrow \text{(Theorems 12 \& 13)}$$

Good idea? No.

Why does Lemma 8.14 imply Silver's Theorem (8.12)? At the top of page 97, Jech says "If GCH holds below κ then the assumptions of Lemma 8.14 are satisfied, and $2^\kappa = \kappa^{\operatorname{cf}\kappa}$. Thus Theorem 8.12 follows from Lemma 8.14." Let's examine this a bit more closely.

Let κ be a singular cardinal of uncountable cofinality, and suppose the GCH holds below κ (i.e., that $2^\alpha = \alpha^+$ for every $\alpha < \kappa$). Now consider two cases: first suppose $\lambda \le \operatorname{cf}\kappa$. Then we have

$$\lambda^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa} = (\operatorname{cf}\kappa)^+ < \kappa,$$

because κ is a limit. Now suppose $\lambda > \operatorname{cf}\kappa$. Then we have

$$\lambda^{\operatorname{cf}\kappa} \le \lambda^\lambda = 2^\lambda = \lambda^+ < \kappa,$$

again because κ is a limit.

Now convince yourself that κ is the limit of a (normal) sequence $(\kappa_\alpha : \alpha < \kappa)$ of regular cardinals. (Recall that successor cardinals are regular.) Now pick such a sequence $(\kappa_\alpha : \alpha < \kappa)$. Because $\kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = 2^{\kappa_\alpha}$ for every $\alpha < \operatorname{cf}\kappa$, the set

$$\left\{\alpha < \operatorname{cf}\kappa : \kappa_\alpha^{\operatorname{cf}\kappa_\alpha} = \kappa_\alpha^+\right\} = \operatorname{cf}\kappa$$

is obviously stationary in $\operatorname{cf}\kappa$. So we can apply Lemma 8.14 to conclude that $\kappa^{\operatorname{cf}\kappa} = \kappa^+$. But you have to look back to page 58 (and note that κ is a strong limit) to see that we have now established the conclusion of Silver's Theorem (8.12): $2^\kappa = \kappa^+$.

What follows is the presentation of Silver's theorem that TF sent us. I have made few changes, none major.

Definition. For α an aleph, $\alpha^{(\mu)}$ is the μth aleph $> \alpha$.

Let us write $T \restriction \alpha$ for $\{a \cap \alpha : a \in T\}$.

The following proof comes from notes taken by Frank Drake of a lecture by Prikry on Silver's theorem.

Theorem 6.1 (Prikry, Silver's theorem)**.** Let κ be a singular cardinal such that $\mathrm{cf}(\kappa) = \lambda > \omega$ and $\alpha < \kappa \to \alpha^\lambda < \kappa$. Suppose $\mu < \lambda$ and $T \subseteq \mathscr{P}(\kappa)$. If $\{\alpha < \kappa : |(T \restriction \alpha)| \leq \alpha^{(\mu)}\}$ is stationary in κ, then $|T| \leq \kappa^{(\mu)}$.

Proof. By induction on μ. Take $C = \{\alpha_\zeta : \zeta < \lambda\}$ a strictly increasing continuous sequence of cardinals with limit κ. Set $S = \{\zeta < \lambda : |T \restriction \alpha_\zeta| \leq (\alpha_\zeta)^{(\mu)}\}$. S is stationary in λ by the hypothesis (since in general A stationary in κ iff $A \cap C$ stationary in κ iff $\{\zeta : \alpha_\zeta \in A\}$ stationary in λ).

Given $\zeta \in S$, let $f_\zeta : T \restriction \alpha_\zeta \to (\alpha_\zeta)^{(\mu)}$ be $1-1$; and given $a \in T$, let $g_a(\zeta) = f_\zeta(a \cap \alpha_\zeta)$ for $\zeta \in S$.

Case $\mu = 0$

(Drake comments that this case is dealt with by Erdős, Hajnal and Milner [**?**].)

Then $g_a(\zeta) < \alpha_\zeta$ for $\zeta \in S$. So if ζ is a limit, $g_a(\zeta) < \alpha_\eta$ for some $\eta < \zeta$ (since C is cts); let $h_a(\zeta)$ be the least such. If $S_0 = \{\zeta \in S : \zeta \text{ is a limit}\}$, then S_0 is stationary too, and $h_a : S_0 \to \lambda$ is regressive. Therefore, by Fodor's theorem, it is constant on a stationary set.

So for each $a \in T$, fix $S_a \subseteq S_0$ and $\eta(a) < \lambda$ such that S_a is stationary in λ and $h_a(\zeta) = \eta(a)$ constantly for $\zeta \in S_a$.

Now there are at most 2^λ pairs $\langle S_a, \eta(a)\rangle$ possible, and $2^\lambda < \kappa$ by assumption. So, given $\langle S', \eta'\rangle$, S' stationary in λ and $\eta' < \lambda$, let

$$T' = \{a \in T : S_a = S' \wedge \eta(a) = \eta'\}$$

We show $|T'| \leq \kappa$, and the result follows because in that case $|T| \leq 2^\lambda \cdot \kappa = \kappa$.

Since $a \in T' \to g_a(\zeta) < \alpha_{\eta'}$ for any $\eta \in S'$, we must have $|T' \restriction \alpha_\zeta| \leq \alpha_{\eta'}$ for $\zeta \in S'$; and if $a, b \in T$ with $a \neq b$ then $\langle a \cap \alpha_\zeta : \zeta \in S'\rangle$ and $\langle b \cap \alpha_\zeta : \zeta \in S'\rangle$ are distinct sequences.

So $|T'| \leq (\alpha_{\eta'})^\lambda < \kappa$ by hypothesis and case $\mu = 0$ follows.

Case $\mu > 0$, μ **limit**

Fix $a \in T$, then $g_a(\zeta) < (\alpha_\zeta)^{(\mu)}$, so $g_a(\zeta) < (\alpha_\zeta)^{(\nu)}$ for some $\nu < \mu$, for each $\zeta \in S$.

So this splits S into $\leq \mu$ pieces, and $\mu < \lambda$, so at least one is stationary. Say $S_a \subseteq S$ is stationary, and $\nu(a) < \nu$, and $\zeta \in S_a \to g_a(\zeta') < (\alpha_\zeta)^{(\nu(a))}$.

Again, fix $S' \subseteq S$ stationary, and $\nu' < \mu$; by the induction hypothesis for case ν', $|\{a \in T : S_a = S' \wedge \nu_a = \nu'\}| \leq \kappa^{(\nu)}$. But again there are $\leq 2^\lambda$ pairs $\langle S', \nu'\rangle$ so $|T| \leq \kappa^{(\mu)}$.

Case $\mu = \nu + 1$ (main case)

First pick an ultrafilter $\mathcal{U}$ on S so that $X \in \mathcal{U} \to X$ is stationary on λ. (The usual proof that a filter extends to an ultrafilter adapts to show that the closed unbounded filter on S extends to such a $\mathcal{U}$: at each stage, make sure that the set added is stationary, using the fact that if $X \sqcup X' = S$ is stationary then X' or X is stationary.) Now think about the ultraproduct

$$(\prod_{\zeta \in S} (\alpha_\zeta)^{(\mu)})/\mathcal{U}.$$

This ultraproduct has a canonical linear order (by Łoś) and this induces an order $\prec$ on T by:

$$a \prec b \text{ iff } \{\zeta \in S : g_a(\zeta) < g_b(\zeta)\} \in \mathcal{U}.$$

This will be a **linear** order on T, since if $a \neq b$ then $a \cap \alpha_{\zeta_0} \neq b \cap \alpha_{\zeta_0}$ for some $\zeta_0 \in S$, and hence $g_a(\zeta) \neq g_b(\zeta)$ for $\zeta > \zeta_0$; i.e., $\{\zeta : g_a(\zeta) \neq g_b(\zeta)\} \in \mathcal{U}$.

Claim: The order-type of T under $\prec$ is $\kappa^{(\mu)}$-like.

Proof of claim: Fix $b \in T$ and look at $T_b = \{a \in T : a \prec b\}$. For $a \in T_b$, let $S_a = \{\zeta \in S : g_a(\zeta) < g_b(\zeta)\}$. Since $S_a \in \mathcal{U}$, S_a is stationary on λ. So fix S' stationary in λ, and let $T' = \{a \in T_b : S_a = S'\}$. Then for $\zeta \in S'$, $g_a(\zeta) < g_b(\zeta) < (\alpha_\zeta)^{(\mu)}$ for each $a \in T'$. But $g_\beta(\zeta)$ has cardinal $\leq (\alpha_\zeta)^{(\mu)}$, since $(\alpha_\zeta)^{(\mu)} = ((\alpha_\zeta)^{(\nu)})^+$, and so $|T' \restriction \alpha_\zeta| \leq (\alpha_\zeta)^{(\nu)}$, for each $\zeta \in S'$. So we can apply the induction hypothesis (case ν) to T', and get $|T'| \leq \kappa^{(\nu)}$. Since (again) there are only 2^λ such S', $|T'| \leq \kappa^{(\nu)}$ too, which is the claim. But any $\kappa^{(\mu)}$-like ordering has cardinal $\leq \kappa^{(\mu)}$. That is to say $|T| \leq \kappa^{(\mu)}$. ∎

Corollary 6.2. Suppose $\omega < \mathrm{cf}(\kappa) = \lambda < \kappa$, $\mu < \lambda$ and $\{\alpha < \kappa : 2^\alpha \leq \alpha^{(\mu)}\}$ is stationary in κ. Then $2^\kappa \leq \kappa^{(\mu)}$.

In particular, if the GCH holds for a stationary set of cardinals below κ, it holds at κ. ∎