INTEGRATION BY PARTS AND TRIG SUBSTITUTION

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1. STANDARD BY-PARTS INTEGRALS

These are the integrals that will be automatic once you have mastered integration by parts. In a typical integral of this type, you have a power of $x$ multiplied by some other function (often $e^x$, $\sin x$, or $\cos x$). Let $u$ be the power of $x$ and $v'$ be the other function so that integrating by parts decreases the power of $x$.

**Example 1.** Compute $\int x \sin x \, dx$.
We use the substitution $u = x$ $v = -\cos x$ $u' = 1$ $v' = \sin x$.

Then integrate by parts:

$$\int x \sin x \, dx = -x \cos x - \int (-\cos x) \, dx = -x \cos x + \sin x + C.$$

Other examples of integrals of this type:

- $\int x^2 e^x \, dx$
- $\int (2x)^2 \cos x \, dx$
- $\int x \sin(2x) \, dx$

Don’t be frightened by the constants. They don’t affect the method at all: you integrate $\int x^2 \cos x \, dx$ and $\int (3x/2)^2 \cos(3x) \, dx$ using the same method; the constants are just different.

2. TRICKY BY-PARTS INTEGRALS

What makes these integrals strange is that setting $v' = 1$ is often a good idea. Also, the integrand is often not a product, as you will see in these examples.

**Example 2.** Compute $\int \ln(x) \, dx$.
We use the substitution $u = \ln(x)$ $v = x$ $u' = \frac{1}{x}$ $v' = 1$.

Then integrate by parts:

$$\int \ln(x) \, dx = x \ln(x) - \int x \frac{1}{x} \, dx = x \ln(x) - \int dx = x \ln(x) - x + C.$$

In that example, somehow the extra factor $x$ you get by integrating $v' = 1$ cancels out with $u' = \frac{1}{x}$ nicely.

**Example 3.** Compute $\int \arcsin(x) \, dx$.
We use the substitution $u = \arcsin x$ $v = x$ $u' = \frac{1}{\sqrt{1 - x^2}}$ $v' = 1$.

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Then integrate by parts:

\[ \int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx. \]

The integral on the right is a typical \( u \)-substitution integral. Set \( u = 1 - x^2 \) to get \( du = -2x \, dx \) and

\[ \int \frac{x}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\frac{1}{2} \int u^{-1/2} = -\sqrt{u} + C = -\sqrt{1-x^2} + C. \]

Plug this result back into equation (1) to get

\[ \int \arcsin x \, dx = x \arcsin x - (-\sqrt{1-x^2}) + C = x \arcsin x + \sqrt{1-x^2} + C. \]

This didn’t work out quite as nicely as Example 2 did, but the \( x \) we got by integrating \( v' = 1 \) served as (part of) the \( du \) in our substitution.

For another tricky by-parts integral, try \( \int (\ln x)^2 \, dx \).

### 3. Sneaky by-parts integrals

The main example of this type of integral is the following:

**Example 4.** Compute \( \int e^x \cos x \, dx \).

We use the substitution

\[ u = e^x \quad v = \sin x \]
\[ u' = e^x \quad v' = \cos x. \]

Then integrate by parts:

\[ \int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx. \]

Someone who’s paying attention to what (s)he is doing at this point might say, ‘Well, we haven’t gotten anywhere, since \( \int e^x \sin x \, dx \) is no easier than the integral we started with!’ That’s a reasonable response, but let’s charge ahead anyway. Use another substitution for the integral on the right:

\[ u = e^x \quad v = -\cos x \]
\[ u' = e^x \quad v' = \sin x. \]

Integrating by parts a second time gives

\[ \int e^x \cos x \, dx = e^x \sin x - (-e^x \cos x + \int e^x \cos x \, dx) = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \]

Here’s where the sneakiness comes in. The integral on the far right is now our original integral, so we can add it to both sides and divide by 2 to get a formula for the original integral!

\[ 2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C, \]

and dividing by 2 gives

\[ \int e^x \cos x \, dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + C. \]

This phenomenon is difficult to replicate (other than in obvious variants of the example, like \( \int e^x \sin x \, dx \) or \( \int e^{(2x)} \sin(3x) \, dx \)). As a result, most problems that require this sneaky trick will look like \( \int e^x \cos x \, dx \) or \( \int e^x \sin x \, dx \) (possibly with extra constants, of course). (One important exception is \( \int \sec^3 x \, dx \), though; see below.)
4. Trig integrals

Before we do some nastier by-parts integrals, we need to learn some trig integrals. First, an example that you’ve known how to do for a while:

**Example 5.** Compute \( \int \sin^3 x \cos x \, dx \).

We notice that the substitution \( u = \sin x \), \( du = \cos x \, dx \) simplifies the integral considerably:

\[
\int \sin^3 x \cos x \, dx = \int u^3 \, du = \frac{u^4}{4} + C = \frac{\sin^4 x}{4} + C.
\]

**Example 6.** Compute \( \int \sec x \, dx \).

5. Extra tricky (and sneaky) by-parts integrals

**Example 7.** Compute \( \int \sec^3 x \, dx \).

6. Exercises

When you’ve mastered the examples in the previous few sections, try these:

1. \( \int \sin \sqrt{x} \, dx \).
2. \( \int x \ln x \, dx \).
3. \( \int \frac{1}{t - \sqrt{1 - t^2}} \, dt \).
4. \( \int \arcsin \sqrt{x} \, dx \).
5. \( \int \frac{1}{x^4 + 4} \, dx \).
6. \( \int \sin(\ln x) \, dx \).
7. \( \int \cos x \ln(\sin x) \, dx \).
8. \( \int \sin x \ln(\sin x) \, dx \).