Chapter 13. Vector geometry

- Vector algebra (Dot product & cross product)

\[ \vec{v} + \vec{w} = (a+c, b+d) \]
\[ \vec{v} - \vec{w} = (a-c, b-d) \]
\[ \vec{v} \cdot \vec{w} = (x, y, z) \]
\[ ||\vec{v}|| = \sqrt{a^2 + b^2} \]

\* Dot product (scalar)

\[ \vec{v} \cdot \vec{w} = ac + bd \]
\[ = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \cos \theta \]

\* Cross product (vector)

\[ \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ w & u & v \end{vmatrix} = (yw - uz, zv - uw, ux - vy) \]

\[ ||\vec{v} \times \vec{w}|| = ||\vec{v}|| \cdot ||\vec{w}|| \cdot \sin \theta \]

Direction: determined by R.H. rule.

Notice: Cross product is only defined for 3D vectors!

If we want to calculate cross of 2D vectors, first expand to 3D. \((a, b, 0)\) and \((c, d, 0)\)
Triangle inequality: \[ \| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \| \]

"\[ = \]" holds only if \[ \theta = 0, \theta = 0 \] or \[ \theta = \pi \]

Prove: \[ \| \mathbf{v} + \mathbf{w} \| = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) \]
\[ = \| \mathbf{v} \| ^2 + 2 \mathbf{v} \cdot \mathbf{w} + \| \mathbf{w} \| ^2 \]
\[ = \| \mathbf{v} \| ^2 + 2\| \mathbf{v} \| \| \mathbf{w} \| \cos \theta + \| \mathbf{w} \| ^2 \]
\[ \leq \| \mathbf{v} \| ^2 + 2\| \mathbf{v} \| \| \mathbf{w} \| + \| \mathbf{w} \| ^2 \]
\[ = (\| \mathbf{v} \| + \| \mathbf{w} \| ) ^2 \]

Equation of line, plane and sphere

- Line: \[ \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{u} \quad (t \in \mathbb{R}) \]
  
  Component form
  \[ x(t) = x_0 + tu, \quad y(t) = y_0 + tv, \quad z(t) = z_0 + tw \]

- Plane: \[ (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0 \]

  \[ (x-x_0, y-y_0, z-z_0) \cdot (a,b,c) = 0 \]

  \[ ax + by + cz = ax_0 + by_0 + cz_0 \]
  \[ = \text{d}. \]

  \[ \mathbf{n} = (a,b,c) \] normal vector.
Consider the two planes \( x+y=3 \) and 
\( x+2y-z=4 \)

(a) Find the parametric equation of the line that is the intersection of the two planes.

(b) Find a plane passing through the origin that is perpendicular to both planes.

(a) \[
\begin{align*}
\begin{cases}
    x+y=3 \\
    x+2y-z=4
\end{cases}
\Rightarrow \begin{cases}
    y=3-x \\
    z=2-x
\end{cases}
\]

\[
\therefore \text{ Take } x=t \text{ as parameter, then } \n(0)= (t, 3-t, 2-t)
\]

(b) perpendicular to both planes

\[
\Rightarrow \text{ parallel to both normal vectors}
\]

\[
\Rightarrow \hat{n}_1 \parallel \hat{n} \times \hat{n}_2
\]

\[
\hat{n}_1 = (1, 1, 0) \quad \hat{n}_2 = (1, 2, -1) \Rightarrow \hat{n} \times \hat{n}_2 = \begin{vmatrix}
    \hat{i} & \hat{j} & \hat{k} \\
    1 & 1 & 0 \\
    1 & 2 & -1
\end{vmatrix} = (-1, 1, 2)
\]

\[
\therefore \text{ Let } \hat{n} = (-1, 1, 1), \text{ passing origin } \Rightarrow d=0
\]

\[
-x+y+z=0
\]

- **Tangent vector, speed and arc-length**

  \[ \vec{r}(t) = (x(t), y(t), z(t)) \quad t \in [a, b] \]

  \[ \vec{r}'(t) = (x'(t), y'(t), z'(t)) \]

  unit tangent vector

  \[ \vec{T}(t) = \frac{\vec{r}'(t)}{|| \vec{r}'(t) ||} \]

  speed \( v(t) = || \vec{r}'(t) || \)

  \[ s(t) = \int_a^t || \vec{r}'(u) || \, du \] arc-length

  \[ \frac{ds}{dt} = v(t) \]

  arc-length parametrization

  \( s = s(t) \quad \rightarrow \quad t = t(s) \)

  \[ \vec{r}(t) \rightarrow \vec{r}_s(s) = \vec{r}(t(s)) \]

  \[ \vec{r}'(t) \rightarrow \vec{r}'_s(s) = \vec{r}'(t(s)) \cdot t'(s) \] (chain rule)

- **Curvature, normal vector and torsion**

  \[ K(s) = || \frac{d^2}{ds^2} \vec{T}(s) || \] (arc-length)

  \[ K(t) = || \frac{d^2}{dt^2} \frac{dt}{ds} \cdot \vec{T}(s) || \]

  \[ = \frac{1}{v(t)} || \vec{T}'(t) || = \frac{|| \vec{T}'(t) ||}{|| \vec{T}(t) ||} \]

  \[ \vec{N}(s) = \frac{\vec{T}'(s)}{|| \vec{T}'(s) ||} \]

  \[ \vec{B}(s) = \vec{T}(s) \times \vec{N}(s) \]

  \[ \vec{T}'(s) = K(s) \vec{N}(s) \]
Another formula for curvature:

\[ K(t) = \frac{||\vec{r}'(t) \times \vec{r}''(t)||}{||\vec{r}'(t)||^3} \]

We can also get curvature from the description of 3D motion:

- Normal vector

  generally \( \vec{N}(s) = \frac{\vec{r}'''(s)}{||\vec{r}'''(s)||} \) (arc-length)

  \( \vec{N}(t) = \frac{\vec{r}'''(t)}{||\vec{r}'''(t)||} \)

A useful trick for 2D case:

First find \( \vec{T}(t) \).

\( \vec{N}(t) \perp \vec{T}(t) \) only 2 choices.

From the plot of curve pick the one on the direction how \( \vec{T}(t) \) changes.

- Torsion

\[ \tau = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{||\vec{r}' \times \vec{r}''||^2} \]
Description of 3D motion
\[ \vec{r}(t) = \vec{P}(t), \quad \vec{v}(t) = \frac{d}{dt} \vec{r}(t) \],
\[ \vec{a}(t) = \frac{d^2}{dt^2} \vec{r}(t) \]
Decompose on \( \vec{T} \) and \( \vec{N} \)
\[ \vec{a}(t) = a_T \vec{T} + a_N \vec{N} \]
\[ a_T = \vec{a} \cdot \vec{T} = \vec{v} \cdot \vec{v}'(t) \]
\[ a_N = \vec{a} \cdot \vec{N} = \| \vec{a} \times \vec{T} \| = k(t) \| \vec{v}(t) \|^2 \]
If we know \( \vec{v}(t) \) and \( a_N \), then \( k(t) = \frac{a_N}{\| \vec{v}(t) \|^2} \) gives the formula for curvature.

\((\text{HW6, Q2})\)
\[ \vec{a} = \frac{q}{m} \frac{\vec{E}}{E} = (0, -1) \quad \vec{v}_0 = (1, 0) \quad \vec{r}_0 = (0, 1) \]
\[ \Rightarrow \vec{v}(t) = \vec{v}_0 + \vec{a}t = (1, -t) \]
\[ \vec{r}(t) = \vec{r}_0 + \int_0^t \vec{v}(u) \, du = (t, 1 - \frac{1}{2} t^2) \]

Solution 2
\[ k(t) = \frac{\| \vec{v}(t) \times \vec{a}(t) \|}{\| \vec{v}(t) \|^3} = \frac{1}{(1 + t^2)^{3/2}} \]
\[ \begin{align*}
\vec{t} &= \frac{\vec{v}(t)}{v(t)} = \frac{1}{\sqrt{1+t^2}}(1, -t) \\
\vec{N} &= \frac{1}{\sqrt{1+t^2}}(-t, -1) \quad \text{(using the trick introduced above)} \\
\vec{a}_T &= \vec{a} \cdot \vec{t} = \frac{t}{\sqrt{1+t^2}} \\
\vec{a}_N &= \vec{a} \cdot \vec{N} = \frac{1}{\sqrt{1+t^2}} \\
K(t) &= \frac{\vec{a}_N \cdot \vec{v}}{v^2} = \frac{1}{(1+t^2)^{\frac{3}{2}}} 
\end{align*} \]
Chapter 15: Differentiation in Several Variables

- Limit & Continuity
  Def: \( \lim_{(x,y) \to (a,b)} f(x,y) = L \)
  For any \( \varepsilon > 0 \), \( \exists \delta > 0 \), s.t. if \( \sqrt{(x-a)^2 + (y-b)^2} < \delta \),
  then \(|f(x,y) - L| < \varepsilon\)

Tests for limit existence:

1. Linear test (necessary but not sufficient)
   \[ y-b = m(x-a) \]
   \[ \lim_{x \to a} f(x, b+m(x-a)) = L \] independent of choice of \( m \). If dependent, then limit doesn't exist.

2. Polar coordinate (necessary and sufficient)
   \[ \begin{align*}
   x &= a + r \cos \theta
   \\
   y &= b + r \sin \theta
   \end{align*} \]
   \[ \lim_{r \to 0} f(a+r \cos \theta, b+r \sin \theta) = L \] independent of choice of \( \theta \) (any function).
\[\text{(1)} \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2+xy^2} = 0\]

\[\text{(2)} \lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4+y^2} \text{ doesn't exist}\]

1. Use polar coordinate
   \[
   \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2+xy^2} = \frac{r^3 \cos^2 \theta (r) \sin \theta (r)}{r^2} = r \cos^2 \theta (r) \sin \theta (r)
   \]
   \[
   \left| \frac{xy^2}{x^2+xy^2} \right| \leq |r| \to 0
   \]
   \[
   \lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2+xy^2} = 0
   \]

2. We can use polar coordinate
   \[
   \frac{x^2y}{x^4+y^2} = \frac{r^3 \cos^2 \theta (r) \sin \theta (r)}{r^4 \cos^4 \theta (r) + r^2 \sin^2 \theta (r)}
   \]
   If we take \(\sin \theta (r) = r\), then \(\cos \theta (r) = 1-r^2\)
   \[
   \Rightarrow \frac{x^2y}{x^4+y^2} = \frac{r^3 (1-r^2) r}{r^4 (1-r^2)^2 + r^4} = \frac{(1-r^2)^2}{(1-r^2)^2 + 1} \to \frac{1}{2} \quad (r > 0)
   \]
   But if \(\theta (r) = 0\), then \(\frac{x^2y}{x^4+y^2} = 0\)
   \(\therefore\) limit depends on choice of \(\theta (r)\), limit doesn't exist
We can also find linear test is not sufficient

\[ y = mx - \frac{m^2 y^2}{x^4 + y^2} = \frac{mx^3}{x^4 + m^2 x^2} \leq \frac{m x^2}{m^2 x^2} = \frac{x}{m} \to 0 \]

but if \( y = m x^2 \)

\[ \frac{\partial^2 y}{\partial x^2} = \frac{m x^4}{(1+m^2) x^4} = \frac{m}{1+m^2} \text{ depends on } m. \]

* Partial derivative & differentiability.

\( f(x, y) \) is differentiable at \((a, b)\)

\( \Leftrightarrow \frac{df}{dx}(a, b) + \frac{df}{dy}(a, b)(y - b) \]

is a good approximation of \( f \) near \((a, b)\)

\[ \lim_{(x, y) \to (a, b)} \frac{f(x, y) - L(x, y)}{\sqrt{(x-a)^2 + (y-b)^2}} = 0 \]

\( \Leftrightarrow \frac{df}{dx}(a, b) \) gives the tangent plane of \( f(x, y) \) at \((a, b)\)

\[ \exists \frac{df}{dx}, \frac{df}{dy} \text{ exist at } (a, b) \text{ but not necessary but not sufficient condition} \]

\[ \frac{df}{dx}, \frac{df}{dy} \text{ continuous near } (a, b) \text{ sufficient but not necessary} \]
Ex. \( f(x,y) = \sqrt{x^2y^2} \) not differentiable at \((0,0)\) but differentiable at other points

If: When \( x \to 0 \) or \( y \to 0 \), \( x^2y^2 \to 0 \) so we can apply chain rule.

\[
\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2y^2}}, \quad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2y^2}} \text{ continuous at } (x,y)
\]

\[
\text{Differentiable at } (x,y) \neq (0,0)
\]

When \( x \to 0 \) and \( y \to 0 \), we need to use limit

\[
\lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\sqrt{h^2-0}}{h} = \lim_{h \to 0} \frac{|h|}{h}
\]

doesn't exist \( \Rightarrow \frac{\partial f}{\partial x} \) doesn't exist at \((0,0)\)

Similarly \( \frac{\partial f}{\partial y} \) doesn't exist at \((0,0)\)

\[
\text{Not differentiable at } (0,0)
\]

- Gradient & directional derivative:

\[
\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})
\]

\[
Df \cdot \vec{u} = \frac{\partial f}{\partial x} u_x + \frac{\partial f}{\partial y} u_y
\]
Derivative along a path $\vec{P}(t)$

$$f'_{t}(t) = Df(\vec{P}(t)), \vec{P}'(t)$$

$$= Df \cdot \|\vec{P}'(t)\|$$

**Example**

1. If $f(x, y), \vec{P}(t) = (x(t), y(t))$ is level curve then $Df \perp \vec{P}'(t)$

2. If $f(x, y, z), Df$ is normal to the tangent plane of level surface at any $P$

Applying chain rule for path $Df \cdot \vec{P}'(t) = 0$  

$\implies Df \perp \vec{P}'(t)$

2. Suppose $\vec{P}(t) = (x(t), y(t), z(t))$ is any curve on level surface passing $P$, then $f(\vec{P}(t)) = c$  

$\implies Df \perp \vec{P}'(t)$

Since $\vec{P}'(t)$ can be any vector on tangent plane. This implies $Df$ normal to tangent plane of level surface at $P$. 
Optimization & Lagrange Multiplier

- Unconstrained optimization:

Find extreme value of \( f(x,y) \) in some domain \( D \).

**Step 1:** Solve \( \nabla f = 0 \), find critical points.

**Step 2:** Check 2nd derivatives \( f_{xx}, f_{xy}, f_{yy} \).
Verify local min/max, or saddle points.

**Step 3:** Compare with boundary value.
(if there is boundary)

- Constrained optimization:

Find extreme value of \( f(x,y) \) with constraint \( g(x,y) = 0 \) in domain \( D \).

**Lagrange Multiplier:**

Solve \[
\begin{align*}
\nabla f &= \lambda \nabla g \\
g(x,y) &= 0
\end{align*}
\]
find critical point

(If \( \lambda(t) \) satisfy \( g(x(t),y(t)) = 0 \), then it becomes to minimize \( f(\lambda(t)) \))
If $g(x,y,z)$ defines a closed and bounded curve, we don't need to check boundary. Only compare the value at critical points we can find max and min.

\[ \text{Ex 8} \] (Final Practice 26)

Solve $\min f(x,y,z) = z$

\[ \begin{cases} x+y+z = 1 \\ x^2+y^2 = 1 \end{cases} \]

Lagrange multiplier

\[ \nabla f = \lambda \nabla g + \mu \nabla h \]

$(0, 0, 1) = \lambda (1, 1, 1) + \mu (2x, 2y, 2z)$

\[ \Rightarrow \begin{cases} \lambda + 2\mu x = 0 \\ \lambda + 2\mu y = 0 \\ \lambda = 1 \end{cases} \Rightarrow \lambda = 1 \\
\lambda = 2\mu \Rightarrow x = y = -\frac{1}{2\mu} \]

\[ x^2 + y^2 = \frac{1}{2\mu} = 1 \Rightarrow \mu = \frac{1}{2} \]

Critical points:

$0 \mu = \frac{1}{2}$, \hspace{1cm} $x = y = -\frac{1}{2}$, \hspace{1cm} $z = |x-y| = 1/2$
$\theta \mu = -\frac{1}{k}, \chi = y = \frac{\sqrt{2}}{2}, \quad z = 1 - x - y = 1 - \sqrt{2}$

Since the intersection of $x + y + z = 1$ and $\chi = y = 1$

it's a closed curve.

It should have max and min at critical points.

$\therefore \text{max } f(x, y) = f\left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right) = 1 + \sqrt{2}$

and $\text{min } f(x, y) = f\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right) = 1 - \sqrt{2}$