**Exercise 1:** Show that the curvature of a function \( y = f(x) \) can be found from the formula
\[
\kappa(x) = \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}.
\]
Apply to curve \( r(x) = (x, f(x)) \)
\[
\begin{align*}
\kappa'(x) &= f''(x) \quad r''(x) = (0, f''(x)) \\
&= 1 \quad f''(x) \\
&= 1 + f'(x)^2 \\
&= 1 + f''(x)^2 \\
&= \frac{|f''(x)|}{(1 + [f'(x)]^2)^{3/2}}
\end{align*}
\]
Recall the curvature formula for a parameterisation \( r(t) \),
\[
\kappa(t) = \frac{||r'(t) \times r''(t)||}{||r'(t)||^3}.
\]
You can’t calculate a cross-product of two vectors in \( \mathbb{R}^2 \), so make the \( z \)-component equal to 0, to put the vectors in \( \mathbb{R}^3 \).

**Exercise 2:** When we find an arc-length parameterisation, the lower limit of the arc-length integral is not particularly important. For example, consider the parametric curve \( r(t) = (\cos(t^3), \sin(t^3)) \). Find the arc-length equation
\[
s = g(t) = \int_a^t ||r'(u)|| \, du,
\]
and hence find an arc-length parameterisation of \( r(t) \). How does changing \( a \) affect the result?

This explains why we can, for example, change \( a \) to \(-\infty\) in this week’s homework.

**Exercise 3:** Curvature has a very intuitive meaning as the reciprocal of the radius of a circle. At any point \( P = r(t) \) on a space curve, there is a circle which is tangent to the curve and has the same curvature, called the osculating circle. The osculating circle is contained in the plane of curvature (the plane spanned by \( T \) and \( N \)) and will have radius \( r_P = 1/\kappa_P \) (the reciprocal of the curvature at \( P \)). The radius \( r_P \) is known as the radius of curvature and the centre of the osculating circle is known as the centre of curvature.

Find a parameterisation of the osculating circle to \( y = x^2 \) at \( x = 1/2 \).

First, find \( \kappa(x) \) using the formula from exercise 1. Then find the centre of the circle, which will a distance \( 1/\kappa(1/2) \) in the direction of \( N \) from the point \( P = (1/2, 1/4) \).

Use this information to parameterise the osculating circle.
\[
\begin{align*}
\vec{r}(t) &= (t, t^2) \\
\kappa(t) &= \frac{2}{(1+4t^2)^{3/2}} \\
\kappa(1/2) &= \frac{1}{2} \\
r_P &= \sqrt{2} \\
\vec{N}(1/2) &= \frac{\vec{r}'(1/2)}{||\vec{r}'(1/2)||} \\
&= \text{obtained by rotation} \\
&= \text{get central point from } r_P \\
&\text{and unit normal vector } N
\end{align*}
\]
\[
\begin{align*}
\vec{P}C &= \vec{x}P \cdot \vec{n} = (-1, 1) \quad \Rightarrow \quad C = (-\frac{1}{2}, \frac{\sqrt{3}}{2}) \\
\therefore \quad \text{The circle } \vec{x}(0) &= \left(-\frac{1}{2} + 2\cos \theta, \frac{\sqrt{3}}{2} \sin \theta \right)
\end{align*}
\]

2D curve Geometry :

<table>
<thead>
<tr>
<th>arc-length s</th>
<th>any parameter t</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{T}''(s) )</td>
<td>( \vec{F}(t) )</td>
</tr>
<tr>
<td>( \vec{T} = \vec{T}'(s) )</td>
<td>( \vec{T} = \frac{\vec{T}'(t)}{</td>
</tr>
<tr>
<td>( K(s) =</td>
<td></td>
</tr>
<tr>
<td>( \vec{N} = \frac{\vec{T}''(s)}{</td>
<td></td>
</tr>
</tbody>
</table>

A useful trick for getting \( \vec{N} \):

1. Calculate \( \vec{T} \)
2. Distinguish \( \vec{T}' \) is on which direction
3. Rotate \( \vec{T} \) on that direction to get \( \vec{N} \)

**Case 1**
\[
\vec{N} = (-b, a)
\]

**Case 2**
\[
\vec{N} = (b, -a)
\]