This is a detailed explanation of a recurrence relation example I started working out in discussion on Thursday, 6 Feb.

Here is the recurrence:

\[
\begin{align*}
t_0 &= 0, \quad t_1 = 1 \\
t_{n+2} &= 2t_n + t_{n+1}.
\end{align*}
\]

First we solve by guessing that \( t_n = r^n \):

\[
r^{n+2} = 2r^n + r^{n+1}.
\]

Divide each side by \( r^n \) and rearrange to get \( r^2 - r - 2 = 0 \), which has solutions \( r = 2 \) and \( r = -1 \). The solution should be of the form \( a2^n + b(-1)^n \), so now we need to use the initial conditions to solve for \( a \) and \( b \).

\[
\begin{align*}
0 &= a2^0 + b(-1)^0 = a + b, \\
1 &= a2^1 + b(-1)^1 = 2a - b.
\end{align*}
\]

Add these two equations to get \( 3a = 1 \) and \( b = -a \), so \( a = \frac{1}{3} \) and \( b = -\frac{1}{3} \). Our solution should be

\[
\frac{1}{3}2^n - \frac{1}{3}(-1)^n.
\]

Let’s prove by induction that this is the solution. That is, let’s prove that if \( s_n \) solves the recurrence (**) and the initial conditions (*), then for every \( n \in \mathbb{N} \), \( s_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n \).

Base case:

\[
\begin{align*}
\frac{1}{3}2^0 - \frac{1}{3}(-1)^0 &= \frac{1}{3} - \frac{1}{3} = 0 = s_0. \quad \checkmark \\
\frac{1}{3}2^1 - \frac{1}{3}(-1)^1 &= \frac{2}{3} + \frac{1}{3} = 1 = s_1. \quad \checkmark
\end{align*}
\]

Inductive step:

Suppose inductively that \( s_m = \frac{1}{3}2^m - \frac{1}{3}(-1)^m \) for all \( m < n \). (This is what’s sometimes called ‘strong induction’.)
In particular,

\begin{align*}
    s_{n-2} &= \frac{1}{3}2^{n-2} - \frac{1}{3}(-1)^{n-2} \\
    s_{n-1} &= \frac{1}{3}2^{n-1} - \frac{1}{3}(-1)^{n-1}.
\end{align*}

We use these two facts and the recurrence and then do some algebraic manipulation to prove that \( s_n = \frac{1}{3}2^n - \frac{1}{3}(-1)^n \).

\begin{align*}
    s_n &= 2s_{n-2} + s_{n-1} \\
         &= 2\left(\frac{1}{3}2^{n-2} - \frac{1}{3}(-1)^{n-2}\right) + \left(\frac{1}{3}2^{n-1} - \frac{1}{3}(-1)^{n-1}\right) \\
         &= \frac{2}{3}2^{n-2} + \frac{1}{3}2^{n-1} - \frac{2}{3}(-1)^{n-2} - \frac{1}{3}(-1)^{n-1} \\
         &= \frac{1}{3}2^{n-1} + \frac{1}{3}2^{n-1} - \frac{2}{3}(-1)^{n-2} + \frac{1}{3}(-1)^{n-2} \\
         &= \frac{2}{3}2^{n-1} - \frac{1}{3}(-1)^{n-2} \\
         &= \frac{1}{3}2^n - \frac{1}{3}(-1)^n,
\end{align*}

as desired. \( \checkmark \)