Ex 5. Def \( f(c,d) = \frac{1}{n} (a_i - c)^2 + (b_i - d)^2 \)

\[ \frac{\partial f}{\partial c} = \frac{1}{n} 2(a_i - c)(-1) = 0 \]
\[ \Rightarrow c = \frac{a_i}{n} \]

\[ \frac{\partial f}{\partial d} = \frac{1}{n} 2(b_i - d)(-1) = 0 \]
\[ \Rightarrow d = \frac{b_i}{n} \]

This is the only critical point of \( f \).

And if we check 2nd derivatives,

\[ \frac{\partial^2 f}{\partial c^2} = \frac{2}{n} > 0, \quad \frac{\partial^2 f}{\partial d^2} = \frac{2}{n} > 0, \quad \frac{\partial^2 f}{\partial c \partial d} = 0 \]

This critical point is local minima.

And since \( f(c,d) \) is the sum of two quadratic functions, it should obtain the global minimum at the local minima.

\[ \text{v.e. } \min_{c,d} f(c,d) = f\left( \frac{a_i}{n}, \frac{b_i}{n} \right) \]
Ex 1. A: local maxima C: local minima B, D: saddle points

Ex 2. Df doesn't exist at (0,0) because \[ \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{1}{h} \] doesn't exist.

\[ \therefore (0,0) \text{ is critical point.} \]

\[ V(x,y) \neq (0,0), \quad f(x,y) > 0 \quad \therefore (0,0) \text{ is a global minima (also local minima).} \]

Ex 3. Let \( f(x,y) = x \)

Move \((x,y)\) along line \(x+y = \frac{1}{2}\) \(\Rightarrow x \in (-\infty, 1]\)

\(\therefore f(x,y)\) can take any value from \(-\infty\) to \(+\infty\)

f is not bounded, therefore, doesn't have global maximum value.

Ex 4. We consider 4 parts

1. Interior critical points

\[ \begin{align*}
\frac{\partial f}{\partial x} &= 2x + 2y \\
\frac{\partial f}{\partial y} &= 2xy + 2y
\end{align*} \]

\(\Rightarrow \begin{align*}
(1) \quad &x = 0, y = 0 \\
(2) \quad &x = -1, y = \frac{\sqrt{2}}{2} \quad \text{outside D}
\end{align*} \]

Obviously \( f(0,0) = 0 \) is global minimum for \( f \)

And \( f(x,0) = 0 \Rightarrow x = 0, y = 0 \). Global minima is unique.

Now we find global maxima. It can only be found on boundary

2. Boundary part I

\( f(x,0) = x^2 \leq f(1,0) = 1 \)

3. Boundary part II

\( f(0,y) = y^2 \leq f(0,1) = 1 \)

4. Boundary part III

\( f(x,1-x) = x^2 + x(1-x)^2 + (1-x)^2 = x^2 - x + 1 \in [g(x)], \)

\( g(x) = 3x^2 - 1 \). \((0, \frac{1}{3}) - (\frac{1}{3}, 1) + \therefore g(x) (0, \frac{1}{3}) \uparrow (\frac{1}{3}, 1) \)

\[ \therefore g(x) \text{ take maximum values at 0 and 1.} \quad g(0) = 2, g(1) = 1 \]

\( \therefore f \) take maximum at \((0,1)\) and \((1,0)\)

Overall, maximum values are taken at \((0,1)\) and \((1,0)\) \( \therefore \max f = 2 \)