PRACTISE PROBLEMS FOR MIDTERM I

These are some practise problems for the midterm in class on 28th April 2017. Also look at homework problems assigned so far.

1. (i). Define the greatest common divisor \(\text{gcd}(a, b)\) of two non-zero integers \(a\) and \(b\).

(ii). Show that if \(a\) is an odd integer and \(b\) is an even integer then \(\text{gcd}(a, b) = \text{gcd}(a+b, a-b)\). Show that the statement may be false if \(a\) and \(b\) are both odd.

2. State the fundamental theorem of arithmetic. Using it prove that the equation \(x^2 = 3\) has no solution in the rationals.

3. If \(n\) is a positive integer, and \(a_1, \ldots, a_{n+1}\) are \(n + 1\) integers, then at least two of them are congruent to each other modulo \(n\). Prove that there are \(n\) integers \(b_1, \ldots, b_n\) that are all incongruent modulo modulo \(n\).

4. Consider the integer \(n = 265\). How many solutions does the congruence \(15x \equiv 3 \pmod{265}\) have modulo 265? Justify your answer.

5. Show that if \(a\) is an integer coprime to 7 (i.e., \(\text{gcd}(a, 7) = 1\)), then for any integer \(b\), \(\text{gcd}(a, b) = \text{gcd}(a, 7b)\). Show that the statement may be false if \(a\) is not coprime to 7.

6. If \(p\) is a prime, and \(a, b, r\) are positive integers, prove that if \(p^r\) divides \(ab\), then \(p^r\) divides either \(a^r\) or \(b^r\) (or both). Show by an example that if \(r > 1\), then in general \(p^r\) may divide neither \(a\) nor \(b\).

7. Let \(n\) be a non-zero integer, and let \(a_1, \ldots, a_m, \ldots\) be an infinite sequence of integers. Then show that it has an infinite subsequence \(a_{i_1}, \ldots, a_{i_m}, \ldots\) consisting of integers that are congruent to each other modulo \(n\).
8. Show that if \( n \) is an odd integer, and \( n = m^2 \) for some \( m \in \mathbb{Z} \), then \( n \) is congruent to 1 mod 8.

9. Prove that if \( a, b \) are integers such that \( \gcd(a^3, b^3) = 1 \) then \( \gcd(a, b) \) is also 1.

10. Consider the integer \( n = 269 \). Discuss if 269 is a prime. How many solutions does the congruence \( 268x \equiv 199 \pmod{269} \) have modulo 269? Justify your answer.

11. Prove by induction that \( n^3 - n \) is divisible by 3 for all integers \( n \).