Exercise 1: Consider the function \( f(x, y) = x^2y \).

1. Calculate the directional derivative of \( f \) at \((1, 1)\) in the direction \( (2, 1) \).

\[
\text{Directional derivative} = \nabla f \cdot \mathbf{v} = \nabla f \left( \frac{\mathbf{v}}{||\mathbf{v}||} \right)
\]

\[
\nabla f(1,1) = (2, 1)
\]

\[
\mathbf{v} = \frac{(2, 1)}{\sqrt{5}}
\]

\[
\frac{\partial f}{\partial \mathbf{v}} = \nabla f \cdot \mathbf{v} = \sqrt{5}
\]

2. Find the rate of change of \( f \) as measured by an object at position \((2, 0)\) travelling with velocity \((0, 3)\).

\[
\frac{df}{dt} = \nabla f \cdot \mathbf{v} = 12
\]

3. Evaluate \( \frac{df}{dt} \) for the path \( \mathbf{r}(t) = (\cos(t), \sin(t)) \).

\[
\mathbf{r}'(t) = (-\sin(t), \cos(t))
\]

\[
\frac{df}{dt} = \nabla f \cdot \mathbf{r}'(t) = -2\sin(t)\cos(t) + \cos^2(t)
\]

4. Find the direction of steepest ascent of \( f \) at \((2, 2)\).

\[
\nabla f(2,2) = (8, 4)
\]

\[
\mathbf{v} = \frac{(8, 4)}{8} = \left( \frac{1}{2}, \frac{1}{2} \right)
\]

5. In what direction(s) will \( f \) have directional derivative 0 at \((2, 2)\)?

\[
\nabla f \cdot \mathbf{v} = 0 \Rightarrow \mathbf{v} = (1, -\frac{1}{2}) \text{ or } (1, \frac{1}{2})
\]

Exercise 2: Find the tangent plane to the surface \( x^2 + y^2 + z^2 = 4 \) at the point \( \left( \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right) \).

\[
\mathbf{r}(t) = \left( \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right) \text{ satisfies } x^2 + y^2 + z^2 = 4
\]

Exercise 3: Prove that the gradient vector is normal to level curves of the function \( f(x, y) \).

Hint: Parameterise the level curve \( f(x(t), y(t)) = c \) by some (unknown) function \( r(t) \).

Suppose \( \mathbf{r}(t) = (x(t), y(t)) \) is the level curve \( f(x(t), y(t)) = c \Rightarrow \frac{df}{dt}(x(t), y(t)) = 0 \).

\[
\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0
\]

\[
\mathbf{r}'(t) = \mathbf{r}'(t) \Rightarrow \mathbf{r}'(t) \perp \nabla f \text{ at every point on } \mathbf{r}(t)
\]

2D: tangent line to level curve should always be normal to \( \nabla f(x, y) \)

3D: tangent plane to level surface should always be normal to \( \nabla f(x, y, z) \)