THE 2D CASE OF THE BOURGAIN-DEMETER-GUTH ARGUMENT

ZANE LI

Let \( e(z) := e^{2\pi iz} \) and for \( g : [0, 1] \to \mathbb{C} \) and \( J \subset [0, 1] \), define the extension operator

\[
E_J g(x) := \int_J g(t)e(tx_1 + t^2x_2) \, dt.
\]

For a positive weight \( \nu : \mathbb{R}^2 \to [0, \infty) \), define the weighted \( L^p \) norm by

\[
\|f\|_{L^p(\nu)} := \left( \int_{\mathbb{R}^2} |f(x)|^p \nu(x) \, dx \right)^{1/p}.
\]

Furthermore for a ball \( B \subset \mathbb{R}^2 \) centered at \( c_B \), let \( w_B(x) := (1 + \frac{|x-c_B|}{R})^{-200} \). We think of \( w_B \) as a smoothed out version of \( 1_B \) and so \( L^p(w_B) \) can be thought of as \( L^p(B) \). Next, let

\[
\|f\|_{L^p(w_B)} := \left( \frac{1}{|B|} \int_{\mathbb{R}^2} |f(x)|^p w_B(x) \, dx \right)^{1/p}
\]

and so thinking \( w_B \) to be basically \( 1_B \) gives that \( \|f\|_{L^p(w_B)} \) is essentially the \( L^p \) norm of \( f \) with respect to the uniform probability measure on \( B \). Finally, let

\[
\text{geom } x_i := (x_1 x_2)^{1/2}
\]

and so \( \text{geom } x_i \) is the geometric mean of \( x_i \).

The goal of this note is to sketch the proof of the following theorem, concentrating on the iteration scheme from [4] in the two dimensional case. This justifies (19) of [4] without an appeal to [1]. We also include some routine applications of Hölder, Minkowski, etc. that are omitted from [4] for completeness. We will prove

**Theorem 1.** Let \( 0 < \delta \leq 1 \). For each ball \( B \subset \mathbb{R}^2 \) of radius \( \delta^{-2} \) and each \( g : [0, 1] \to \mathbb{C} \), we have

\[
\|E_{[0,1]}g\|_{L^p(w_B)} \lesssim \delta^{-\varepsilon} \left( \sum_{J \subset [0,1]} \|E_J g\|_{L^p(w_B)}^2 \right)^{1/2}
\]

where the implicit constant is independent of \( \delta, B, \) and \( g \).

Here and throughout, \( \delta^{-1} = N \in \mathbb{Z} \) and \( \sum_{J \subset [0,1], |J| = \delta} \) is a sum when \( J \) ranges over the intervals \( [j\delta, (j+1)\delta), 0 \leq j \leq N-1 \). Other sums over intervals are defined similarly.

Denote by \( V_p(\delta) = V_{p,2}(\delta) \) the smallest constant such that

\[
\|E_{[0,1]}g\|_{L^p(w_B)} \leq V_p(\delta) \left( \sum_{J \subset [0,1]} \|E_J g\|_{L^p(w_B)}^2 \right)^{1/2}.
\]

It is useful to think of \( V_p(\delta) \) as a power of \( \delta^{-1} \). Our goal will be to show that \( V_{6,2}(\delta) \lesssim \delta^{-\varepsilon} \). We defer applications of estimates of this type to Vinogradov’s Mean Value Theorem to [4] and [5].
1. Main Tools

We mention the main tools used in the iteration scheme to prove Theorem 1. We will only at most briefly mention how these results are proven and refer the reader to [4], [5], and [6]. In a few instances, the tool follows from a routine application Hölder or Minkowski and its proof was omitted from [4], we will write out the full details in those instances for completeness.

For some $K > 4$, let $I_1, I_2$ be intervals of length $1/K$ in $[0, 1]$ which are separated by at least $1/K$.

1.1. Ball inflation and $L^2$ decoupling. We record the two dimensional analogue of the key ball inflation theorem. Ball inflation allows us to inflate from $L^{p/2}_\#$ (or $L^p_\#$) norms over the smaller balls ($\Delta$ below) to $L^{p/2}_\#$ (or $L^p_\#$) norms over bigger balls ($B$ below).

**Theorem 2** (Ball inflation, Theorem 5.6 in [4]). Let $\rho < 1/K$ and fix $p \geq 2$. Let $B$ be an arbitrary ball in $\mathbb{R}^2$ with radius $\rho^{-2}$ and let $B$ be a finitely overlapping cover of $B$ with balls $\Delta$ of radius $\rho^{-1}$. Then for each $g : [0, 1] \to \mathbb{C}$,

$$\frac{1}{|B|} \sum_{\Delta \in B} \text{geom}(\sum_{J, |J| = \rho} \|E_J, g\|_{L^{p/2}_\#(w_{\Delta})}^2)^{p/2} \lesssim \varepsilon, K \rho \varepsilon \text{geom}(\sum_{J, |J| = \rho} \|E_J, g\|_{L^{p/2}_\#(w_{\Delta})}^2)^{p/2}$$

where the implicit constant is independent of $g$, $\rho$, and $B$.

**Proof.** The proof will use the uncertainty principle, bilinear Kakeya, and dyadic pigeonholing. It will be useful to observe that the Fourier transform of $E_J, g$ is supported on $f(t, t^2) : t \in J$. We refer the reader to [4] or Section 3 of [6].

We shall refer to both (1) and (2) as “ball inflation.” Since the proof of (2) is only briefly mentioned in [4], we include the full details at the end of this section. We will also use the following result that will allow us to decouple at a scale equal to the inverse of the radius of the ball we are analyzing.

**Proposition 3** ($L^2$ decoupling, Lemma 7.1 of [4]). Let $\{J_i\}_i$ be arbitrary collections of pairwise disjoint intervals $J$ with length equal to an integer multiple of $R^{-1}$. Then
The proof of this proposition essentially just uses that the Fourier transform
Given a function
By how our weight is defined, note that
Proof of (2). By how our weight is defined, note that
\[ \sum_{\Delta \in B} w_{\Delta} \lesssim w_B. \]
Then
\[ \sum_{\Delta \in B} \text{geom}( \sum_{J \subseteq I, |J| = \rho} \| E_{J,t}g \|^2_{L^p(w_{\Delta})})^{p/2} \leq \text{geom} \sum_{\Delta \in B} (\sum_{J \subseteq I, |J| = \rho} \| E_{J,t}g \|^2_{L^p(w_{\Delta})})^{p/2} \]
\[ \lesssim \text{geom} \sum_{J \subseteq I, |J| = \rho} \| E_{J,t}g \|^2_{L^p(w_B)} \]
where the first inequality is by Hölder and the second inequality is by Minkowski (in the sense of \( \sum_k \| f_k \|^2_{L^p(w_{\Delta})} \)) and that \( \sum_{\Delta} \Delta w_{\Delta} \lesssim w_B \) (this last inequality is one of the reasons why we work with \( w_B \) instead of \( 1_B \) since \( \sum_{\Delta} \Delta \) is not necessarily \( \lesssim 1_B \), for example consider balls that cover the boundary of \( B \)). To change from \( L^p \) to \( L^p_k \), multiply everything by \( (|B| \cdot |\Delta|)^{-1} \) and use that this is \( \sim |B|^{-1} \). This completes the proof of (2).

1.2. Hölder’s inequality and averaging. Given a function \( g : [0,1] \to \mathbb{C} \), a ball \( B^r \) with radius \( \delta^{-r} \) in \( \mathbb{R}^n \), \( t \geq 1 \), and \( 1 \leq s \leq r \), define
\[ D_t(q, B^r) = \text{geom} \sum_{J \subseteq I, |J, s| = \delta^q} \| E_{J,s}g \|^2_{L^p(w_{B^r})} \]
and given a finitely overlapping cover \( B_s(B^r) \) of \( B^r \) with balls \( B^s \), define
\[ A_p(q, B^r, s) = \left( \frac{1}{|B_s(B^r)|} \sum_{B^s \in B_s(B^r)} D_2(q, B^s)^p \right)^{1/p}. \]
At times to emphasize the dependence of \( A_p(q, B^r, s) \) on \( g \) and \( \delta \), we will sometimes instead write \( A_p,q,\delta(q, B^r, s) \) or \( A_p(q, B^r, s, g) \). We note that
\[ A_p(q, B^r, r) = D_2(q, B^r). \]
We also note that \( A_p(q, B^r, s) \) will also depend on \( B_s(B^r) \).
We first will prove a Hölder type inequality for \( D_t \), more specifically,
\[ D_t(q, B^r) \leq D_{p_1}(q, B^r)^{1-\alpha} D_{p_2}(q, B^r)^{\alpha} \]
for \( \frac{1}{\alpha} = \frac{1-\alpha}{p_1} + \frac{\alpha}{p_2} \). For simplicity, we will ignore the weights \( w_B \) and simply replace them with \( 1_B \). Note that the factor \( 1/|B| \) in the definition of \( D_t \) balances out by
how \( \alpha \) is defined and hence we may replace \( L^1_{\#}, L^p_\# \), and \( L^p \) with \( L^t \), \( L^p \), and \( L^p \), respectively. Next, it suffices to prove that

\[
\sum_{j \in I} \| E_j g \|_{L^t(B')}^2 \leq \left( \sum_{j \in I} \| E_j g \|_{L^p(B')}^2 \right)^{1 - \alpha} \left( \sum_{j \in I} \| E_j g \|_{L^p(B')}^2 \right)^\alpha
\]

for \( I = I_1 \) and \( I = I_2 \). The proof of the above equation will just use H"older’s inequality twice and so we will write \( \sum_j \) in place of \( \sum_{j \in I} \). Let \( a, b \) be such that \( ar = p_1, bs = p_2, a + b = t \) and \( 1/r + 1/s = 1 \). Then

\[
\sum_j \| E_j g \|_{L^t}^2 = \sum_j \| (E_j g)^a (E_j g)^b \|_{L^t}^{2/t} \leq \sum_j \| (E_j g)^a \|_{L^p}^{2/p_1} \| (E_j g)^b \|_{L^p}^{2/p_2} = \sum_j \| E_j g \|_{L^p}^{2p_1/(tr)} \| E_j g \|_{L^p}^{2p_2/(ts)}.
\]

Now we apply H"older’s inequality to the sum. Since \( \frac{1}{r} + \frac{1}{s} = 1 \),

\[
\sum_j \| E_j g \|_{L^t}^{2p_1/(tr)} \| E_j g \|_{L^t}^{2p_2/(ts)} \leq \left( \sum_j \| E_j g \|_{L^p}^{2p_1/(tr)} \right)^{1 - \alpha} \left( \sum_j \| E_j g \|_{L^p}^{2p_2/(ts)} \right)^\alpha.
\]

Thus it remains to check that \( tr(1 - \alpha) = p_1 \) and \( ts \alpha = p_2 \). But this follows from solving the following three equations

\[
\frac{1}{r} + \frac{1}{s} = 1, \quad \frac{1 - \alpha}{p_1} + \frac{\alpha}{p_2} = \frac{1}{t}, \quad \frac{p_1}{r} + \frac{p_2}{s} = t.
\]

Finally, we record a simple averaging result that follows from the definition of \( A_p \). Fix arbitrary positive reals \( r < s < t \). Suppose \( B_\#(B') \) is a finitely overlapping cover of a ball \( B' \) of radius \( \delta^{-s} \) by balls \( B' \) of radius \( \delta^{-r} \) and similarly define \( B_\#(B') \). Finally, define

\[
B_\#(B') := \{ B': B' \in B_\#(B'), B' \in B_\#(B') \}.
\]

Then we have the following averaging result.

**Proposition 4.**

\[
\frac{1}{|B_\#(B')|} \sum_{B' \in B_\#(B')} A_p(r, B', r)^p = A_p(r, B', r)^p.
\]

**Proof.** This will follow immediately from the definition of \( B_\#(B') \) and \( A_p \). Expanding the left hand side, we have

\[
\frac{1}{|B_\#(B')|} \sum_{B' \in B_\#(B')} A_p(r, B', r)^p
\]

\[
= \frac{1}{|B_\#(B')|} \sum_{B' \in B_\#(B')} 1_{B_\#(B')} \sum_{B' \in B_\#(B')} D_p(r, B')^p
\]

\[
= \frac{1}{|B_\#(B')|} \sum_{B' \in B_\#(B')} D_p(r, B')^p = A_p(r, B', r)^p.
\]

This completes the proof of Proposition 4. \( \square \)

We also remark that ball inflation (1) and (2) will be useful in bounding averages of \( D_l(q, B')^p \) when \( B' \) runs over some finitely overlapping cover \( B \).
1.3. Rescaling and multilinear equivalence. We will also need to make use of two previous results from two different Bourgain-Demeter papers ([2], [3]). Towards the end of our iteration scheme (specifically the proof of Theorem 7), we will need to use the following rescaling lemma.

**Proposition 5** (Section 7 of [2]). Let $0 < \rho \leq 1$. For each interval $I \subset [0, 1]$ of length $\delta^\rho$ and each ball $B \subset \mathbb{R}^2$ of radius $\delta^{-2}$,

$$
\|E_I g\|_{L^p(\mathbb{R}^2)} \lesssim V_p(\delta^{1-\rho})(\sum_{|J| = \delta} \|E_J g\|_{L^p(\mathbb{R}^2)}^2)^{1/2}.
$$

It will happen in Section 2 that to analyze $V_p(\delta)$, it will be easier to analyze a multilinear version of $V_p$. With $I_1, I_2$, and $K$ as defined at the beginning of this section, let $V_p(\delta, K)$ denote the smallest constant such that the inequality

$$
\|\text{geom } E_{I_1, I_2} g\|_{L^p(\mathbb{R}^2)} \leq V_p(\delta, K)\text{geom}(\sum_{|J| = \delta} \|E_J g\|_{L^p(\mathbb{R}^2)}^2)^{1/2}
$$

holds for each ball $B \subset \mathbb{R}^2$ of radius $\delta^{-2}$ and each $g : [0, 1] \to \mathbb{C}$. By Hölder’s inequality, $V_p(\delta, K) \leq V_p(\delta)$ (without using rescaling, just extend $g$ by 0). We will make use that the reverse inequality is essentially true.

**Proposition 6** (Theorem 4.1 of [3]). Fix arbitrary $0 < \delta \leq 1$. For each $K$, there exists $\varepsilon_p(K)$ with $\lim_{K \to \infty} \varepsilon_p(K) = 0$ such that

$$
V_p(\delta) \lesssim_{p,K} \delta^{-\varepsilon_p(K)} V_p(\delta, K).
$$

With all our tools now set up, we next will prove Theorem 1 given we know a certain inequality is true regarding $A_p(u, B^2, u)$. The proof of this inequality will use all the tools in this section.

2. Proving $V_{6, 2}(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon}$

Using the results from Section 1, we can prove the following theorem which is the two dimensional analogue of Theorem 7.3 and Theorem 8.1 in [4].

**Theorem 7.** Let $p < 6$ be sufficiently close to 6. For each $W > 0$, for each sufficiently small $u > 0$, the following inequality holds for each $g : [0, 1] \to \mathbb{C}$, $0 < \delta \leq 1$ and for each ball $B^2$ in $\mathbb{R}^2$ of radius $\delta^{-2}$

$$
A_p(u, B^2, u, g) \lesssim_{\varepsilon} \delta^{-\varepsilon} V_{p, 2}(\delta)^{1-uW} D_p(1, B^2, g).
$$

We will prove this theorem in the next two sections. We first see how this theorem with Proposition 6 work together to prove $V_{6, 2}(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon}$. The proof will be exactly as in Section 8 in [4] and so we will not dwell too much on the details here.

Essentially an application of Hölder’s inequality will show that $V_{6, 2}(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon}$ if $V_{p, 2}(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon}$ for all $p < 6$ sufficiently close to 6 (see the proof of Lemma 8.2 in [4]). Fix $p < 6$ sufficiently close to 6 so that Theorem 7 holds. Let $\eta > 0$ be the such that

$$
\lim_{\delta \to 0} V_{p, 2}(\delta)\delta^{\eta+\sigma} = 0 \text{ for each } \sigma > 0
$$

and

$$
\lim_{\delta \to 0} V_{p, 2}(\delta)\delta^{\eta-\sigma} = \infty \text{ for each } \sigma > 0.
$$
Note that (5) combined with (6) show that \( \log V_{p,2}(\delta) / \log \delta \to \eta \) as \( \delta \to 0 \). Such an \( \eta \) exists since \( V_{p,2}(\delta \epsilon) \leq V_{p,2}(\delta)V_{p,2}(\epsilon) \) and so \( \log V_{p,2} \) is subadditive.

We claim that \( \eta = 0 \). We will use the multilinear equivalence. Let \( B^2 \) be an arbitrary ball of radius \( \delta^{-2} \) in \( \mathbb{R}^2 \). By Cauchy-Schwarz,

\[
\| \text{geom } E_{I_i} g \|_{L^p_n(w_{R^2})} \leq \delta^{-u/2} \| \text{geom} \left( \sum_{J_i \subset I_i \mid |J_i|=\delta^s} |E_{J_i} g|^2 \right)^{1/2} \|_{L^p_n(w_{R^2})}.
\]

Expanding the integral and using that \( \| B_n(B^2) \|_{|B^u|} \sim |B^2| \) and Minkowski’s inequality the right hand side of the above is dominated by

\[
\delta^{-u/2} \left( \frac{1}{|B_n(B^2)|} \sum_{B^s \in B_n(B^2)} D_p(u, B^u, g)^p \right)^{1/p}.
\]

Since \( D_p(u, B^u) \lesssim D_2(u, B^u) \), the above is dominated by \( A_p(u, B^2, u) \) and so

\[
\| \text{geom } E_{I_i} g \|_{L^p_n(w_{R^2})} \leq \delta^{-u/2} A_p(u, B^2, u, g).
\]

Now we are in a position to use Theorem 7 and Proposition 6. Combining the above equation with Theorem 7 and (5) gives that for each \( \sigma > 0 \),

\[
\| \text{geom } E_{I_i} g \|_{L^p_n(w_{R^2})} \lesssim \text{geom}( \sum_{J_i \subset I_i \mid |J_i|=\delta^s} \| E_{J_i} g \|_{L^p_n(w_{R^2})}^2 )^{1/2}.
\]

By using that \( V_p(\delta, K) \) is the best constant for the multilinear inequality (4), Proposition 6 and (6), and the fact that the above inequality holds uniformly for all \( g : [0,1] \to \mathbb{C} \) and balls \( B^2 \), we have

\[
\delta^{-\eta + \sigma + \varepsilon_p(K)} \lesssim \text{geom}( \sum_{J_i \subset I_i \mid |J_i|=\delta^s} \| E_{J_i} g \|_{L^p_n(w_{R^2})}^2 )^{1/2}.
\]

Since this is true for arbitrarily small \( \delta \), the exponent of \( \delta \) on the left hand side must be \( \leq \) the exponent of \( \delta \) on the right hand side. Rearranging gives

\[
\eta \leq \frac{1}{2W} + \frac{\varepsilon + \varepsilon_p(K) + \sigma(2 - uW)}{uW}
\]

and using that \( uW < 1 \) and the above inequality is true for all \( \varepsilon, \sigma > 0 \) and \( K \geq 4 \), it follows that \( \eta \leq 1/(2W) \). Letting \( W \to \infty \) gives that \( \eta = 0 \).

Since \( \eta = 0 \), \( \log V_{p,2}(\delta) / \log \delta \to 0 \) as \( \delta \to 0 \). Therefore \( V_{p,2}(\delta) \lesssim \delta^{-\varepsilon} \) for \( p < 6 \) sufficiently close to 6 which proves that \( V_{p,2}(\delta) \lesssim \delta^{-\varepsilon} \). It now remains to prove Theorem 7.

3. The iteration for dyadic balls

To analyze \( A_p(u, B^2, u) \) with \( u \) sufficiently small, we will first analyze \( A_p(1, B, 1) \) where \( B \) is a ball of radius \( \delta^{-2} \), \( r = 1, 2, \ldots \) and then by redefining \( \delta \) we can trade “large \( r \)” for “small \( u \).” For each \( n = 0, 1, 2, \ldots \), choose a finitely overlapping cover \( B_{2^n}(B^{2^n+1}) \) of a ball \( B^{2^n+1} \) of radius \( \delta^{-2^{n+1}} \) by balls \( B^n \) of radius \( \delta^{-2^{n}} \). With these
covers, define a finitely overlapping cover $B_1(B^{2^n})$ of $B^{2^n}$ by balls $B^1$ as follows (this is analogous to how $B_1(B^{3^n})$ is defined in Proposition 7.5 of [4])

$$B_1(B^{2^n}) = \{ B^1 : B^{2^i} \in B_2(B^{2^{i+1}}) \text{ for } i = 0, 1, \ldots, n - 1 \}.$$

For $p \geq 4$, let $\alpha$ be defined by

$$\frac{1}{p/2} = \frac{\alpha}{p} + \frac{1-\alpha}{2}.$$ 

Then

$$A_p(1, B^2, 1) \lesssim_{\varepsilon, K} \delta^{-\varepsilon} D_p/2 \langle 1, B^2 \rangle \lesssim_{\varepsilon, K} \delta^{-\varepsilon} D_p(1, B^2)^\alpha A_p(2, B^2, 2)^{1-\alpha}$$

where the first inequality is by ball inflation (1) combined with the following application of Hölder (here we use $p \geq 4$)

$$\|E_{I_1}g\|_{L^2_{w_{B^1}}} \leq \|E_{I_1}g\|_{L^{p/2}_{w_{B^1}}}$$

and the second inequality is by Hölder’s inequality for $D_t$, the observation that $A_p(q, B^r, r) = D_2(q, B^r)$, and $L^2$ decoupling (Proposition 3) to increase from $q = 1$ to $q = 2$. Raise (7) to the power $p$ and then average the resulting inequality over balls $B^2 \in B_2(B^4)$. By Hölder’s inequality,

$$\frac{1}{|B_2(B^4)|} \sum_{B^2 \in B_2(B^4)} A_p(1, B^2, 1)^p \lesssim_{\varepsilon, K} \delta^{-\varepsilon^p} \left( \frac{1}{|B_2(B^4)|} \sum_{B^2 \in B_2(B^4)} D_p(1, B^2)^p \right)^\alpha \times \left( \frac{1}{|B_2(B^4)|} \sum_{B^2 \in B_2(B^4)} A_p(2, B^2, 2)^p \right)^{1-\alpha}.$$ 

Applying Lemma 4 for the left hand side of the above equation (here we have also have used how $B_1(B^{2^n})$ is defined) and ball inflation (2) for the first term on the right hand side (note that the second term on the right hand side does not need Lemma 4, however we will need this Lemma at the next step of the iteration to prove (8)), we obtain

$$A_p(1, B^4, 1) \lesssim_{\varepsilon, K} \delta^{-\varepsilon} D_p(1, B^4)^\alpha A_p(2, B^4, 2)^{1-\alpha}.$$ 

Iterating this $r$ times, we have

$$A_p(1, B^{2^r}, 1) \lesssim_{\varepsilon, K, r} \delta^{-\varepsilon} D_p(1, B^{2^r})^\alpha A_p(2, B^{2^r}, 2)^{1-\alpha}.$$ 

4. Finishing the iteration

By raising (8) to the power $p$ and summing over finitely overlapping families of balls, the above inequality holds for all balls $B$ of radius $\geq \delta^{-2^r}$ (once again using ball inflation and Lemma 4). Then

$$A_p(1, B, 1) \lesssim_{\varepsilon, K, r} \delta^{-\varepsilon} D_p(1, B)^\alpha A_p(2, B, 2)^{1-\alpha}.$$ 

To emphasize the dependence of $A_p$ on $\delta$, we will write $A_{p, \delta}$ to emphasize this dependence.
Fix a small $\delta > 0$. We will prove an inequality about $A_{p,\delta}(u, B^2, u)$ for small $u$ depending on $\delta$. Let $\delta' := \delta^u$. Then $A_{p,\delta}(u, B^2, u) = A_{p,\delta'}(1, B^{2/u}, 1)$ since

$$A_{p,\delta'}(1, B^{2/u}, 1)^p = \frac{1}{|B_1(B^{2/u})|} \sum_{B^1 \in B_1(B^{2/u})} D_2(1, B^1)^p$$

$$= \frac{1}{|B_1(B^{2/u})|} \sum_{B^1 \in B_1(B^{2/u})} \text{geom}(\sum_{J_{i, u} \subseteq J} \|E_{J,i, u}g\|_{L^2_{\text{geom}}(B^1)}^2)^{p/2}$$

where $B_1(B^{2/u})$ is a finitely overlapping cover of $B^{2/u}$ (a ball of radius $\delta'^{-2}/u$) by balls of radius $\delta'^{-1}$ and

$$A_{p,\delta}(u, B^2, u)^p = \frac{1}{|B_u(B^2)|} \sum_{B^2 \in B_u(B^2)} \text{geom}(\sum_{J_{i, u} \subseteq J} \|E_{J,i, u}g\|_{L^2_{\text{geom}}(B^2)}^2)^{p/2}$$

where $B_u(B^2)$ is a finitely overlapping cover of $B^2$ (a ball of radius $\delta^{-2}$) by balls of radius $\delta^{-u}$. Similarly, $A_{p,\delta}(2u, B^2, 2u) = A_{p,\delta'}(2, B^{2/u}, 2)$ and $D_{p,\delta}(u, B^2) = D_{p,\delta'}(1, B^{2/u})$.

Since (9) is true for all balls $B$ of radius $\geq \delta'^{-2}$, if $u > 0$ is such that $2^r \leq 2/u$, then

$$A_{p,\delta'}(1, B^{2/u}, 1) \lesssim_{\varepsilon, K, r} \delta'^{-\varepsilon} D_{p,\delta'}(1, B^{2/u})^\alpha A_{p,\delta'}(2, B^{2/u}, 2)^{1-\alpha}$$

and hence

$$A_{p,\delta}(u, B^2, u) \lesssim_{\varepsilon, K, r} \delta^{-\varepsilon} D_p(u, B^2)^\alpha A_{p}(2u, B^2, 2u)^{1-\alpha}$$

We summarize this result in the following proposition.

**Proposition 8.** Let $p \geq 4$ and $u > 0$ such that $2^{r-1}u \leq 1$. Then for each ball $B^2$ of radius $\delta^{-2}$, we have

$$A_p(u, B^2, u) \lesssim_{\varepsilon, K, r} \delta^{-\varepsilon} D_p(u, B^2)^\alpha A_p(2u, B^2, 2u)^{1-\alpha}$$

We now iterate Proposition 8. If $2^{r}u \leq 1$, then $u' = 2u$ satisfies the condition $2^{r-1}u' \leq 1$ and hence

$$A_p(2u, B^2, 2u) \lesssim_{\varepsilon, K, r} \delta^{-\varepsilon} D_p(2u, B^2)^\alpha A_p(4u, B^2, 4u)^{1-\alpha}$$

(note that key feature is that the ball $B^2$ stays fixed so we can now iterate) and hence

$$A_p(u, B^2, u) \lesssim_{\varepsilon, K, r} \delta^{-\varepsilon} D_p(u, B^2)^\alpha D_p(2u, B^2)^\alpha(1-\alpha) A_p(4u, B^2, 4u)^{(1-\alpha)^2}$$

Iterating this process $M$ times gives the following result.

**Proposition 9.** Let $p \geq 4$. Given integers $r, M \geq 1$ and let $u > 0$ be such that $2^{r-1+M}u \leq 1$. Then for each ball of radius $\delta^{-2}$, we have

$$A_p(u, B^2, u) \lesssim_{\varepsilon, K, r, M} \delta^{-\varepsilon} \prod_{i=0}^{M} D_p(2^i u, B^2)^\alpha(1-\alpha)^i A_p(2^{M+1}u, B^2, 2^{M+1}u)^{(1-\alpha)^M+1}$$

By how $V_p(\delta)$ is defined,

$$D_p(2^i u, B^2) \leq V_p(\delta) D_p(1, B^2)$$

(10)
for $0 \leq i \leq M$. This inequality requires no rescaling. We illustrate how to prove such a result by just considering the case when $i = M$, the result for other values of $i$ follow similarly. To show (10), it is enough to show that for all $g : [0, 1] \to \mathbb{C}$,

$$
( \sum_{J \subseteq I_i} \|E_J g\|_{L^{p}(w_{B^2})}^2 )^{1/2} \leq V_p(\delta)( \sum_{J' \subseteq I_i} \|E_{J'} g\|_{L^{p}(w_{B^2})}^2 )^{1/2}.
$$

Thus it is enough to show that for all $J \subseteq I_i$ with $|J| = \delta^{2M}u$ we have

$$(11) \quad \|E_J g\|_{L^{p}(w_{B^2})} \leq V_p(\delta)( \sum_{J' \subseteq J, |J'| = \delta} \|E_{J'} g\|_{L^{p}(w_{B^2})}^2 )^{1/2}$$

for all $g : [0, 1] \to \mathbb{C}$.

Now by how $V_p(\delta)$ is defined, we have

$$(12) \quad \|E_{[0,1]} h\|_{L^{p}(w_{B^2})} \leq V_p(\delta)( \sum_{J' \subseteq [0,1], |J'| = \delta} \|E_{J'} h\|_{L^{p}(w_{B^2})}^2 )^{1/2}$$

for all $h : [0, 1] \to \mathbb{C}$. In particular, this is true for all functions $h$ which equal 0 outside the interval $J$. Since $E_{[0,1]}(gJ) = E_J g$, (11) then follows from (12).

Thus inserting (10) into Proposition 9 gives

$$A_p(u, B^2, u) \lesssim_{\varepsilon, K, r, M} \delta^{-\varepsilon} V_p(\delta)^{1-(1-\alpha)M+1} D_p(1, B^2)^{1-(1-\alpha)M+1} A_p(2^{M+1}u, B^2, 2^{M+1}u)^{(1-\alpha)M+1}. $$

Since $\alpha = (4 - p)/(2 - p)$, $2(1 - \alpha) > 1$ for $p < 6$ and for each $W > 0$, there exists an $M$ sufficiently large such that $2^{M+1}(1 - \alpha)M+1 > W$.

We can now prove Theorem 7.

**Proof of Theorem 7.** Fix an arbitrary $W > 0$. Let $u$ be so small and choose $r$ and $M$ (depending on $W$) such that $uW < 1$, $2^{M+1}(1 - \alpha)M+1 > W$ and $2^{r+M-1}u \leq 1$. Hölder’s inequality applied to $\sum_{B_2^{M+1}u \in B_2^{M+1}u} (B^2)$ and using that $|B_2^{M+1}u(B^2)| |B_2^{2M+1}u| \sim |B^2|$ yields that

$$A_p(2^{M+1}u, B^2, 2^{M+1}u) \lesssim D_p(2^{M+1}u, B^2).$$

Rescaling as in Proposition 5 gives $D_p(2^{M+1}u, B^2) \lesssim V_{p,2}(\delta^{1-2M+1}u) D_p(1, B^2)$ and thinking that $V_{p,2}(\delta)$ as a power of $\delta$ (strictly speaking we cannot do this by our definition of $V_{p,2}(\delta)$, however, we just want to prove $V_{p,2}(\delta) \lesssim \delta^{-\varepsilon}$ and to make this completely rigorous one could go through the argument and replace $V_{p,2}(\delta)$ by $\delta^{a_{p,2}}$ and analyze the $a_{p,2}$ instead) shows that

$$D_p(2^{M+1}u, B^2) \lesssim V_{p,2}(\delta)^{1-2M+1}u D_p(1, B^2)$$

and hence

$$A_p(u, B^2, u) \lesssim_{\varepsilon, K, W} \delta^{-\varepsilon} V_p(\delta)^{1-2(1-\alpha)M+1} D_p(1, B^2) \lesssim_{\varepsilon, K, W} \delta^{-\varepsilon} V_{p,2}(\delta)^{1-uW} D_p(1, B^2).$$

This completes the proof of Theorem 7. $\Box$
REFERENCES