ON THE MUMFORD-TATE CONJECTURE OF ABELIAN FOURFOLDS

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Abstract. We prove the Mumford-Tate conjecture for absolutely simple abelian fourfolds with trivial endomorphism algebras.

The main goal of this paper is to prove the Mumford-Tate conjecture for certain abelian fourfolds. Let $A/F$ be an abelian variety defined over a number field $F$ of dimension $n$. Fix an algebraic closure $\bar{F}$ of $F$ and a complex embedding $\bar{F} \to \mathbb{C}$. Let $V = H_1(A/F, \mathbb{Q})$ be the first singular homology group of $A/F$ with coefficients in $\mathbb{Q}$. Then we denote by $MT(A/F)$ (resp. $Hg(A/F)$) the Mumford-Tate group (resp. Hodge group) associated to the natural Hodge structure of $V$.

On the other hand, for any rational prime $l$, let $T_\ell A(\bar{F})$ be the $\ell$-adic Tate module of $A$ and set $V_i = T_\ell A(\bar{F}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$. Then we have a Galois representation

$$\rho_l : \text{Gal}(\bar{F}/F) \to \text{Aut}_{\mathbb{Q}_l}(V_i).$$

We define an algebraic group $G_{l/\mathbb{Q}_l}$ as the Zariski closure of the image of $\rho_l$ inside the algebraic group $\text{Aut}_{\mathbb{Q}_l}(V_i)$, and let $G_{l/\mathbb{Q}_l}^0$ be its identity component. By comparison theorem, we have an isomorphism $V \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong V_i$. Under this isomorphism, the Mumford-Tate conjecture states that:

Conjecture 0.1. For any prime $l$, we have the equality $G_{l/\mathbb{Q}_l}^0 = MT(A) \times_{\mathbb{Q}} \mathbb{Q}_l$.

In this paper, we are interested in the case that $A/F$ is an absolutely simple abelian fourfolds (so in particular $n = 4$).

Let $\mathfrak{g}/\mathbb{Q}$ (resp. $\mathfrak{g}_{l/\mathbb{Q}_l}$) be the Lie algebra of the algebraic group $MT(A)/\mathbb{Q}$ (resp. $G_{l/\mathbb{Q}_l}$). Then let $\mathfrak{h}/\mathbb{Q}$ (resp. $\mathfrak{h}_{l/\mathbb{Q}_l}$) be the subalgebra of $\mathfrak{g}/\mathbb{Q}$ (resp. $\mathfrak{g}_{l/\mathbb{Q}_l}$) consisting of elements of trace 0. So we have $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{Q} \cdot \text{Id}$ (resp. $\mathfrak{g}_l = \mathfrak{h}_l \oplus \mathbb{Q}_l \cdot \text{Id}$). In [9], Moonen and Zarhin computed the Lie algebras $\mathfrak{h}/\mathbb{Q}$ and $\mathfrak{h}_{l/\mathbb{Q}_l}$. From their result, the endomorphism $\text{End}^0(A/F)$ together with its action on the Lie algebra $\text{Lie}(A/F)$ determines the Lie algebras $\mathfrak{h}/\mathbb{Q}$ and $\mathfrak{h}_{l/\mathbb{Q}_l}$ uniquely except in the case that $\text{End}^0(A/F) = \mathbb{Q}$. When $\text{End}^0(A/F) = \mathbb{Q}$, we have two possibilities for $\mathfrak{h}$: either $\mathfrak{h} = \mathfrak{sp}_4$ over $\mathbb{Q}$ or $\mathfrak{h} = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\mathbb{Q}$. Similarly we have two possibilities for $\mathfrak{h}_l$: either $\mathfrak{h}_l = \mathfrak{sp}_4$ over $\mathbb{Q}_l$, or $\mathfrak{h}_l = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\mathbb{Q}_l$. In the first case happens when $A/F$ comes from a ‘generic’ element in the Siegel moduli space while the second happens when $A/F$ comes from an analytic family of abelian varieties constructed by Mumford in [11].

On the other hand, Deligne proved the inclusion $\mathfrak{h}_l \subseteq \mathfrak{h} \otimes_{\mathbb{Q}} \mathbb{Q}_l$. To prove the Mumford-Tate conjecture for the abelian variety $A/F$ with $\text{End}^0(A/F) = \mathbb{Q}$, it is sufficient to prove the following:

Theorem 1. If $\mathfrak{h}_l = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\mathbb{Q}_l$, then $A/F$ comes from a Shimura curve constructed by Mumford in [11].

This theorem is the main result in this paper. The argument is based on two results: one is to use local Galois representation to determine the Serre-Tate coordinates of an ordinary abelian variety; the other is formal linearity of Shimura variety of Hodge type. Here we give a sketch of the proof.

Suppose that the abelian variety $A/F$ satisfies $\mathfrak{h}_l = \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathfrak{sl}_2$ over $\mathbb{Q}_l$. By a theorem of Pink, there exists a set $V$ of finite places of $F$ of density 1 at which the abelian variety $A/F$ has good ordinary reduction. For $v \in V$, let $k_v$ be the residue field of $F$ at $v$ with characteristic $p = p_v$ and we use $A_{k_v}$ to denote the reduction of $A/F$ at $v$. If the abelian variety $A/F$ comes from a Shimura curve $Z \to A_{4,1,n}$

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where $A_{v/k_v}$ is the Siegel moduli space of principally polarized 4-dimensional abelian varieties with a suitable level structure, then $A_{v/k_v}$ gives a closed ordinary point $x_v$ of $Z$. As $Z$ is a Shimura variety of Hodge type, from a result of Noot ([8] Theorem 4.2), it is formally linear at $x_v$, i.e. the formal completion of $Z$ at $x_v$ is a formal torus of rank 1. Since the abelian variety $A_{IF}$ gives a non-torsion point on this formal torus, this torus can be determined by the Serre-Tate coordinates of the abelian variety $A_{IF}$.

On the other hand, we prove a general fact that the local Galois representation at $v$ attached to the $p = p_v$-adic Tate module of $A_{IF}$ determines the Serre-Tate coordinates of $A_{IF}$ (notice that we prove this fact without any assumption on the Galois representation attached to $A_{IF}$). To be more precise, let $D_v$ (resp. $I_v$) be the decomposition (resp. inertia) group of $\text{Gal}(\bar{F}/F)$ at $v$. After choosing a suitable symplectic $\mathbb{Z}_p$-basis of the $p$-adic Tate module $T_pA(F)$, the local Galois representation is of the shape:

$$
\rho_p : I_v \rightarrow \text{GSp}_8(\mathbb{Z}_p),
\sigma \mapsto \begin{pmatrix}
\chi_p(\sigma) & I_n \\
0 & I_n
\end{pmatrix},
$$

where $I_4$ is the $4 \times 4$ identity matrix, $B(\sigma) = (b_{ij}(\sigma))_{1 \leq i,j \leq 4}$ is a symmetric $4 \times 4$ matrix depending on $\sigma$ and $\chi_p : I_v \rightarrow \mathbb{Z}_p^\times$ is the $p$-adic cyclotomic character. By direct calculation, the maps $b_{ij} : I_v \rightarrow \mathbb{Z}_p$ are 1-cocycles valued in $\mathbb{Z}_p(\chi_p)$ for all $1 \leq i,j \leq 4$ and give elements in the cohomology group $H^1(I_v, \mathbb{Z}_p(\chi_p))$. We prove that these elements give the Serre-Tate coordinates of $A_{IF}$ by Kummer theory. In fact, we prove that the Serre-Tate coordinates of $A_{IF}$ sits in a rank 1 formal torus defined by linear equations determined by $b_{ij}$ forcing $B(\sigma)$ to have image in a rank 1 subspace of $M_{4\times 4}(\mathbb{Q}_p)$.

In [15], Noot gave a detailed analysis of the isogeny types of the abelian variety $A_{v/k_v}$ and the local Galois representation $\rho_p : I_v \rightarrow \text{GSp}_8(\mathbb{Z}_p)$. He proved that for any Frobenius element $\text{Frob}_v \in D_v$, the element $\rho_p(\text{Frob}_v) \in G_p(\mathbb{Q}_p)$ generates a maximal torus of $G_p(\mathbb{Q}_p)$. Also he got a control on the image $\rho_p(I_v)$. This information imposes restrictive conditions on the 1-cocycles $b_{ij}$’s and then we get an explicit description on the Serre-Tate coordinates of $A_{IF}$.

Finally we consider the torsion points on the formal torus we get in the previous step. These points correspond to the quasi-canonical CM liftings of the abelian variety $A_{v/k_v}$ in the sense of [8] Definition 2.9. Given the analysis of the Serre-Tate coordinates explained as above, we can use the Mumford-Tate groups of these abelian varieties to generate a candidate of the Mumford-Tate group of $A_{IF}$ and then construct a Shimura curve which contains all these quasi-canonical liftings of $A_{v/k_v}$. Then by formal linearity of Shimura varieties of Hodge type, we see that $A_{IF}$ comes from this Shimura curve and then we conclude.

**Organization of this paper:** In section 1, we review the deformation theory of ordinary abelian varieties and the construction of Serre-Tate coordinates. In section 2, we prove a result about how to use local Galois representation to determine the Serre-Tate coordinates of an ordinary abelian variety. In section 3 and 4, we recall general facts of Shimura varieties and known results for Mumford-Tate conjecture. In section 5, we introduce the notion of an abelian variety with Galois representation of Mumford’s type and recall Noot’s results on the study of isogeny types of the reductions of such abelian varieties. In section 6, we use theorems developed in previous sections to determine the Serre-Tate coordinates of abelian varieties with Galois representations of Mumford’s type and in section 7, we finish the proof of the main theorem.

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8. Full Serre-Tate coordinates
1. Serre-Tate deformation theory

1.1. Cartier duality theorem. In this subsection we recall the Cartier duality theorem for abelian schemes. For more details, see [10] section 14, 15 when the base is the spectrum of an algebraically closed field and [12] Chapter 1 for the general base.

Theorem 2. (See [12] Theorem 1.1) Let \( \mathbb{A} \) and \( \mathbb{B} \) be two abelian schemes over a scheme \( S \), and \( f : \mathbb{A} \to \mathbb{B} \) be an \( S \)-isogeny. Let \( \mathbb{A}^! \) (resp. \( \mathbb{B}^! \)) be the dual of \( \mathbb{A} \) (resp. \( \mathbb{B} \)) and \( f^! : \mathbb{B}^! \to \mathbb{A}^! \) be the dual of \( f \). Then there is a pairing of finite flat group schemes over \( S \):

\[
\langle \cdot, \cdot \rangle_f : \ker(f) \times \ker(f^!) \to G_{m/S},
\]

which is non-degenerate, bilinear and compatible with arbitrary base change.

Now let \( S \) be an arbitrary scheme and \( \mathbb{A}_{/S} \) be an abelian scheme. For any integer \( N > 0 \), the multiplication by \( N \) map \([N] : \mathbb{A} \to \mathbb{A} \) is an \( S \)-isogeny. In this case we denote the pairing \( \langle \cdot, \cdot \rangle_{[N]} \) in Theorem 2 by

\[
E_{\mathbb{A}_{/S},N} : \mathbb{A}[N] \times \mathbb{A}^![N] \to G_{m/S}.
\]

Let \( A_{/k} \) be an ordinary abelian variety over an algebraically closed field \( k \) with characteristic \( p > 0 \). Suppose that we are given a lifting \( \mathbb{A}_{/R} \) of \( A_{/k} \) to a local artinian ring \( R \) with residue field \( k \). In section 1.2, we will define a pairing:

\[
e_{\mathbb{A}_{/R},p^n}(x, y) = E_{\mathbb{A}_{/R},p^n}(x, y),
\]

for each \( n \geq 1 \). Then following lemma will be used frequently in our argument:

Lemma 1.1. Given \( x \in \mathbb{A}(R)[p^n] \) and \( y \in A^!(R)[p^n] \), suppose that there exist an artinian local ring \( R' \) (finite flat over \( R \)) with \( Y \in \mathbb{A}^!(R')[p^n] \) which lifts \( y \), then we have an equality in \( \hat{G}_{m}(R') \):

\[
e_{\mathbb{A}_{/R'},p^n}(x, y) = E_{\mathbb{A}_{/R'},p^n}(x, y).
\]

1.2. Serre-Tate coordinate. Let \( k = \bar{k} \) be an algebraically closed field and \( W = W(k) \) be the ring of Witt vectors with coefficients in \( k \). Let \( K^{ur} \) be the quotient field of \( W \).

Fix an ordinary abelian variety \( A_{/k} \) and let \( R \) be a local artinian ring with maximal ideal \( m \) and residue field \( k \). Let \( A'_{/k} \) be the dual of \( A_{/k} \) and \( T_p A(k) \) (resp. \( T_p A^!(k) \)) be the \( p \)-adic Tate module of \( A_{/k} \) (resp. \( A'_{/k} \)). From Serre-Tate deformation theory, we have a bijection

\[
\{ \text{isomorphism classes of liftings of } A_{/k} \text{ to } R \} \to \text{Hom}_{\bar{k}}(T_p A(k) \otimes_{\mathbb{Z}_p} T_p A^!(k), \hat{G}_{m}(R)).
\]

For later argument we briefly recall the construction of the above bijection. We refer [6] Chapter 2 for more details. For any lifting \( \mathbb{A}_{/R} \) of \( A_{/k} \), we have an extension of Barsotti-Tate groups over \( R \):

\[
0 \to \mathbb{A} \to \mathbb{A}[p^n] \to T_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to 0.
\]

From the appendix in [7], this extension is obtained from the standard extension

\[
0 \to T_p A(k) \to T_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to T_p A(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p / \mathbb{Z}_p \to 0.
\]
by pushing out along a unique homomorphism \( \varphi_{k/R} : T_p A(k) \to \hat{\aleph}(R) \). This homomorphism can be constructed as follows: as \( R \) is artinian and local, there exists an integer \( n \), such that \( m^{n+1} = (0) \), every element in \( \hat{\aleph}(R) \) is killed by \( p^n \) by Lemma 1.1.2 in [6]. We define a homomorphism

\[
p^n : A(k)[p^n] \to \hat{\aleph}(R), \ x \mapsto p^n x,
\]

where \( x \) is any element in \( A(R) \) which lifts \( x \). Then \( \varphi_{k/R} \) can be defined as the composition

\[
T_p A(k) \to A(k)[p^n] \xrightarrow{p^n} \hat{\aleph}(R).
\]

For any \( m > 0 \), we have a natural pairing:

\[
E_{A/k, p^m} : A[p^m] \times A^t[p^m] \to \mu_{p^m}.
\]

The restriction of \( E_{A/k, p^m} \) to \( A[p^m] \times A^t(k)[p^m] \) gives rise to an isomorphism of \( k \)-groups:

\[
\hat{A}[p^m] \to \text{Hom}_Z(A^t(k)[p^m], \mu_{p^m}).
\]

As \( \hat{A} \) is multiplicative and \( R \) is local artinian, this isomorphism extends uniquely to an isomorphism of \( R \)-groups:

\[
\hat{A}[p^m] \to \text{Hom}_Z(A^t(k)[p^m], \mu_{p^m}),
\]

and hence we have a perfect pairing

\[
E_{\hat{A}/R, p^m} : \hat{A}[p^m] \times A^t(k)[p^m] \to \mu_{p^m}.
\]

By taking limit for variable \( m \), we get a perfect pairing:

\[
E_{\hat{A}/R} : \hat{A} \times T_p A^t(k) \to \hat{G}_{\mathfrak{m}}.
\]

Then the Serre-Tate coordinate can be defined as:

\[
q(\hat{A}/R; \alpha, \alpha_t) = E_{\hat{A}/R}(\varphi_{\hat{A}/R}(\alpha), \alpha_t) \in \hat{G}_{\mathfrak{m}}(R),
\]

for any \( \alpha \in T_p A(k) \) and \( \alpha_t \in T_p A^t(k) \).

**Remark 1.2.** For any \( \alpha \in T_p A(k) \) (resp. \( \alpha_t \in T_p A^t(k) \)), let \( \alpha_n \) (resp. \( \alpha_{t,n} \)) be the image of \( \alpha \) (resp. \( \alpha_t \)) under the projection \( T_p A \to A(k)[p^n] \) (resp. \( T_p A^t \to A^t(k)[p^n] \)). Let \( \tilde{\alpha}_n \in \hat{\aleph}(R) \) be an arbitrary lifting of \( \alpha_n \in A(k)[p^n] \). Then by definition \( \varphi_{\hat{A}/R}(\alpha) = p^n \tilde{\alpha}_n \in \hat{\aleph}(R) \). Since the group \( \hat{\aleph}(R) \) is killed by \( p^n \), we have \( \tilde{\alpha}_n \in \hat{A}[p^{2n}](R) \) and \( q(\hat{A}/R; \alpha, \alpha_t) = E_{\hat{A}/R, p^n}(p^n \tilde{\alpha}_n, \alpha_{t,n}) \).

1.3. Section of the connected-étale sequence and \( p \)-th power roots of the Serre-Tate coordinate. We keep the notations in the previous subsection. Let \( R \) be an artinian local ring with maximal ideal \( \mathfrak{m} \) and residue field \( k \). Assume that \( \mathfrak{m}^{n+1} = (0) \). Let \( \hat{\aleph}/R \) be a lifting of \( A/\bar{k} \). Then for each integer \( m > 0 \), we have an exact sequence of finite group schemes over \( R \):

\[
0 \to \hat{\aleph}[p^m] \to \hat{\aleph}[p^n] \to A(k)[p^m] \to 0. \tag{1.3}
\]

By Cartier duality, we get an exact sequence:

\[
0 \to \hat{\aleph}^t[p^m] \to \hat{\aleph}^t[p^n] \to A^t(k)[p^m] \to 0 \tag{1.4}
\]

over \( R \). The splitting of the above two exact sequences are equivalent.

The sequence (1.3) does not necessarily split over \( R \) in general. The splitting of (1.3) is equivalent to the existence of an étale subgroup of \( \hat{\aleph}[p^m] \) which lifts \( A^t(k)[p^m] \). Hence the exact sequence (1.3) splits after an fpf extension of \( R \). Now we fix some integer \( m \geq n \) in the following discussion. Then we can find an artinian local ring \( R' \) finite flat over \( R \) such that each \( \alpha_{t,m} \in A^t(k)[p^m] \) is lifted to some \( \tilde{\alpha}_{t,m} \in \hat{\aleph}[p^m](R') \). From [11] we have the following equality in \( \hat{G}_{\mathfrak{m}}(R') \):

\[
q(\hat{A}/R; \alpha, \alpha_t) = E_{\hat{A}/R, p^m}(\varphi_{\hat{A}/R}(\alpha), \alpha_{t,m}) = E_{\hat{A}/R, p^m}(\varphi_{\hat{A}/R}(\alpha), \tilde{\alpha}_{t,m}) = E_{\hat{A}/R, p^m}(p^m \tilde{\alpha}_m, \tilde{\alpha}_{t,m}),
\]

where \( \tilde{\alpha}_m \in \hat{\aleph}(R) \) is a lifting of \( \alpha_m \in A(k)[p^m] \). As \( \tilde{\alpha}_n \in A[p^{2m}](R) \) by Remark [12] we have

\[
q(\hat{A}/R; \alpha, \alpha_t) = E_{\hat{A}/R, p^m}(p^m \tilde{\alpha}_m, \tilde{\alpha}_{t,m}) = E_{\hat{A}/R, p^{2m}}(\tilde{\alpha}_m, \tilde{\alpha}_{t,m}).
\]
For any \(s \geq 0\), we assume further that \(\alpha_{t,m+s} \in A'(k)[p^{m+s}]\) lifts to some \(\tilde{\alpha}_{t,m+s} \in A'[p^{m+s}](R')\) and \(p^s\tilde{\alpha}_{t,m+s} = \alpha_{t,m}\). Then

\[
q(\tilde{A}/R; \alpha, \alpha_t) = \tilde{E}_{A'/R}(\alpha_{t,m}, p^s\tilde{\alpha}_{t,m+s}) = E_{A'/R}(\alpha_{t,m}, \alpha_{t,m+s})^p.
\]

In other words, \(E_{A'/R}(\alpha_{t,m}, \alpha_{t,m+s})\) is a \(p^s\)-th root of the Serre-Tate coordinate \(q(\tilde{A}/R; \alpha, \alpha_t)\). The element \(E_{A'/R}(\alpha_{t,m}, \alpha_{t,m+s})\) definitely depends on the choice of \(\alpha_{t,m+s}\). In the following we want to determine how it depends on the choice of the integer \(m\) and the lifting \(\alpha_{t,m}\).

First let \(\tilde{\alpha}_m \in \tilde{A}(R)\) be another lifting of \(\alpha_m \in A(k)[p^m]\), then \(\tilde{\beta}_m = \alpha_m - \tilde{\alpha}_m \in \tilde{A}(R)\), and hence \(\tilde{\beta}_m\) is killed by \(p^m\). When \(s + n \leq m\), by Lemma \([1, 1]\) we have

\[
E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m}) = E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m}) = E_{A'/R}(\alpha_{t,m}, p^s\alpha_{t,s+m+n}) = E_{A'/R}(\alpha_{t,m}, p^s\alpha_{t,s+m+n}) = 1.
\]

Hence when \(0 \leq s \leq m - n\), the element \(E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m})\) does not depend on the choice of \(\tilde{\alpha}_m\).

Now let \(m' \geq m\) be another integer and assume that \(0 \leq s \leq m - n\). Let \(\tilde{\alpha}'_m\) be a lifting of \(\alpha_{m'} \in A(k)[p^{m'}]\). Then we have

\[
E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m}) = E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m}) = E_{A'/R}(\alpha_{t,m}, p^s\alpha_{t,s+m+n}) = \tilde{E}_{A'/R}(\alpha_{t,m}, p^s\alpha_{t,s+m+n}).
\]

As \(m^s - m(\alpha_{m'}) = \alpha \in A(k)[p^m]\), from the previous argument, we see that

\[
E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m}) = E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m+n}) = 1.
\]

In other words, the element \(E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m})\) does not depend on the choice of \(m\).

Since the integer \(m\) can be as an arbitrary integer greater than \(n\), from the above discussion we see that for every integer \(s \geq 1\), there exists a \(p^s\)-th root \(E_{A'/R}(\alpha_{t,m}, \alpha_{t,s+m})\) of the Serre-Tate coordinate \(q(\tilde{A}/R; \alpha, \alpha_t)\) as long as we choose a compatible lifting \((\tilde{\alpha}_{t,m})_m\) of \((\alpha_{t,m})_m\).

1.4. Extension to more general bases. Let \(CL/W\) be the category of complete local \(W\)-algebras with residue field \(k\). Fix an object \(R\) in \(CL/W\) with maximal ideal \(m\). For each \(n \geq 0\), set \(R_n = R/m^{n+1}\), which is an artinian local ring with residue field \(k\) and \(R = \lim_{\rightarrow} R_n\). As before we fix an ordinary abelian variety \(A_{jk}\). By passing to the projective limit, we have a bijection:

\[
\{\text{isomorphism classes of liftings of } A_{jk} \text{ to } R\} \rightarrow \text{Hom}_{Z_p}(T_pA(k) \otimes Z_p T_p A'(k), \tilde{G}_m(R)),
\]

\[A_{jk} \mapsto q(A_{jk}^R; -,-)\]

such that for any \(\alpha \in T_pA(k)\) and \(\alpha_t \in T_pA'(k)\),

\[q(A_{jk}^R; \alpha, \alpha_t) = \lim_{\rightarrow} q(A_{n/R_n}; \alpha, \alpha_t),\]

where \(A_n = A \otimes R_n\).

For any lifting \(A_{jk}^R\) of \(A_{jk}\) to \(R\), we have the connect-étale exact sequence of Barsotti-Tate groups over \(R\):

\[0 \rightarrow \hat{A}[0^\infty] \rightarrow A[p^\infty] \rightarrow A[p^\infty] \rightarrow 0\]

Suppose that the above exact sequence splits after a faithfully flat extension of \(R\), i.e. there exist a \(W\)-algebra \(R'\) finite and flat over \(R\), and a morphism of Barsotti-Tate groups over \(R'\) \(j : T_pA(k) \otimes Z_p Q_p/Z_p \rightarrow A[p^\infty]\), such that \(\pi \circ j = id\).

For each \(n \geq 1\), set \(R'_n = R' \otimes R_n = R'/m^{n+1}R'\). Then we have a split exact sequence of Barsotti-Tate groups over \(R'_n\):

\[0 \rightarrow \hat{A}_n[p^\infty] \rightarrow A_n[p^\infty] \rightarrow T_pA(k) \otimes Z_p Q_p/Z_p \rightarrow 0\]

By the discussion in the previous section, for any \(\alpha \in T_pA(k)\), \(\alpha_t \in T_pA'(k)\) and \(m \geq 0\), we have a \(p^m\)-th root of the Serre-Tate coordinate \(q(A_{n/R_n}; \alpha, \alpha_t)\) in \(R'_n\), which is denoted by \(t_{m,n} \in \tilde{G}_m(R'_n)\). By taking projective limit, we have that \(t_m = \lim_{\rightarrow} t_{m,n} \in \tilde{G}_m(R')\) is a \(p^m\)-th root of the Serre-Tate coordinate \(q(A_{R}; \alpha, \alpha_t)\) in \(R'\).
1.5. Fröbenius action on the Serre-Tate coordinate. In this subsection we take \( k \) to be an algebraic closure of the prime field \( \mathbb{F}_p \). Recall that \( W = W(k) \) is the ring of Witt vectors with coefficients in \( k \) and \( K^{ur} \) is the quotient field of \( W \), which can be identified with the \( p \)-adic completion of the maximal unramified extension \( \mathbb{Q}_p^{ur} \) of \( \mathbb{Q}_p \). Let \( \sigma \) (resp. \( \Sigma \)) be the absolute Fröbenius automorphism of \( k \) (resp. \( W \)).

As before we fix an ordinary abelian variety \( A_{/k} \). Consider the functor

\[
\text{Def}_{A_{/k}} : CL/W \to \text{Sets}
\]

\[
\mathcal{R} \mapsto \{ \text{isomorphism classes of liftings of } A_{/k} \to \mathcal{R} \}.
\]

The Serre-Tate deformation theorem tells us that the functor \( \text{Def}_{A_{/k}} \) is represented by some object \( \mathcal{R}^{\text{univ}} \) in \( CL/W \), and the Serre-Tate coordinate gives us an isomorphism of functors:

\[
q(-; -) : \text{Def}_{A_{/k}} \to \text{Hom}(T_p A(k) \otimes_{\mathbb{Z}_p} T_p A'(k), \hat{\mathbb{G}}_m).
\]

Set \( \hat{\mathcal{M}}_{A_{/k}} = \text{Spf}(\mathcal{R}^{\text{univ}}) \) which is a formal \( W \)-torus. Define formal \( W \)-torus \( \hat{\mathcal{M}}^{(\Sigma)}_{A_{/k}} \) by the following Cartesian diagram:

\[
\hat{\mathcal{M}}^{(\Sigma)}_{A_{/k}} \xrightarrow{\text{Spec}(W)} \hat{\mathcal{M}}_{A_{/k}} \xrightarrow{\text{Spec}(W)} \text{Spec}(W),
\]

and define the abelian variety \( A^{(\sigma)}_{/k} \) by the following Cartesian diagram:

\[
A^{(\sigma)}_{/k} \xrightarrow{\text{Spec}(\sigma)} A \xrightarrow{\text{Spec}(\sigma)} \text{Spec}(k).
\]

By [6] Lemma 4.1.1, we have

**Lemma 1.5.** There is a canonical isomorphism of formal \( W \)-torus: \( \hat{\mathcal{M}}^{(\sigma)}_{A_{/k}} \to \hat{\mathcal{M}}^{(\Sigma)}_{A_{/k}} \), under which \( \Sigma \) (resp. \( \sigma \)) corresponds to \( \sigma(\alpha), \sigma(\alpha_t) \).

Since the abelian variety \( A_{/k} \) is projective, it is defined over a finite field \( \mathbb{F}_q \) inside \( k \), where \( q = p^l \) for some integer \( l \geq 1 \). Let \( \sigma_1 \) (resp. \( \Sigma_1 \)) be the \( l \) times composition of \( \sigma \) (resp. \( \Sigma \)) with itself. Then \( A^{(\sigma)}_{/k} \cong A_{/k} \). Then Lemma 1.5 indicates that \( \Sigma_t \) induces an automorphism of the deformation space \( \hat{\mathcal{M}}_{A_{/k}} \) which sends the Serre-Tate coordinate \( q(\hat{A}_{/k}; \alpha, \alpha_t) \) to \( q(\hat{A}^{(\Sigma)}_{/k}; \sigma_t(\alpha), \sigma_t(\alpha_t)) \).

For later argument, we need the following result:

**Lemma 1.6.** Let \( K/K^{ur} \) be a finite Galois representation inside \( \mathbb{C}_p \). Write \( \mathcal{O}_K \) for the valuation ring of \( K \). Let \( A_{/K} \) be a lifting of an ordinary abelian variety \( A_{/k} \) to \( \mathcal{O}_K \). For any \( \sigma \in \text{Gal}(K/K^{ur}) \), define an abelian scheme \( A^{(\sigma)}_{/\mathcal{O}_K} \) by the following Cartesian diagram:

\[
\hat{A}^{(\sigma)}_{/\mathcal{O}_K} \xrightarrow{\text{Spec}(\mathcal{O}_K)} \hat{A} \xrightarrow{\text{Spec}(\mathcal{O}_K)} \text{Spec}(\mathcal{O}_K).
\]

Then \( A^{(\sigma)}_{/\mathcal{O}_K} \) is also a lifting of \( A_{/k} \) and we have the equality:

\[
\sigma(q(\hat{A}^{(\sigma)}_{/\mathcal{O}_K}; \alpha, \alpha_t)) = q(\hat{A}_{/\mathcal{O}_K}; \alpha, \alpha_t),
\]

for any \( \alpha \in T_p A(k) \) and \( \alpha_t \in T_p A'(k) \).

The above lemma is nothing but the functorial property of Serre-Tate coordinates so we omit the proof here.
1.6. Partial Serre-Tate coordinates. In this section we write $R$ for a complete noetherian local ring with maximal ideal $m$ and residue field $k$ and an abelian variety $A_{/k}$ which is not necessarily ordinary. Suppose that the Barsotti-Tate group $A[p^{\infty}]_{/k}$ is not local-local, i.e. the slopes of $A[p^{\infty}]_{/k}$ contains 0 and 1 (see [11] for the definition of slopes of Barsotti-Tate groups over an arbitrary field) or equivalently, the Tate module $T_p(A)(k)$ is nontrivial.

We say that a Barsotti-Tate group $G_{/R}$ is multiplicative if its dual $G^t_{/R}$ is ind-étale. We say that $G_{/R}$ is local-local if both $G_{/R}$ and $G^t_{/R}$ are connected. As $k$ is algebraically closed, we can decompose the Barsotti-Tate group $A[p^{\infty}]_{/k}$ as $A[p^{\infty}] = A[p^{\infty}]^{ord} \times A[p^{\infty}]^{ll}$, where $A[p^{\infty}]^{ord}$ is a product of a multiplicative Barsotti-Tate group and an ind-étale Barsotti-Tate group, and $A[p^{\infty}]^{ll}$ is local-local.

Now we assume that $R$ is artinian and $m^{n+1} = (0)$ for some $n \geq 1$. Let $A_{/R}$ be a lifting of $A_{/k}$ to $R$. We assume further that we have a decomposition of Barsotti-Tate groups over $R$:

$$A[p^{\infty}] = A[p^{\infty}]^{ord} \times A[p^{\infty}]^{ll},$$

where $A[p^{\infty}]^{ord}$ (resp. $A[p^{\infty}]^{ll}$) lifts $A[p^{\infty}]^{ord}$ (resp. $A[p^{\infty}]^{ll}$). This is equivalent to saying that $A[p^{\infty}]$ can be decomposed as $A[p^{\infty}]^{mult} \times A[p^{\infty}]^{ll}$ over $R$, where $A[p^{\infty}]^{mult}$ is multiplicative and $A[p^{\infty}]^{ll}$ is local-local.

Then we have an exact sequence of Barsotti-Tate groups over $R$:

$$0 \rightarrow A[p^{\infty}]^{mult} \rightarrow A[p^{\infty}]^{ord} \rightarrow T_p(A)(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \rightarrow 0.$$  

Similar with the ordinary case we define a homomorphism

$$p^n : A[p^n](k) \rightarrow \hat{A}(R),$$

where $\hat{x}$ is an arbitrary lifting of $x$ in $\hat{A}(R)$. Write $\varphi_{\overline{A}/R}$ as the composition

$$T_p(A)(k) \rightarrow A[p^n](k) \xrightarrow{p^n} \hat{A}(R) \rightarrow A[p^{\infty}]^{mult}(R).$$

On the other hand we have a perfect pairing of $k$-group schemes $A^{mult}[p^n] \times A[p^n] \rightarrow \mu_{p^n}$, which can be lifted uniquely to a perfect pairing of $R$-group schemes $\hat{A}[p^{\infty}]^{mult} \times A[p^n](k) \rightarrow \mu_{p^n}$. By taking limit, we have a perfect pairing

$$e_{\hat{A}/R} : \hat{A}[p^{\infty}]^{mult} \times T_p A^{r}(k) \rightarrow \hat{G}_m$$

over $R$. Then we can define the partial Serre-Tate coordinate by the formula:

$$q(\hat{A}/R; \alpha, \alpha_t) : T_p(A)(k) \otimes_{\mathbb{Z}_p} T_p(A^{r})(k) \rightarrow \hat{G}_m(R)$$

$$\alpha \otimes \alpha_t \mapsto e_{\hat{A}/R}(\varphi_{\overline{A}/R}, \alpha_t),$$

for any $\alpha \in T_p(A)(k)$ and $\alpha_t \in T_p(A^{r})(k)$.

As in the ordinary case, we assume that the exact sequence

$$0 \rightarrow A[p^{\infty}]^{mult} \rightarrow A[p^{\infty}]^{ord} \rightarrow T_p(A)(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \rightarrow 0$$

splits after some faithfully flat extension $R'$ of $R$. Then for any $\alpha_t = (\alpha_{t,n}) \in T_p A^{r}(k)$, we choose a compatible lifting ($\tilde{\alpha}_{t,n}$) of $(\alpha_{t,n})$ in $A[p^{\infty}]^{ord}(R')$, i.e. $\tilde{\alpha}_{t,n} \in A[p^{\infty}]^{ord}(R')$ and $p(\tilde{\alpha}_{t,n+1}) = \alpha_{t,n}$. Then for any $s > 0$ and $m \geq n + s$, the element $q_{\hat{A}/R}^{s}(\alpha_{t,n})$ is a $p^s$-th root of the partial Serre-Tate coordinate $q(\hat{A}/R; \alpha, \alpha_t)$, where $\pi : \hat{A} \rightarrow \hat{A}[p^{\infty}]^{mult}$ is the natural projection and $\tilde{\alpha}_m \in \hat{A}(R)$ is an arbitrary lifting of $\alpha_m \in A[p^n](k)$.

Also similar to section 1.4 by taking projective limit, we can extend the above result to $R \in CL/JW$.

2. Application to Galois representations

2.1. The case of elliptic curves. Fix an algebraically closed field $k$ with characteristic $p > 0$ and an ordinary elliptic curve $E_{/k}$. Let $K$ be a finite extension of $\mathbb{Q}_p$ with valuation ring $\mathcal{O}_K$ and recall that $K^{ur}$ is the quotient field of the ring $W = W(k)$. Let $\Omega$ be the algebraic closure of $K^{ur}$ inside $\mathbb{C}_p$. Then we have the isomorphism of Galois groups: $\text{Gal}(\Omega/K^{ur}) \cong \text{Gal}(\mathbb{Q}_p/\mathbb{Q}_p)$. As $K$ and $K^{ur}$ are linearly disjoint over $\mathbb{Q}_p$, we take $L$ to be the composite of $K$ and $K^{ur}$ over $\mathbb{Q}_p$, which is the $p$-adic completion of $K$ inside $\Omega$. Let $\mathcal{O}_L$ be the valuation ring of $L$. 
Suppose that $E_{/\mathcal{O}_K}$ is a lifting of $E_{/k}$ to $\mathcal{O}_K$. Since $E_{/k}$ is an elliptic curve, it is naturally isomorphic to its dual $E'_{/k}$, and hence we have a Serre-Tate coordinate $\varphi(E_{/\mathcal{O}_K}; -,-) : T_p E(k) \otimes_{\mathbb{Z}_p} T_p E(k) \to \hat{G}_m(\mathcal{O}_K)$.

From the exact sequence of Barsotti-Tate groups over $\mathcal{O}_K$:
$$0 \to \hat{E} \to E[p^\infty] \to T_p E(k) \otimes_{\mathbb{Z}_p} (\hat{Q}_p / \mathbb{Z}_p) \to 0,$$
we have an exact sequence of Tate modules:
$$0 \to T_p \hat{E}(\hat{Q}_p) \xrightarrow{\varphi} T_p E(\hat{Q}_p) \xrightarrow{\rho} T_p E(k) \to 0.$$ 

We identify $T_p \hat{E}(\hat{Q}_p)$ as a submodule of $T_p E(\hat{Q}_p)$ under $i$. Then we can choose a $\mathbb{Z}_p$-basis $\{v^e, v^t\}$ of $T_p E(\hat{Q}_p)$ such that $\rho$ is the basis of the $\mathbb{Z}_p$-module $T_p \hat{E}(\hat{Q}_p)$ and $v^t$ is mapped to a basis $u = (u_n)$ of $T_p E(k)$ under the map $\varphi$ with $u_n \in E[p^n](k)$. Then set $t = q(E_{/\mathcal{O}_K}; u, u)$.

Under the $\mathbb{Z}_p$-basis $\{v^e, v^t\}$ of $T_p E(\hat{Q}_p)$, we have a Galois representation attached to $T_p E(\hat{Q}_p)$:
$$\rho : \text{Gal}(\hat{Q}_p / K) \to \text{GL}_2(\mathbb{Z}_p),$$
$$\sigma \mapsto \left( \begin{array}{cc} \chi(\sigma) & b(\sigma) \\ 0 & 1 \end{array} \right),$$

where $\chi : \text{Gal}(\hat{Q}_p / K) \to \mathbb{Z}_p^\times$ is the $p$-adic cyclotomic character.

On the other hand, for each integer $n \geq 1$, the element $v^t_n \in E[p^n](\hat{Q}_p)$ generates an étale subgroup of $E[p^n]$ which lifts the constant group scheme $E[p^n]_{/k}$. Thus we can find a (possibly infinite) extension $\hat{K}$ of $K$ inside $\hat{Q}_p$, such that $v^t_n$ is defined over $\mathcal{O}_K$ for all $n (\mathcal{O}_K$ is the valuation ring of $K$). Replacing $\hat{K}$ by its Galois closure inside $\hat{Q}_p$, we can assume that $\hat{K} / K$ is Galois. Let $L$ be the composite of $\hat{K}$ and $K^{ur}$ over $\mathcal{O}_L$, and $\mathcal{O}_L$ be the valuation ring of $L$. Under the above notations, the exact sequence of Barsotti-Tate groups:
$$0 \to \hat{E} \to E[p^\infty] \to T_p E(k) \otimes_{\mathbb{Z}_p} (\hat{Q}_p / \mathbb{Z}_p) \to 0,$$

splits when base change to $\mathcal{O}_L$. By the discussion in Chapter 1, for any integer $s \geq 1$, we have a unique $p^s$-th root $\sqrt[s]{t} \in \mathcal{O}_L$ of the Serre-Tate coordinate $t$ which depends only on $v^t = (v^t_n)$. Our main result is the following:

**Theorem 3.** For any $\sigma \in \text{Gal}(\hat{Q}_p / K)$ and integer $s \geq 1$, under the isomorphism $\text{Gal}(\hat{Q}_p / K) \cong \text{Gal}(\Omega / L)$, we have the equality:
$$\sigma(\sqrt[s]{t}) = E_{\mathbb{Z}/\mathcal{O}_L,p^s}(v^t_n, v^e_n)^{b(\sigma)}.$$ 

**Proof.** For any integer $n \geq 1$, let $W_n = \mathcal{O}_K/m_n^{-1}\mathcal{O}_K = \mathcal{O}_L/m_n^{-1}\mathcal{O}_L$ ($m_L$ resp. $m_L$) is the maximal ideal of $\mathcal{O}_K$ (resp. $\mathcal{O}_L$) and $W_n = W_n \otimes_{\mathcal{O}_K} \mathcal{O}_K$, which is an artinian local ring faithfully flat over $W_n$. Set $E/W_n = E \times_{\mathcal{O}_K} W_n$ and $t_n = q(E/W_n; u, u)$. Then we have $t = \lim t_n \in \hat{G}_m(\mathcal{O}_L)$. From the discussion in Section 1.3, for each $m \geq n + s$, we have a unique $p^s$-th root of $t_n$, which is given by the formula:
$$\sqrt[s]{t_n} = E_{\mathbb{Z}/W_n,p^m}(\tilde{\alpha}_m, v^t_{m+s}) \in \mu_{2^m}(\mathbb{W}_n),$$

where $\tilde{\alpha}_m \in E(W_n)$ is an arbitrary lift of $u_m \in E[p^m](k)$, and $v^t_{m+s}$ is the reduction of $v^t_n$ in $E[p^m + s](W_n)$. From the discussion in Section 1.4, we have $\sqrt[s]{t} = \lim \sqrt[s]{t_n} \in \hat{G}_m(\Omega_L)$.

Now for any $\sigma \in \text{Gal}(\hat{Q}_p / K)$, since $K / K$ is Galois, $\sigma$ induces an automorphism of $\hat{K}$, and hence induces an automorphism of $\hat{W}_n$ for each $n$. We still denote this automorphism by $\sigma$.

Since $\tilde{\alpha}_m \in E(W_n)$ and $\sigma$ fixes $K$, $\sigma(\tilde{\alpha}_m) = \tilde{\alpha}_m$. By our assumption on the expression of the Galois representation $\rho$, we have $\sigma(v^t) = b(\sigma)v^e + v^t$, and hence $\sigma(v^t_{m+s}) = b(\sigma)v^t_{m+s} + v^e_{m+s}$.

As the pairing $E_{\mathbb{Z}/W_n,p^m}(\tilde{\alpha}_m, v^e_{m+s})$ is compatible with arbitrary base change, we have
$$\sigma'(\sqrt[s]{t_n}) = \sigma(E_{\mathbb{Z}/W_n,p^m}(\tilde{\alpha}_m, v^e_{m+s})) = E_{\mathbb{Z}/W_n,p^m}(\sigma(\tilde{\alpha}_m), \sigma(v^t_{m+s}))$$
$$= E_{\mathbb{Z}/W_n,p^m}(\alpha_m, b(\sigma)v^e + v^t) = E_{\mathbb{Z}/W_n,p^m}(\tilde{\alpha}_m, v^e_{m+s})^{b(\sigma)}.$$
Now we analyze the term $\mathbb{E}_{\mathbb{E}/\mathbb{W}_n, p^m}(\tilde{\alpha}_{m, n}, \tilde{v}_{m+s}^o)$. As $\tilde{v}_{m+s}^o \in \tilde{E}(\mathbb{W}_n)$ and $\tilde{\alpha}_{m, n} \in \mathbb{E}[p^m](k)$, from Lemma 1.1, we have

$$\mathbb{E}_{\mathbb{E}/\mathbb{W}_n, p^m}(\tilde{\alpha}_{m, n}, \tilde{v}_{m+s}^o) = \mathbb{E}_{\mathbb{E}/\mathbb{W}_n, p^m}(u_m, \tilde{v}_{m+s}^o) = \mathbb{E}_{\mathbb{E}/\mathbb{W}_n, p^m}(\tilde{v}_{m+s}^o, \tilde{v}_{m+s}^o) = \mathbb{E}_{\mathbb{E}/\mathbb{W}_n, p^m}(p^{\tilde{v}_{m+s}^o}, \tilde{v}_{m+s}^o) = \mathbb{E}_{\mathbb{E}/\mathbb{W}_n, p^m}(\tilde{v}_{m+s}^o, \tilde{v}_{m+s}^o).$$

The last term is the projection of $\mathbb{E}_{\mathbb{E}/\mathbb{O}_K, p^{m+s}}(v_{m+s}^o, v_{m+s}^o)$ under the base change $\mathbb{O}_K \to \tilde{\mathbb{W}}_n$, which is denoted by $\mathbb{E}_{\mathbb{E}/\mathbb{O}_K, p^o}(v_{m+s}^o, v_{m+s}^o)$.

By a direct computation, we have

$$\mathbb{E}_{\mathbb{E}/\mathbb{O}_K, p^{m+s}}(v_{m+s}^o, v_{m+s}^o) = \mathbb{E}_{\mathbb{E}/\mathbb{O}_K, p^o}(p^m v_{m+s}^o, p^m v_{m+s}^o) = \mathbb{E}_{\mathbb{E}/\mathbb{O}_K, p^o}(v_{m+s}^o, v_{m+s}^o).$$

Hence we have

$$\sigma(\sqrt[\mathbb{n}]{T_n}) = \sqrt[\mathbb{n}]{T_n}: (\mathbb{E}_{\mathbb{E}/\mathbb{O}_K, p^o}(v_{m+s}^o, v_{m+s}^o))^{b(\mathbb{\sigma})}.$$ 

By taking projective limits for $n$, we have the desired equality:

$$\sigma(\sqrt[\mathbb{n}]{T}) = \sqrt[\mathbb{n}]{T}: \mathbb{E}_{\mathbb{E}/\mathbb{O}_K, p^o}(v_{m+s}^o, v_{m+s}^o)^{b(\mathbb{\sigma})}.$$ 

2.2. Generalization to higher dimensions. We keep the same notations as in section 2.1. In this section we want to generalize the result in the previous section to higher dimensions. Let $K/\mathbb{Q}_p$ be a finite extension inside $\mathbb{Q}_\mathbb{p}$ with valuation ring $\mathbb{O}_K$. The field $L$ is defined to be the composite of $K$ and $K_{ur}$ over $\mathbb{Q}_p^{ur}$.

Fix an algebraic closure $k$ of the prime field $\mathbb{F}_p$. Let $A_{/k}$ be an abelian variety and $\mathbb{A}_{/\mathbb{O}_K}$ be a lifting of $A_{/k}$ to $\mathbb{O}_K$. Then we have a connected-étale exact sequence of Barsotti-Tate groups over $\mathbb{O}_L$:

$$0 \to \mathbb{A} \to \mathbb{A}[p] \to T_p A(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0.$$ 

For every integer $n \geq 1$, we have a perfect pairing:

$$\mathbb{e}_{p^n}: \mathbb{A}[p^n] \times A[p^n](k) \to \mathbb{G}_m$$

over $\mathbb{O}_L$. Taking projective limits, we have a perfect pairing:

$$\mathbb{e}_{p^n}: T_p \mathbb{A}(\mathbb{C}_p) \times T_p A^l(k) \to T_p \mathbb{G}_p$$

over $\mathbb{O}_L$. For later argument, we fix a basis $\mathbb{C}_p = (\mathbb{C}_p^n)$ of the $\mathbb{Z}_p$-module $T_p \mathbb{G}_p$. 

Now suppose that we are given a polarization $\lambda: \mathbb{A} \to \mathbb{A}^l$ of $\mathbb{A}_{/\mathbb{O}_K}$ whose degree is prime to $p$. This polarization induces isomorphisms of $p$-adic Tate modules $T_p \mathbb{A}(\mathbb{C}_p) \to T_p \mathbb{A}^l(\mathbb{C}_p)$ and $T_p A(k) \to T_p A^l(k)$, which are still denoted by $\lambda$. We can take a $\mathbb{Z}_p$-basis $\{v_1, \ldots, v_n\}$ of the Tate module $T_p A(k)$ such that:

1. $\{v_1, \ldots, v_n\}$ is a $\mathbb{Z}_p$-basis of $T_p \mathbb{A}(\mathbb{C}_p)$;
2. $\{v_1, \ldots, v_n\}$ is a lifting of a basis $\{u_1, \ldots, u_n\}$ of the Tate module $T_p A(k)$;
3. under the pairing $\mathbb{e}_{p^n}$ and the isomorphism $\lambda: T_p A(k) \to T_p A^l(k)$, we have $\mathbb{e}_{p^n}(v_i^o, \lambda(u_j)) = 1$ if $i \neq j$ and $\mathbb{e}_{p^n}(v_i^o, \lambda(u_j)) = \zeta_p^n$ if $i = j$.

Under the basis $\{v_1, \ldots, v_n, v_1^o, \ldots, v_n^o\}$, the Galois representation attached to the Tate module $T_p \mathbb{A}(\mathbb{C}_p)$ is of the shape:

$$\rho: \mathbb{Gal}(\mathbb{Q}_p/K) \to \mathbb{Gsp}_{2n}(\mathbb{Z}_p)$$

$$\sigma \mapsto \left( \chi_p(\sigma) \cdot I_n \quad B(\sigma) \right),$$

where $\chi_p: \mathbb{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p$ is the $p$-adic cyclotomic character, $I_n$ is the $n \times n$ identity matrix and $B = (b_{ij})_{1 \leq i, j \leq n}: \mathbb{Gal}(\mathbb{Q}_p/K) \to M_{n \times n}(\mathbb{Z}_p)$ is a map valued in the set of $n \times n$ symmetric matrices.
Now we consider the Serre-Tate coordinates $t_{ij} = q(\mathfrak{a}_{ij}; u_i, \lambda(u_j))$. From the discussion in section 1.3, the lifting $\nu_{ij}$ of $u_i$ gives a compatible sequence of $p$-th power roots $\{\sqrt[p]{T_{ij}}\}_{i=1,2,...}$ of the Serre-Tate coordinates $t_{ij}$, for $1 \leq i, j \leq n$.

Under the above notations, we have:

**Theorem 4.** For any $\sigma \in \text{Gal}(\mathbb{Q}_p/K)$ and integer $s \geq 1$, under the isomorphism $\text{Gal}(\mathbb{Q}_p/K) \cong \text{Gal}(\Omega/L)$, we have the equality:

$$\sigma(\sqrt[p]{T_{ij}}) = \hat{b}_{ij}(\sigma).$$

### 2.3. Extension to the decomposition group.

In this section we want to extend the previous results to the decomposition group, i.e. we want to use the Serre-Tate coordinates to study the Galois representation of the decomposition group. First we give a cohomological interpretation of Theorem 4.

**Remark 2.1.** Recall that we have a Galois representation

$$\rho : \text{Gal}(\mathbb{Q}_p/K) \to \text{Gsp}_{2n}(\mathbb{Z}_p),$$

$$\sigma \mapsto \left( \begin{array}{cc} \chi_p(\sigma) \cdot I_n & B(\sigma) \\ 0 & I_n \end{array} \right).$$

By direct calculation, for every pair $1 \leq i, j \leq n$, the map $b_{ij} : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p$ is a 1-cocycle if we define the action of $\text{Gal}(\mathbb{Q}_p/K)$ on $\mathbb{Z}_p$ by the $p$-adic cyclotomic character $\chi_p$. Hence we have an element in the cohomology group $H^1(\text{Gal}(\mathbb{Q}_p/K), \mathbb{Z}_p(\chi_p))$, which is denoted by $\hat{b}_{ij}$.

On the other hand, under the basis $\zeta_p^{\infty}$ of $\mathbb{T}_p\mu_p^{\infty}(\mathbb{C}_p)$, we have an isomorphism of $\text{Gal}(\mathbb{Q}_p/K)$-modules $\mathbb{Z}_p(\chi_p) \to \mathbb{T}_p\mu_p^{\infty}(\mathbb{C}_p)$ which sends 1 to $\zeta_p^{\infty}$. So we have an isomorphism of cohomology groups:

$$H^1(\text{Gal}(\mathbb{Q}_p/K), \mathbb{Z}_p(\chi_p)) \cong H^1(\text{Gal}(\mathbb{Q}_p/K), \mathbb{T}_p\mu_p^{\infty}(\mathbb{C}_p)).$$

Now by Kummer theory, we have an isomorphism:

$$H^1(\text{Gal}(\mathbb{Q}_p/K), \mathbb{T}_p\mu_p^{\infty}(\mathbb{C}_p)) \to \hat{K}^\times,$$

where $\hat{K}^\times$ is the pro-$p$-completion of the multiplicative group $K^\times$. As $\hat{G}_m(\mathcal{O}_L) = 1 + m_L$, we can regard $\hat{G}_m(\mathcal{O}_L)$ as a subgroup of $\hat{K}^\times$. Then Theorem 4 tells us that under the isomorphism

$$H^1(\text{Gal}(\mathbb{Q}_p/K), \mathbb{Z}_p(\chi_p)) \to \hat{K}^\times,$$

the element $\hat{b}_{ij}$ coming from the Galois representation $\rho$ corresponds to the Serre-Tate coordinate $t_{ij} \in \hat{G}_m(\mathcal{O}_L) \subseteq \hat{K}^\times$.

We start the discussion concerning elliptic curves. Let $K'/\mathbb{Q}_p$ be a finite extension with valuation ring $\mathcal{O}_K$ and residue field $\mathbb{F}_q$ ($q = p^r$ for some integer $r \geq 1$) and let $E/F_p$ be an elliptic curve whose special fiber $\hat{E}_{/\mathbb{F}_q}$ is ordinary. Recall that we fix an algebraic closure $\mathbb{Q}_p$ (resp. $k$) of $\mathbb{Q}_p$ (resp. $\mathbb{F}_q$). Under the ordinary assumption, we have an exact sequence of the $p$-adic Tate modules:

$$0 \to T_p\hat{E}(\mathbb{Q}_p) \xrightarrow{\nu} T_pE(\mathbb{Q}_p) \xrightarrow{\nu_{st}} T_pE(k) \to 0.$$ 

As in section 2.1, we choose a $\mathbb{Z}_p$-basis $\{\nu^i, \nu^{st}\}$ of $T_p\hat{E}(\mathbb{Q}_p)$ such that $\nu^i$ is a basis of $T_p\hat{E}(\mathbb{Q}_p)$ and $\nu^{st}$ is mapped to a basis $u$ of $T_pE(k)$ under the reduction map. Under this basis, we have a Galois representation attached to $T_pE(\mathbb{Q}_p)$:

$$\rho : \text{Gal}(\mathbb{Q}_p/K) \to \text{GL}_2(\mathbb{Z}_p),$$

$$\sigma \mapsto \left( \begin{array}{cc} \chi_p(\sigma) \cdot \eta^{-1}(\sigma) & b(\sigma) \\ 0 & \eta(\sigma) \end{array} \right),$$

where $\chi_p : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p^\times$ is the $p$-adic cyclotomic character and $\eta : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p^\times$ is an unramified character. Now we define a map $c : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p$ by setting $c(\sigma) = \eta^{-1}(\sigma)b(\sigma)$ for all $\sigma \in \text{Gal}(\mathbb{Q}_p/K)$. As $\rho$ is a representation, a direct calculation shows that $c : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p$ is a 1-cocycle valued in $\mathbb{Z}_p(\chi_p\eta^{-2})$. If we choose a different lifting $\nu^{st} \in T_pE(\mathbb{Q}_p)$ of $u \in T_pE(k)$, the 1-cocycle $c : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p$
Proof. The proof is quite similar with that of Theorem 3, so we do not give all the details here.

We assume that there is a Galois extension $\overline{K}/L$ such that the exact sequence

$$0 \rightarrow \hat{E} \rightarrow \mathbb{E}[p^\infty] \rightarrow T_pE(k) \otimes_{\mathbb{Z}_p} (\mathbb{Q}_p/\mathbb{Z}_p) \rightarrow 0$$

splits over $O_{\overline{K}}$. As in the proof of Theorem 3, we can define $W_n, \tilde{W}_n$. Then for $m \geq s + n$,

$$\sqrt[p^m]{t} = E_{E/\tilde{W}_n, p^m} (\tilde{\alpha}_{m,n}, t_{m+s}).$$

For $\sigma \in \text{Gal}(\hat{Q}_p/K)$, we have

$$\sigma(\sqrt[p^m]{t}) = E_{E/\tilde{W}_n, p^m} (\sigma(\tilde{\alpha}_{m,n}), \sigma(t_{m+s})).$$

As $\tilde{\alpha}_{m,n} \in E(W_n)$ is a lifting of $u_m \in E[p^m](k)$, the element $\sigma(\tilde{\alpha}_{m,n}) \in E(W_n)$ is a lifting of $\sigma(u_m) = \eta(\sigma) \cdot u_m \in E[p^m](k)$. From the argument in section 1.3, the element $\sqrt[p^m]{t}$ is independent of the choice of the lifting of $u_m$. Thus we have

$$\sigma(\sqrt[p^m]{t}) = E_{E/\tilde{W}_n, p^m} (\sigma(\tilde{\alpha}_{m,n}), \sigma(t_{m+s})) = E_{E/\tilde{W}_n, p^m} (\sigma(\tilde{\alpha}_{m,n}), b(\sigma)t_{m+s} + \eta(\sigma)t_{m+s}).$$

By the same analysis on the term $E_{E/\tilde{W}_n, p^m} (\tilde{\alpha}_{m,n}, t_{m+s})$ as in the proof of Theorem 3, and taking limit for various $n$, we have the desired equality:

$$\sigma(\sqrt[p^m]{t}) = \big( \sqrt[p^m]{t} \big) \eta(\sigma) \cdot E_{E/O_{\overline{K}}, p^m}(v_{s}^{\sigma}, v_{s}^{\sigma}) b(\sigma) \eta(\sigma) = \big( \sqrt[p^m]{t} \big) \eta(\sigma) \cdot E_{E/\mathbb{Q}_p, p^m}(v_{s}^{\sigma}, v_{s}^{\sigma}) b(\sigma) \eta(\sigma).$$

Taking the $\eta^{-2}(\sigma)$-th power on both sides, we get the desired equality. 

\[ \square \]

Remark 2.2. As in Remark 2.1, we can give a cohomological interpretation of Theorem 5. Let $K^{ur}$ be the maximal unramified extension of $K$ in $\hat{Q}_p$, and let $I = \text{Gal}(\hat{Q}_p/K^{ur}) \subseteq \text{Gal}(\hat{Q}_p/K) = G$ be the inertia group. Then we have the inflation-restriction exact sequence:

$$0 \rightarrow H^1(G/I, \mathbb{Z}_p(\chi_p \eta^{-2})) \rightarrow H^1(G, \mathbb{Z}_p(\chi_p \eta^{-2})) \rightarrow H^1(I, \mathbb{Z}_p(\chi_p \eta^{-2})) \rightarrow H^2(G/I, \mathbb{Z}_p(\chi_p \eta^{-2})).$$

As the character $\eta$ is unramified, the inertia group $I$ acts on $\mathbb{Z}_p(\chi_p \eta^{-2})$ by the $p$-adic cyclotomic character. Hence $\mathbb{Z}_p(\chi_p \eta^{-2})^I = 0$. So the restriction map induces an isomorphism:

$$H^1(G, \mathbb{Z}_p(\chi_p \eta^{-2})) \rightarrow H^1(I, \mathbb{Z}_p(\chi_p))^I.$$

From Remark 2.1, under the isomorphism of $I$-modules $\mathbb{Z}_p(\chi_p) \rightarrow T_p\mu_p(\mathbb{C}_p)$ which sends 1 to $\lim E_{E/\mathbb{Q}_p, p^m}(v_{s}^{\sigma}, v_{s}^{\sigma})$ and the isomorphism $H^1(I, T_p\mu_p(\mathbb{C}_p)) \cong (K^{ur})^\times$, the image of $c$ in $H^1(I, \mathbb{Z}_p(\chi_p))$ corresponds to the Serre-Tate coordinate $t \in (K^{ur})^\times$.

On the other hand, it is easy to check that the map

$$\text{Gal}(\hat{Q}_p/K) \rightarrow T_p\mu_p(\mathbb{C}_p),$$

$$\sigma \mapsto \lim \frac{\sigma(\sqrt[p^m]{t}) \eta^{-2}(\sigma)}{\sqrt[p^m]{t}},$$

is changed by a 1-coboundary valued in $\mathbb{Z}_p(\chi_p \eta^{-2})$. Hence to determine the Galois representation $\rho$ (up to isomorphism), it is enough to determine the corresponding element of $c : \text{Gal}(\mathbb{Q}_p/K) \rightarrow \mathbb{Z}_p$ in the cohomology group $H^1(\text{Gal}(\mathbb{Q}_p/K), \mathbb{Z}_p(\chi_p \eta^{-2}))$. In fact, we have the following relation:

**Theorem 5.** For any $\sigma \in \text{Gal}(\hat{Q}_p/K)$ and integer $s > 0$, we have the equality:

$$\frac{\sigma(\sqrt[p^m]{t}) \eta^{-2}(\sigma)}{\sqrt[p^m]{t}} = E_{E/\hat{Q}_p, p^m}(v_{s}^{\sigma}, v_{s}^{\sigma}) \psi^{\sigma}(\sigma).$$
is a 1-cocycle valued in $H^1(G, T_p\mu_2 \otimes (\mathbb{C}_p)(\chi_p)^{t-2})$ whose restriction to the inertia group corresponds to the Serre-Tate coordinate $t$ under the above isomorphism. Using this cohomological interpretation, we get another proof of Theorem 3

Moreover, from the restriction map, we see that the image of $c$ in $H^1(I, \mathbb{Z}_p(\chi_p))$ is invariant under the action of $G$. Let $f : I \to T_p\mu_2 \otimes (\mathbb{C}_p), \sigma \mapsto \lim g(\sqrt[p^n]{T})$ be the 1-cocycle corresponding to the Serre-Tate coordinate $t$. For any $g \in G$, the action of $g$ on the cocycle $f$ is given by the formula:

$$f^g(\sigma) = g \cdot f(g^{-1} \sigma g).$$

Hence

$$f^g(\sigma) = (\lim\sigma(\sqrt[p^n]{T}))_{\chi_p} = (\lim\sigma(\sqrt[p^n]{T}))_{\chi_p} = (\lim\sigma \cdot g(\sqrt[p^n]{T}))_{\chi_p}.$$  

As $\{\sqrt[p^n]{T}\}$ is a compatible $p$-th power roots of $t$, $\{g(\sqrt[p^n]{T})\}$ is a compatible $p$-th power roots of $g(t)$. Under the isomorphism induced by Kummer theory, the 1-cocycle $\sigma \mapsto (\lim\sigma \cdot g(\sqrt[p^n]{T}))_{\chi_p}$ corresponds to $g(t)^{n-2}$. So we have the equality $g(t) = t^{n-2}$. The cohomological interpretation gives another proof of Lemma 3.5.

The case of higher dimensions is more complicated. Let $\mathbb{A}_{\mathcal{O}_K}$ be an abelian scheme of relative dimension $n$ whose special fiber $A_{/\mathbb{Z}_p}$ is ordinary. As in section 2.2, we can choose a $\mathbb{Z}_p$-basis of the $p$-adic Tate module $T_p\mathbb{A}_{\mathbb{Z}_p} = T_p\mathbb{A}_{\mathbb{Z}_p}$ under which the Galois representation attached to $T_p\mathbb{A}_{\mathbb{Z}_p}$ is of the shape:

$$\rho : \text{Gal}(\mathbb{Q}_p/K) \to \text{GSp}_{2n}(\mathbb{Z}_p) \quad \sigma \mapsto \begin{pmatrix} \chi_p(\sigma) \cdot T(\sigma) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} B(\sigma) \\ T(\sigma)^{-1} \end{pmatrix},$$

here again $\chi_p : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p^\times$ is the $p$-adic cyclotomic character, $B = (b_{ij})_{1 \leq i, j \leq n} : \text{Gal}(\mathbb{Q}_p/K) \to M_{n \times n}(\mathbb{Z}_p)$ is a map, and $T(\cdot) : \text{Gal}(\mathbb{Q}_p/K) \to \text{GL}_n(\mathbb{Z}_p)$ is the unramified homomorphism which sends (any) Frobenius element in $\text{Gal}(\mathbb{Q}_p/K)$ to a matrix $X \in \text{GL}_n(\mathbb{Z}_p)$.

Under the above setting, we define a map $C : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p$ by requiring that $C(\sigma) = B(\sigma) \cdot T(\sigma)^t$ for any $\sigma \in \text{Gal}(\mathbb{Q}_p/K)$. A direct calculation shows that $C : \text{Gal}(\mathbb{Q}_p/K) \to M_{n \times n}(\mathbb{Z}_p)$ is a 1-cocycle if we define the $\text{Gal}(\mathbb{Q}_p/K)$-action on $M_{n \times n}(\mathbb{Z}_p)$ by the formula: $\sigma : M = \chi_p(\sigma)T(\sigma) \cdot M \cdot T(\sigma)^t$. Let $I \subseteq \text{Gal}(\mathbb{Q}_p/K)$ be the inertia group. Again the inflation-restriction exact sequence tells us that the restriction map induces an isomorphism:

$$H^1(I, \text{Gal}(\mathbb{Q}_p/K), M_{n \times n}(\mathbb{Z}_p)) \to H^1(I, (\mathbb{Q}_p/K), (\mathbb{Z}_p)^{\text{Gal}(\mathbb{Q}_p/K)/I}).$$

For $1 \leq i, j \leq n$, the restriction of the map $b_{ij} : \text{Gal}(\mathbb{Q}_p/K) \to \mathbb{Z}_p$ to the inertia group $I$ is a 1-cocycle valued in $\mathbb{Z}_p(\chi_p)$. From Theorem 3.4 and Remark 2.1, under the isomorphism

$$H^1(I, M_{n \times n}(\mathbb{Z}_p)(\chi_p)) \cong H^1,I, (\mathbb{Q}_p/K) \cong (\mathbb{Z}_p)^{\text{Gal}(\mathbb{Q}_p/K)/I},$$

the images of $b_{ij}$'s correspond to the Serre-Tate coordinates $t_{ij}$'s. Hence the Serre-Tate coordinates $t_{ij}$'s determine the images of the 1-cocycle $C$ in the cohomological group $H^1(\text{Gal}(\mathbb{Q}_p/K), M_{n \times n}(\mathbb{Z}_p))$ and hence determine the Galois representation $\rho$ (up to isomorphism).

Since we know little about the matrix $A$, we cannot expect to get an explicit expression of the 1-cocycle $C$ as in Theorem 3.5. For later argument, we consider a special case: suppose that there exists a finite extension $L/\mathbb{Q}_p$ with valuation ring $\mathcal{O}_L$ and a matrix $W \in \text{GL}_n(\mathcal{O}_L)$ such that $WXW^{-1} = D = \text{diag}(d_1, \ldots, d_n)$ is a diagonal matrix in $\text{GL}_n(\mathcal{O}_L)$. Hence $(W_i)^{-1}X_i^{-1}W_i = D_i^{-1}$. Now we consider a conjugation of the Galois representation $\rho$:

$$\rho' = \begin{pmatrix} W & 0 \\ 0 & (W^{-1})^{-1} \end{pmatrix} \rho \begin{pmatrix} W^{-1} & 0 \\ 0 & W^t \end{pmatrix} : \text{Gal}(\mathbb{Q}_p/K) \to \text{GSp}_{2n}(\mathcal{O}_L) \quad \sigma \mapsto \begin{pmatrix} \chi_p(\sigma) \cdot T'(\sigma) \\ 0 \end{pmatrix} = \begin{pmatrix} B'(\sigma) \\ (T'(\sigma)^{-1})^t \end{pmatrix},$$

where $\sigma' = \begin{pmatrix} \chi_p(\sigma) \cdot T'(\sigma) \\ 0 \end{pmatrix} = \begin{pmatrix} B'(\sigma) \\ (T'(\sigma)^{-1})^t \end{pmatrix}$.
where $T' : \text{Gal}(\overline{\mathbb{Q}}_p/K) \to \text{GL}_n(\mathfrak{O}_L)$ is the unramified homomorphism sending (any) Frobenius element to the matrix $D \in \text{GL}_n(\mathfrak{O}_L)$. By direct calculation, $B'(\sigma) = WB(\sigma)W'$. So for any $1 \leq i, j \leq n$, the map $b'_{ij} : \text{Gal}(\overline{\mathbb{Q}}_p/K) \to \mathfrak{O}_L$ is an $\mathfrak{O}_L$-linear combination of $b_k$'s. From our previous discussion, the Serre-Tate coordinates $t_k$'s of $A_{\mathbb{Q}_K}$ determine the images of $b_k$'s in $H^1(I, \mathcal{O}_L(\chi_p))$, and hence determine the images of $b'_{ij}$'s in $H^1(I, \mathcal{O}_L(\chi_p)) = H^1(\mathbb{I}, \mathcal{O}_L(\chi_p)) \otimes_{\mathbb{Z}_p} \mathcal{O}_L$. On the other hand, if we define $\eta_j : \text{Gal}(\overline{\mathbb{Q}}_p/K) \to \mathcal{O}_L$ as the unramified character which sends (any) Frobenius element to $d_j \in \mathcal{O}_L$, then the map $c'_{ij} = \eta_j \cdot b'_{ij} : \text{Gal}(\overline{\mathbb{Q}}_p/K) \to \mathcal{O}_L$ is a 1-cocycle valued in $\mathcal{O}_L(\chi_p)$. Again the restriction map gives us an isomorphism

$$H^1(\text{Gal}(\overline{\mathbb{Q}}_p/K), \mathcal{O}_L(\chi_p \eta_j)) \to H^1(I, \mathcal{O}_L(\chi_p))^{\text{Gal}(\overline{\mathbb{Q}}_p/K)/I}.$$ 

So in this way, the Serre-Tate coordinates determines the images of $c'_{ij}$ in $H^1(\text{Gal}(\overline{\mathbb{Q}}_p/K), \mathcal{O}_L(\chi_p \eta_j))$.

3. Siegel modular Shimura varieties

In this section we recall basic results on Siegel modular Shimura varieties and Serre-Tate deformation theory. Our main reference is [3].

Fix a positive integer $d$ and a prime $p$. Let $\mathbb{Z}(p)$ be the localization of $\mathbb{Z}$ at $(p)$. Let $G/\mathbb{Q} = \text{GSp}(2d)/\mathbb{Q}$ be the symplectic similitude group over $\mathbb{Q}$, i.e. for any $\mathbb{Q}$-algebra $R$, we have $G(R) = \{ X \in \text{GL}_{2d}(R)|X^tJ_dX = \nu(X)J_d, \text{ for some } \nu(A) \in R^\times \}$,

where $J_d = \begin{pmatrix} 0 & -1_d \\ 1_d & 0 \end{pmatrix}$. Define the Siegel upper half space

$$\mathcal{H}_d = \{ Z = X + iY \in M_{d \times d}(\mathbb{C}) | Z = Z^t, Y > 0 \}.$$ 

Set $X = \mathcal{H}_d \cup \overline{\mathcal{H}}_d$. For any integer $N > 0$, define

$$\tilde{\mathcal{H}}(N) = \{ \alpha \in \text{GSp}_{2d}(\tilde{\mathbb{Z}})|\alpha \equiv 1 \text{ mod } N \}.$$ 

Let $W = \mathbb{Q}^{2d}$ with the alternating form $\psi(x, y) = x^tJ_dy$. For each $\mathbb{Q}$-algebra $R$, $G(R) = \text{GSp}_{2d}(R)$ acts on $W \otimes_{\mathbb{Q}} R$ in the natural way, preserving the alternating form $\psi$ up to scalar multiplication. Set $\tilde{L} = L \otimes_{\mathbb{Z}} \tilde{\mathbb{Z}}$ and $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $W_p = W \otimes_{\mathbb{Q}} \mathbb{Q}_p$ for each rational prime $p$. Let $\{e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_{2d} = (0, 0, \ldots, 1)\}$ be the standard $\mathbb{Z}_p$-basis of $L_p$.

For $N \geq 3$, consider the following moduli problem:

$$\mathcal{E}_N : \text{Sch}_{/\mathbb{Z}[\frac{1}{N}]} \rightarrow \text{Sets},$$ 

$$S \mapsto \mathcal{E}_N(S) = \{(A, \lambda, \eta_N)\}_{/\mathbb{Z}}.$$ 

such that for each $\mathbb{Z}[\frac{1}{N}]$-scheme $S$, $\mathcal{E}_N(S)$ is the set of isomorphism classes of the triples $(A, \lambda, \eta_N)$ consisting of:

(1) an abelian scheme $A_{/S}$ of relative dimension $d$;

(2) a principal polarization $\lambda : A \to A^t$ of $A$;

(3) a level $N$ structure $\eta_N : (\mathbb{Z}/N\mathbb{Z})^{2d} = \mathcal{L}/\mathcal{N} \cong A[N](k(s))$, under which the symplectic pairing $< , >$ on $\mathcal{L}/\mathcal{N}$ is sent to the Weil pairing on $A[N]$ induced by the polarization $\lambda$, and $s : \text{Spec}(k(s)) \to S$ is a geometric point of $S$.

It is well known that the moduli problem $\mathcal{E}_N$ is represented by a scheme $\mathcal{A}_{1,N}[\frac{1}{N}]$. Moreover, the $\mathbb{C}$-valued point of $\mathcal{A}_{1,N}$ is given by

$$\mathcal{A}_{1,N}(\mathbb{C}) = G(\mathbb{C}) \backslash (X \times G(\mathcal{A}_I))/\tilde{\mathcal{H}}(N).$$ 

Then we define two pro-schemes:

$$\text{Sh}_{/\mathbb{Q}} = \varprojlim_{N} \mathcal{A}_{1,N}[\frac{1}{N}], \text{Sh}_{/\mathbb{Q}}(p) = \varprojlim_{(p,N)=1} \mathcal{A}_{1,N}[\frac{1}{N}]$$ 

Take a closed point $x_p = (A_0, \lambda_0, \eta_0(p)) / \mathbb{F}_p \in \text{Sh}_{/\mathbb{Q}}(\mathbb{F}_p)$ such that the abelian variety $A_0 / \mathbb{F}_p$ is ordinary. Under this assumption, the endomorphism algebra $D = \text{End}^\vee(A_0 / \mathbb{F}_p)$ is a matrix algebra over a CM algebra (i.e. a finite product of CM fields) $M$. The CM algebra $M$ is generated by the Frobenius endomorphism
of $A_0/F_p$ over $\mathbb{Q}$. Let $R$ be the order of $M$ generated by the Frobenius map of $A_0/F_p$ over $\mathbb{Z}$. Let $R_{(p)} = R \otimes_{\mathbb{Z}} \mathbb{Z}(p)$. Define a torus $T_{/\mathbb{Z}(p)}$ by setting

$$T(\mathbb{Z}(p)) = \{ a \in R_{(p)}^* \mid x \in \mathbb{Z}^\times(p) \}.$$ 

For each $g \in G(\mathbb{A}^{(p, \infty)})$, it acts on the moduli problem $\mathcal{E}(p)$ by sending a triple $(A, \bar{\lambda}, \eta(p)) \in \mathcal{E}(p)(S)$ to the triple $(A, \bar{\lambda}, \eta(p) \circ g) \in S$. By universal property, $g$ induces an automorphism of the Shimura variety $Sh_{/\mathbb{Z}(p)}^{(p)}$, which is still denoted by $g$.

Define a homomorphism $\hat{\rho} : T(\mathbb{Z}(p)) \to G(\mathbb{A}^{(p, \infty)})$ by the formula $a \circ \eta_0(p) = \eta_0(p) \circ \hat{\rho}(a)$, for $a \in T(\mathbb{Z}(p))$. The image of $T(\mathbb{Z}(p))$ under $\hat{\rho}$ stabilizes the closed point $x_p$ under the action of $G(\mathbb{A}^{(p, \infty)})$ on $Sh_{/\mathbb{Z}(p)}^{(p)}$ explained as above.

Let $\hat{S}_p/W_p$ be the formal completion of the Shimura variety $Sh_{/\mathbb{Z}(p)}^{(p)}$ along the closed point $x_p$, where $W_p = W(\mathbb{F}_p)$ is the ring of Witt vectors with coefficients in $\mathbb{F}_p$. As the abelian variety $A_0/F_p$ is ordinary, by Serre-Tate deformation theory, we have an isomorphism:

$$\hat{S}_p \cong \text{Hom}_{\mathbb{Z}_p}(\text{Sym}(T_pA_0(\mathbb{F}_p) \otimes_{\mathbb{Z}_p} T_pA_0(\mathbb{F}_p)), \hat{\mathbb{G}}_m).$$

Each $a \in T(\mathbb{Z}(p))$ gives an automorphism on the Serre-Tate deformation space $\hat{S}_p$. In terms of the Serre-Tate coordinates, this action is given by the formula:

$$a \circ t = (t \circ (a \otimes a)^{-1})^a,$$

for $t \in \text{Hom}_{\mathbb{Z}_p}(\text{Sym}(T_pA_0(\mathbb{F}_p) \otimes_{\mathbb{Z}_p} T_pA_0(\mathbb{F}_p)), \hat{\mathbb{G}}_m)$.

For simplicity, we assume that the abelian variety $A_0/F_p$ is simple. But the following results can be generalized to non-simple cases without any difficulty. Under this assumption, $M$ is a CM field and if $A_0/F_p$ is defined over a finite field $\mathbb{F}_q$ ($q$ is a power of $p$), then $M$ is generated by $\mathbb{F}_q$ over $\mathbb{Q}$. Let $F$ be the maximal totally real subfield of $M$. We make another assumption that the degree of $M$ over $\mathbb{Q}$ is $2d$. We choose embeddings $\varphi_1, \ldots, \varphi_d : M \to \mathbb{Q}$, such that all the embeddings of $M$ into $\mathbb{Q}$ are given by the set $\{\varphi_1, \ldots, \varphi_d, \bar{\varphi}_1, \ldots, \bar{\varphi}_d\}$, where $\bar{\cdot}$ means a complex conjugation in $\mathbb{Q}$, and $M$ acts on the rational Tate module $T_pA_0(\mathbb{F}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the character $\Pi_{\varphi}^{ \pm 1} \bar{\varphi}$. Then we have chosen the embeddings $\varphi_1, \ldots, \varphi_d$ so that the deformation space $\hat{S}_p$ has canonical coordinates $t_{i,j}$ on which the group $T(\mathbb{Z}(p))$ acts through the character $\varphi_i \cdot \bar{\varphi}_j, 1 \leq i, j \leq d$.

4. Known results towards the Mumford-Tate conjecture

In this section we summarize results on the Mumford-Tate conjecture.

Let $A$ be an abelian variety of dimension $d$ over a number field $F$. Fix an embedding $F \hookrightarrow \mathbb{C}$ and an algebraic closure $\bar{F}$ of $F$.

The singular homology group $V = H_1(A(\mathbb{C}), \mathbb{Q})$ is a $2d$-dimensional vector space over $\mathbb{Q}$. Then we have the Hodge decomposition $V_\mathbb{C} = V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1}$, such that $V^{1,0} = \mathbb{C}^*$. We define a cocharacter $\mu_\infty : G_{\text{an}} \to \text{Aut}_{\mathbb{C}}(V_\mathbb{C})$ such that any $z \in \mathbb{C}^*$ acts on $V^{1,0}$ by multiplication by $z^{-1}$ and acts trivially on $V^{0,1}$.

**Definition 4.1.** The Mumford-Tate group of the abelian variety $A/F$ is the smallest algebraic subgroup $\text{MT}(A) \subset \text{Aut}_{\mathbb{Q}}(V)$ defined over $\mathbb{Q}$ such that the cocharacter $\mu_\infty$ factors through $\text{MT}(A) \times_{\mathbb{Q}} \mathbb{C}$.

For any rational prime $l$, let $T_lA(F)$ be the $l$-adic Tate module of $A$ and set $V_l = T_lA(F) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, which is a $2d$-dimensional vector space over $\mathbb{Q}_l$. Then we have a Galois representation:

$$\rho_l : \text{Gal}(\bar{F}/F) \to \text{Aut}_{\mathbb{Q}_l}(V_l).$$

We define $G_{l/\mathbb{Q}_l}$ as the Zariski closure of the image of $\rho_l$ inside $\text{Aut}_{\mathbb{Q}_l}(V_l)$ and let $G_{l/\mathbb{Q}_l}$ be its identity component. From Faltings’ theorem, the group $G_{l/\mathbb{Q}_l}$ is reductive.

The Mumford-Tate conjecture predicts that

**Conjecture 4.2.** For any prime $l$, we have the equality $G_{l/\mathbb{Q}_l} = \text{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_l$.

Deligne proved the following:

For all $p$ such that $p \nmid \text{deg}(\rho)$, we have

$$G_{l/\mathbb{Q}_l} = \text{MT}(A) \times_{\mathbb{Q}} \mathbb{Q}_l.$$
Theorem 6. For any prime $l$, we have the inclusion $G_{l/Q_l}^0 \subseteq MT(A) \times \mathbb{Q}_{l}^\times$.

From now on we assume that $d = 4$ and the abelian variety $A_{/F}$ is absolutely simple. Then the absolute endomorphism algebra $\text{End}^0(A_{/F})$ is a division algebra. In [9], Moonen and Zarhin proved that in almost all cases, the endomorphism algebra $\text{End}^0(A_{/F})$ together with its action on the Lie algebra $\text{Lie}(A_{/F})$ uniquely determines the Lie algebras of the Mumford-Tate group $MT(A)/\mathbb{Q}$ and the reductive group $G_{l/Q_l}^0$. Then only exception happens when $\text{End}^0(A_{/F}) = \mathbb{Q}$. In this case, there are two possibilities for the Lie algebra of $MT(A)/\mathbb{Q}$ together with its action on $V$ (resp. the Lie algebra of $G_{l/Q_l}^0$ together with its action on $V_l$):

1. $\mathfrak{c} \oplus \mathfrak{sp}_4$ with the standard representation, where $\mathfrak{c}$ is the 1-dimensional center $\mathfrak{c}$ of the Lie algebra;
2. $\mathfrak{c} \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, with the 1-dimensional center $\mathfrak{c}$, and the representation of $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is the tensor product of the standard representation of $\mathfrak{sl}_2$.

Together with Theorem [5], to prove the Mumford-Tate conjecture for simple 4-dimensional abelian varieties, it is enough to prove the following:

Theorem 7. Let $A_{/F}$ be an abelian variety of dimension 4 over a number field $F$. Suppose that $\text{End}(A_{/F}) = \mathbb{Z}$. If for some prime $l$, the group $G_{l/Q_l}^0$ together with its action on $V_l$ belongs to the second case listed above, then the same is true for the the group $MT(A)/\mathbb{Q}$ together with its action on $V$, i.e. the Mumford-Tate conjecture holds for $A_{/F}$.

5. Reductions of abelian varieties with Galois representations of Mumford’s type

Definition 5.1. Let $K$ be a field of characteristic 0 and fix an algebraic closure $\overline{K}$ of $K$. Let $G_{/K}$ be an algebraic group and let $V$ be a finite dimensional $K$-vector space with a faithful representation of $G$. We say that the pair $(G,V)$ is of Mumford’s type if the following three conditions are satisfied:

1. $\text{Lie}(G)$ has one dimensional center $\mathfrak{c}$;
2. $\text{Lie}(G)_K \cong \mathfrak{c}_K \oplus \mathfrak{sl}_2^\mathfrak{t}_K$;
3. $\text{Lie}(G)_K$ acts on $V_K$ by the tensor product of the standard representations of $\mathfrak{sl}_2,K$.

For any semisimple group $G_{/K}$, there exist (up to isomorphism) a simply connected group $\tilde{G}$ (resp. adjoint group $G^{\text{ad}}$) such that there exists central isogenies $\tilde{G} \to G$ (resp. $G \to G^{\text{ad}}$) over $K$.

Let $F$ be a number field and $A_{/F}$ be a four dimensional abelian variety. Let $G_F = \text{Gal}(\overline{F}/F)$ be the Galois group of $F$ and we use $v$ to indicate a finite place of $F$ and use $p_v$ to denote its residue characteristic. Let $F_v$ be the completion of $F$ at $v$ and $G_{F_v} \subseteq G_F$ be the decomposition group at $v$. Let $k_v$ be the residue field whose cardinality is $q_v$. Fix a Frobenius element $\text{Frob}_v$ at $v$. Let $l$ be any prime number. Recall that we have the Galois representation $\rho_l : G_F \to \text{Aut}_{\mathbb{Q}_l}(V_l)$ and $G_l$ is the Zariski closure of the image of $\rho_l$ in $\text{Aut}_{\mathbb{Q}_l}(V_l)$. From a result of Serre ([13] Theorem 3.6), replacing $F$ by a finite extension if necessary, we can assume that the algebraic group $G_{l,Q_l}$ is connected for every prime $l$ and it is a reductive group by a result of Faltings ([13] Corollary 5.8).

From [15] Lemma 1.3, we know that if the pair $(G_l,V_l)$ is of Mumford’s type for one prime $l$, then the same is true for all primes, and we have $\text{End}(A_{/F}) = \mathbb{Z}$.

Definition 5.2. If the abelian variety $A_{/F}$ has the property that the pair $(G_l,V_l)$ is of Mumford’s type for some prime $l$, we say that $A_{/F}$ is an abelian variety with Galois representation of Mumford’s type.

From [15] Corollary 2.2, if $A_{/F}$ is an abelian variety with Galois representation of Mumford’s type, it has potentially good reduction at all places of $F$. Hence replacing $F$ by a finite extension, we can assume that $A_{/F}$ has good reduction everywhere.

For any finite place $v$ of $F$, we choose a semisimple element $t_v \in \text{GL}_d(\mathbb{Q}_l)$ such that its characteristic polynomial is equal to the characteristic polynomial of the element $p_l(\text{Frob}_v)$. By Weil’s theorem, the conjugacy class of element $t_v$ in $\text{GL}_d(\mathbb{Q}_l)$ exists and does not depend on $l$. Let $T_v \subseteq \text{GL}_{8,\mathbb{Q}_l}$ be the Zariski closure of the subgroup generated by $t_v$, which is unique up to conjugation in $\text{GL}_{8,\mathbb{Q}_l}$. From [13] Theorem 3.7, we have:
Theorem 8. There exists a set $V_{\text{max}}$ of finite places of $F$ of Dirichlet density 1, such that for all $v \in V_{\text{max}}$, we have:

1. the group $T_{v,\mathcal{Q}_l}$ is connected and hence a torus;
2. for any $l \neq p_v$, the torus $T_{v,\mathcal{Q}_l}$ is conjugate to a maximal torus of $G_{l/\mathcal{Q}_l}$ under $\text{GL}_3(\mathbb{Q}_l)$.

As $A_{/F}$ is an abelian variety with Galois representation of Mumford’s type, for each prime $l$, the root system of the simple factors of $G_{l/\mathcal{Q}_l}$ has type $A_1$. In particular, the abelian variety $A_{/F}$ satisfies the hypothesis in [13] Theorem 7.1 and it follows that there exists a subset $V_{\text{good}} \subseteq V_{\text{max}}$ of finite places of $F$ with Dirichlet density 1 such that $A_{/F}$ has ordinary reduction at $v$ for all $v \in V_{\text{good}}$.

Fix a place $v \in V_{\text{good}}$. First we want to study the isogeny type of the reduction $A_{v/k_v}$ of $A_{/F}$ at $v$. From Lemma 1.3, there exist infinitely many primes $l$’s such that the derived group $G^\text{der}_{l}$ of $G_{l}$ is $\mathbb{Q}_l$-simple. We fix such a prime $l \neq p_v$. Then we have:

Proposition 5.3. Let $k_v$ be an algebraic closure of $k_v$. Then the reduction $A_{v/k_v}$ is either simple or isogenous to a product of an elliptic curve and a simple abelian threefold. In particular, the eigenvalues of the Frobenius $\text{Frob}_v$ on $V_1$ are all distinct.

The above proposition is an immediate consequence of [15] Proposition 4.1. But to establish notations used in our later argument, we give a sketchy proof here.

Proof. Let $\rho_{v,l} : G_{F_v} \to G_{l}(\mathbb{Q}_l)$ be the local $l$-adic Galois representation attached to $V_1$. From the proof of [15] Proposition 4.1, replacing $F$ by a finite extension if necessary, we can assume the following conditions:

1. the cardinality $q_v$ of the residue field $k_v$ is a perfect square;
2. for any $\sigma \in G_{F_v}$, we have the congruence $\rho_{v,l}(\sigma) \equiv 1_8 (\text{mod } l^2)$ in $G_{l}(\mathbb{Z}_l)$;
3. all the simple factors of $A_{v/k_v}$ are defined over $k_v$.

Recall that $\tilde{G}_{l/\mathcal{Q}_l}$ is the simply connected group with a central isogeny $\tilde{G}_l \to G_l$. From the second assumption above, the representation $\rho_{v,l} : G_{F_v} \to G_l(\mathbb{Z}_l)$ lifts uniquely to a representation $\tilde{\rho}_{v,l} : G_{F_v} \to \tilde{G}_{l}(\mathbb{Z}_l)$.

Now set $\pi = \tilde{\rho}_{v,l}(\text{Frob}_v) \in \tilde{G}(\mathbb{Q}_l)$, and let $\tilde{T}$ be the Zariski closure of the subgroup of $\tilde{G}(\mathbb{Q}_l)$ generated by $\pi$, which is a connected torus. We can assume that the residue field $k_v$ has even degree over its prime field and hence its cardinality $q_v$ is a perfect square. Set $\alpha = \frac{1}{\sqrt{q_v}} \in \tilde{G}(\mathbb{Q}_l)$ and let $\tilde{T}'$ be the Zariski closure of the subgroup of $\tilde{G}(\mathbb{Q}_l)$ generated by $\alpha$. Then $\tilde{T} \cong \mathbb{G}_{m,\mathbb{Q}_l} \times \tilde{T}'$ for some torus $\tilde{T}'$ of the derived subgroup $\tilde{G}^\text{der}$ of $\tilde{G}$. Let $\tilde{T}_{Q_l}$ be a maximal torus of $\tilde{G}^\text{der}$ containing $\tilde{T}'$.

As the pair $(G_l, V_1)$ is of Mumford’s type, from the above construction, the torus $\tilde{T}_{Q_l}$ has rank 3 and we have an isomorphism $X(\tilde{T}) \cong \mathbb{Z}^3$ such that the weights of the representation of $\tilde{T}_{Q_l}$ on $V_l$ correspond to $\pm(1, \pm 1, \pm 1) \in \mathbb{Z}^3$. The evaluation at the element $\alpha \in \tilde{T}(\mathbb{Q}_l)$ gives an additive map $ev : X(\tilde{T}) \to (\mathbb{Q}_l)^*$. As $\tilde{T}'$ is a subtorus of $\tilde{T}$, the restriction gives a natural surjection $X(\tilde{T}) \to X(\tilde{T}')$, whose kernel is the same as the kernel of the map $ev$. Hence we have an injective map $ev' : X(\tilde{T}') \to (\mathbb{Q}_l)^*$. By construction, the values $ev((\pm 1, \pm 1, \pm 1))$ are exactly the eigenvalues of $\alpha$ on $V_l$ and hence they are all in $\mathbb{Q}$ and have absolute value 1. The injection $X(\tilde{T}') \to (\mathbb{Q}_l)^*$ gives an action of $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ which extends the $\text{Gal}(\mathbb{Q}_l/\mathbb{Q}_l)$-action. It follows that actually the torus $\tilde{T}'$ is defined over $\mathbb{Q}$ and the map $ev' : X(\tilde{T}') \to (\mathbb{Q}_l)^*$ takes values in $(\mathbb{Q}_l)^*$ and is $\text{Gal}(\mathbb{Q}/\mathbb{Q})$-equivariant. The decomposition group $G_{Q_l}$ acts on the character group $X(\tilde{T}) \cong \mathbb{Z}^3$ through the group $\{\pm 1\}^3 \times S_3$ and similarly the Galois group $G_{Q_l}$ acts on the character group $X(\tilde{T}')$ in a similar way.

We fix an embedding $i_{p_v} : \bar{\mathbb{Q}} \to \mathbb{Q}_{p_v}$ which induces a $p_v$-adic valuation $v_{p_v}$ on $\bar{\mathbb{Q}}$, normalized by $v_{p_v}(q_v) = 1$ and define $\phi_v = v_{p_v} \circ ev' : X(\tilde{T}') \to \mathbb{Q}$, which is $\mathbb{Z}$-linear.

When the reduction $A_{v/k_v}$ is ordinary, from the argument in [15] Proposition 4.1, we see that $\ker(ev)$ is trivial, i.e. $X(\tilde{T}) = X(\tilde{T}')$ and hence $\tilde{T} = \tilde{T}'$. Under the isomorphism $X(\tilde{T}) \cong \mathbb{Z}^3$, the Galois action of $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ on $X(\tilde{T}')$ permutes the set $\{(\pm 1, \pm 1, \pm 1)\}$ and induces a group homomorphism $h_v : \text{Gal}(\mathbb{Q}/\mathbb{Q}) \to \{\pm 1\}^3 \times S_3 = \text{Aut}(X(\tilde{T}'))$. 
As the derived group $G_{1/\overline{Q}}^{der}$ is assumed to be $\overline{Q}_l$-simple, the image of the Galois group $\text{Gal}(\overline{Q}/Q_l)$ under $h_v$ contains a cycle in $S_3$ of length 3. Hence the Galois group $\text{Gal}(\overline{Q}/Q)$ acts transitively on the set \(\{(1, 1, -1), (1, -1, 1), (-1, 1, 1)\} \subset X(T')\). On the other hand, any complex conjugation in $\text{Gal}(\overline{Q}/Q)$ acts on $X(T')$ by multiplication by $-1$. So we have the following possibilities:

1. the action of $\text{Gal}(\overline{Q}/Q)$ on the set $\{ev((\pm 1, \pm 1, \pm 1))\} \subset \overline{Q}$ is transitive. In this case, the abelian variety $\mathcal{A}_{/\overline{K}}$ is simple and $ev((1, 1, 1)) \in \overline{Q}$ generates a CM field of degree 8 over $\overline{Q}$;

2. the action of $\text{Gal}(\overline{Q}/Q)$ on the set $\{ev((\pm 1, \pm 1, \pm 1))\} \subset \overline{Q}$ has two orbits: $\{ev((1, 1, 1), ev((-1, 1, -1))$ and $\{ev((1, 1, -1), ev((1, -1, 1)), ev((-1, 1, 1)), ev((-1, 1, -1))\}$. In this case the abelian variety $\mathcal{A}_{/\overline{K}}$ is isogenous to a product of an elliptic curve and a simple abelian threefold. The element $ev((1, 1, 1)) \in \overline{Q}$ generates a quadratic field over $Q$ and $ev((1, 1, -1)) \in \overline{Q}$ generates a CM field of degree 6 over $\overline{Q}$.

We keep the notation as in the above proof. Since the abelian variety $\mathcal{A}_{/\overline{K}}$ is ordinary, its slopes are 0 and 1, each of which has multiplicity 4. On the other hand, the slopes of $\mathcal{A}_{/\overline{K}}$ are given by the values $\{v_p, (\sqrt{Q_p}) \cdot \phi_v((\pm 1, \pm 1, \pm 1))\}$. Hence the set $\{\phi_v((\pm 1, \pm 1, \pm 1))\}$ takes values in the set $\{\pm \frac{1}{2}\}$. Then we can choose an isomorphism $X(T) \cong \mathbb{Z}^3$ such that $\phi_v((1, 1, 1)) = \frac{1}{2}$. As the map $\phi_v : T \to X(T')$ is additive, we have

$$\phi_v((1, 1, -1)) + \phi_v((1, -1, 1)) + \phi_v((-1, 1, 1)) = \frac{1}{2}.$$ 

It follows that one of the three numbers $\phi_v((1, 1, -1))$, $\phi_v((1, -1, 1))$, $\phi_v((-1, 1, 1))$ is $-\frac{1}{2}$ and the other two are $\frac{1}{2}$. Without loss of generality, we can assume that $\phi_v((1, 1, -1)) = -\frac{1}{2}$. Then $\phi_v((1, 0, 0)) = \frac{1}{2}$ and $\phi_v((0, 0, 1)) = \phi_v((0, 0, 1)) = 0$.

Now consider the composition:

$$\tilde{h}_v : \text{Gal}(\overline{Q}/Q) \ni \{\pm 1\}^3 \to S_3 \to S_3,$$

where the second map the the natural projection. Define a number field $K(v)$ in $\overline{Q}$ as the fixed field of the group $H_v = \tilde{h}_v^{-1}((id, (23))) \subseteq \text{Gal}(\overline{Q}/Q)$ ($H_v$ is the subgroup of $\text{Gal}(\overline{Q}/Q)$ which fixes the first component of $X(T) \cong \mathbb{Z}^3$). As the image of $h_v$ contains a cycle of length 3 in $S_3$, $K(v)$ is a cubic field. Since the image of any complex conjugation in $\text{Gal}(\overline{Q}/Q)$ under $h_v$ is $(1, -1, -1, id) \in \{\pm 1\}^3 \times S_3$, the field $K(v)$ is necessarily totally real.

If we consider another place $v' \in V_{good}$, we can get another totally real cubic field $K(v')$ by the same construction as above. The fields $K(v)$ and $K(v')$ are isomorphic. In fact, from the proof of [5,3], we see that $\tilde{T}'_{/\overline{Q}_l} \subset \tilde{G}'_{l/\overline{Q}_l}$ is a maximal torus and we can consider the associated reduced root system $\Psi$. As the torus $\tilde{T}$ can be defined over $\overline{Q}$, we have the continuous group homomorphism $h_v : \text{Gal}(\overline{Q}/Q) \rightarrow \text{Aut}(\Psi) \cong \{\pm 1\}^3 \times S_3$. If we consider another place $v' \in V_{good}$, we have another maximal torus $\tilde{T}'_{/\overline{Q}_l} \subset \tilde{G}'_{l/\overline{Q}_l}$ which can be defined over $\overline{Q}$. As the tori $\tilde{T}$ and $\tilde{T}'$ are conjugate over $\overline{Q}_l$ inside $\tilde{G}'_{l/\overline{Q}_l}$, it induces an isomorphism between the root data associated to these two tori. Such an isomorphism is unique up to conjugation by elements in the Weyl group $W(\Psi)$ of $\Psi$. Let $\text{Out}(\Psi) = \text{Aut}(\Psi)/W(\Psi)$ be the outer automorphism group of $\Psi$, which is isomorphic to $S_3$. Then the composite $\tilde{h}_v : \text{Gal}(\overline{Q}/Q) \rightarrow \text{Out}(\Psi)$ does not depend on the choice of the maximal torus $\tilde{T}$. Hence $\tilde{h}_v$ is independent of $v$ and so is the cubic field $K(v)$. In the following, we just denote this field by $K$.

Set $H'(v) = \tilde{h}_v^{-1}((1, \pm 1, \pm 1) \times (id, (23)))) \subset \text{Gal}(\overline{Q}/Q)$, and let $L(v)/K$ be the fixed field of $H'(v)$ inside $\overline{Q}$. Then $L(v)/K$ is necessarily a quadratic extension. Moreover, any complex conjugation in $\text{Gal}(\overline{Q}/Q)$ acts on $X(T) \cong \mathbb{Z}^3$ by inversion, one can check that $L(v)$ is a CM field by direct calculation.

As the torus $\tilde{T} = G_{\overline{Q}_l} \times \tilde{T} = G_{\overline{Q}_l}$ is generated by $\pi = \tilde{h}_v(Frob_{\overline{Q}_l})$, from the above construction, we see that $\tilde{T}'_{/\overline{Q}_l} = \tilde{T}'_{/\overline{Q}_l} \cong G_{\overline{Q}_l} \times T'_{/\overline{Q}_l}$. Here $T'_{/\overline{Q}_l} \subset \text{Gal}(\overline{Q}/Q)$ acts on $X(T)$ by $-1$. Since the subset $V_{good}$ of finite places has Dirichlet density 1, we can find a place $v \in V_{good}$ over a rational prime $p = p_v$ such that $p$ splits completely in the cubic totally real field $K$ and for simplicity
we write $L = L(v)$. So there are three different places $v = v_1, v_2, v_3$ of $K$ lying over $p$. Since we fix an embedding $i_p = i_{p_v} : \bar{Q} \to \bar{Q}_p$, we have three embeddings $\varphi_1, \varphi_2, \varphi_3 : K \to \bar{Q}$ such that $\varphi_i$ induces the place $v_i$ for $i = 1, 2, 3$. As $L/K$ is a totally imaginary quadratic extension, we can denote the embeddings of $L$ to $\bar{Q}$ by $\psi_i, \bar{\psi}_i : L \to \bar{Q}, i = 1, 2, 3$ such that $\psi_i, \bar{\psi}_i$ extend the embedding $\varphi_i$ for $i = 1, 2, 3$.

Now recall that in the proof of Proposition 3.3 we considered the element $\alpha = \frac{q}{K^{1/2}} \in \bar{T}'(\bar{Q}) = T_L(\bar{Q}) \subseteq L^\times$, which satisfies:

$$v_p(\psi_1(\alpha)) = \frac{1}{2}, \quad v_p(\bar{\psi}_1(\alpha)) = -\frac{1}{2},$$

and

$$v_p(\psi_i(\alpha)) = v_p(\bar{\psi}_i(\alpha)) = 0$$

for $i = 2, 3$. This implies that the place $v = v_1$ of $K$ splits into two different places $w_1, \bar{w}_1$ of $L$. Since $p$ splits in $K$, we see that the $w_1$-adic (resp. $\bar{w}_1$-adic) completion of $L$ is isomorphic to $\bar{Q}_p$. We keep the choice of the place $v$ and the above property will be used in later argument.

6. Linear relations of the Serre-Tate coordinates

Fix the place $v$ as in the preceding section and set $p = p_v$. We then have the Galois representation attached to the $p$-adic Tate module of $A/F^p$: $\rho_p : \text{Gal}(\bar{F}/F) \to G_p \hookrightarrow \text{Aut}_{\bar{Q}_p}(V_p)$. In this section we want to study the local Galois representation $\rho_{v,p} : \text{Gal}(\bar{F}_v/F_v) \to G_p \hookrightarrow \text{Aut}_{\bar{Q}_p}(V_p)$ and its restriction to the inertia group $I_v \subset D_v$. As the abelian variety $A/F$ has good reduction at $v$, the representation $\rho_{v,p}$ is crystalline with Hodge-Tate weight $0$ and $1$.

6.1. Filtered modules and Newton cocharacters. First we recall the notions of filtered modules and Newton cocharacters.

Let $\text{Rep}_{D_v}$ be the tannakian category of all finite dimensional continuous representation of the decomposition group $D_v$ over $\bar{Q}_p$ and let $((V_p))$ be the full tannakian subcategory of $\text{Rep}_{D_v}$ generated by $V_p$. Let $\text{Vec}_{\bar{Q}_p}$ be the category of finite dimensional $\bar{Q}_p$-vector spaces, and we have the forgetful functor $\omega_{V_p} : ((V_p)) \to \text{Vec}_{\bar{Q}_p}$, which is a fiber functor of the tannakian categories. The automorphism group $H_{V_p} = \text{Aut}^\otimes(\omega_{V_p})$ of the fiber functor $\omega_{V_p}$ can be identified with the Zariski closure of the image of the local Galois representation $\rho_{v,p}$.

Let $\sigma : F_v \to F_v$ be the Frobenius automorphism. By $p$-adic Hodge theory, one can associate a filtered module $M_p$ to the crystalline representation $V_p$. The filtered module $M_p$ is a finite dimensional $F_v$-vector space with a $\sigma$-linear automorphism $F_{\rho_M} : M_p \to M_p$. Let $\text{MF}_{F_v}$ be the tannakian category of weakly admissible filtered modules over $F_v$, and let $((M_p))$ be the full tannakian subcategory of $\text{MF}_{F_v}$ generated by $M_p$. Then we have the forgetful functor $\omega_{M_p} : ((M_p)) \to \text{Vec}_{\bar{Q}_p}$, which is a fiber functor of the tannakian categories. Let $H_{M_p} = \text{Aut}^\otimes(\omega_{M_p}) \subset \text{Aut}_{F_v}(M_p)$ be the automorphism group of the fiber functor $\omega_{M_p}$ defined over $F_v$. Then from [2] Theorem 3.2, the algebraic group $H_{M_p}$ is an inner form of $H_{V_p} \times_{Q_p} F_v$. Hence we can identify $(H_{M_p})_{Q_p}$ with $(H_{V_p})_{Q_p}$.

Let $m_v = [F_v : Q_p]$. Then the morphism $F_{\rho_{M_p}} : M_p \to M_p$ is $F_v$-linear, and it gives a $Q$-grading $M_p = \bigoplus_{i \in Q} M_{p,i}$, such that the eigenvalues of $F_{\rho_{M_p}}$ on $M_{p,i}$ has valuation $m_v i$ (the valuation on $Q_p$ is normalized so that the valuation of $p$ is $1$). Then we can define the Newton cocharacter of $M_p$:

$$\mu_{M_p,F_v} : G_{m,F_v} \to H_{M_p,F_v},$$

such that $G_{m,F_v}$ acts on $M_{p,i}$ by $(\cdot)^{m_v i}$. 

6.2. Application to the study of local Galois representations. As mentioned above, the algebraic group $H_{M_p,F_v}$ is an inner form of $H_{\mathbb{T}_v \times \mathbb{Q}_p,F_v}$, the cocharacter $\mu_{M_p,\mathbb{Q}_p} = \mu_{M_p,F_v} \times F_v \bar{\mathbb{Q}}_p$ gives a cocharacter

$$\mu : \mathbb{G}_{m,\mathbb{Q}_p} \rightarrow H_{\mathbb{T}_v,\mathbb{Q}_p} \hookrightarrow G_{p,\mathbb{Q}_p}.$$ 

As we have the central isogenies $\tilde{G}_{p,\mathbb{Q}_p} \cong \mathbb{G}_{m,\mathbb{Q}_p} \times (\text{SL}_2,\mathbb{Q}_p)^3 \rightarrow G_{p,\mathbb{Q}_p}$, there exists a positive integer $k$ such that $\mu^k : \mathbb{G}_{m,\mathbb{Q}_p} \rightarrow G_{p,\mathbb{Q}_p}$ can be lifted to a homomorphism:

$$\tilde{\mu} = (\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \tilde{\mu}_3) : \mathbb{G}_{m,\mathbb{Q}_p} \rightarrow \tilde{G}_{p,\mathbb{Q}_p},$$

where $\tilde{\mu}_0 : \mathbb{G}_{m,\mathbb{Q}_p} \rightarrow \mathbb{G}_{m,\mathbb{Q}_p}$ and $\tilde{\mu}_1 : \mathbb{G}_{m,\mathbb{Q}_p} \rightarrow \text{SL}_2,\mathbb{Q}_p$, $i = 1, 2, 3$ are homomorphism of algebraic groups.

Hence there exists $n_0, n_1, n_2, n_3 \in \mathbb{Q}$, such that the slopes of $M_p$ are given by the numbers $n_0 \pm n_1 \pm n_2 \pm n_3$. From the argument in the proof of Theorem 3.2 in [15], we have $n_0 = \frac{1}{2}$. Moreover, when $A_{i/F}$ has good ordinary reduction at $v$, i.e., the Newton polygon of $M_p$ is $4 \times 0, 4 \times 1$, we have that one of the three numbers $n_1, n_2, n_3$ is $\frac{1}{2}$ and the other two are 0. Without loss of generality, we can assume that $n_1 = \frac{1}{2}$ and $n_2 = n_3 = 0$.

On the other hand, by a theorem of Katz-Messing ([5], 1.3.5) we have the following:

**Theorem 9.** The characteristic polynomial of $F_{M_p}^{\tilde{\mu}_0}$ on $M_p$ is equal to the characteristic polynomial of $\rho_l(Frob_v)$ for any $l \neq p$.

First we assume that all the eigenvalues of $\rho_l(Frob_v)$ are in $\mathbb{Z}_p$. In this case we have an explicit expression for the Serre-Tate coordinates of $A_{i/F}$. By this assumption and our choice of the place $v$ together with the above theorem, we see that the 8 eigenvalues of $F_{M_p}^{\tilde{\mu}_0}$ on $M_p$ are all distinct an lie in $\mathbb{Z}_p$. As the reduction of $A_{i/F}$ at $v$ is ordinary, we can choose a symplectic basis $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$ of the $p$-adic Tate module $T_{p,\mathbb{A}}(\bar{\mathbb{Q}})$, under which the local Galois representation is of the shape:

$$\rho_{v,p} : T_v = \text{Gal}(\bar{F}_v/F_v) \rightarrow \text{GSp}_8(\mathbb{Z}_p),$$

$$\sigma \mapsto \begin{pmatrix} T_1(\sigma) & B(\sigma) \\ 0 & T_2(\sigma) \end{pmatrix},$$

where $B(\sigma) = (b_{i,j}(\sigma))_{1 \leq i,j \leq 4}$ is a matrix in $M_{4 \times 4}(\mathbb{Z}_p)$ depending on $\sigma$, and $T_1(\sigma), T_2(\sigma)$ are diagonal matrices of the shapes:

$$T_1(\sigma) = \begin{pmatrix} (\chi_p^{-1}v_{(-1,1,1)})(\sigma) & 0 & 0 & 0 \\ 0 & (\chi_p^{-1}v_{(-1,1,1)})(\sigma) & 0 & 0 \\ 0 & 0 & (\chi_p^{-1}v_{(-1,1,1)})(\sigma) & 0 \\ 0 & 0 & 0 & (\chi_p^{-1}v_{(-1,1,1)})(\sigma) \end{pmatrix},$$

and

$$T_2(\sigma) = \begin{pmatrix} \psi_{(-1,1,1)}(\sigma) & 0 & 0 & 0 \\ 0 & \psi_{(-1,1,1)}(\sigma) & 0 & 0 \\ 0 & 0 & \psi_{(-1,1,1)}(\sigma) & 0 \\ 0 & 0 & 0 & \psi_{(-1,1,1)}(\sigma) \end{pmatrix}.$$ 

Here $\chi_p : D_v \rightarrow \mathbb{Z}_p^\times$ is the $p$-adic cyclotomic character and $\psi_{(i,j,k)} : D_v \rightarrow \mathbb{Z}_p^\times$ is the unramified character which sends $Frob_v$ to the element $\sqrt{q_v} \cdot ev(i,j,k)$ for $(i,j,k) \in \{\pm 1, \pm 1, \pm 1\}$ defined in section 3.

Now we consider the Hodge cocharacter $\mu_{HT} : \mathbb{G}_{m,\mathbb{C}_p} \rightarrow G_{p,\mathbb{C}_p}$ associated to the Galois representation $V_p$. From Sen’s theory ([17], Theorem 2), the Zariski closure of the image of $\mu_{HT}$ in $G_p$ over $\mathbb{Q}_p$ is equal to the Zariski closure of $\rho_{v,p}(I_v)$ inside $G_{/\mathbb{Q}_p}$, which is denoted by $H'_{V_p/\mathbb{Q}_p}$.

Consider the representation $\rho^{ad}_{v,p} : I_v \xrightarrow{\rho_{v,p}} G_p(\mathbb{Q}_p) \rightarrow G^{ad}_p(\mathbb{Q}_p)$.

From [15], Proposition 3.5, the representation $\rho^{ad}_{v,p}$ projects $I_v$ nontrivially to exactly one of the $\mathbb{Q}_p$-simple factor of $G^{ad}_p(\mathbb{Q}_p)$, which is denoted by $G^{ad}_{p,1/\mathbb{Q}_p}$. From [15], Proposition 3.6, when $A_{i/F}$ has good ordinary reduction at $v$, we have an isomorphism $(G^{ad}_{p,1/\mathbb{Q}_p})_{\mathbb{Q}_p} \cong \text{PSL}_2,\mathbb{Q}_p$. Hence the root system of $H'_{V_p/\mathbb{Q}_p}$ is $(\pm 2, 0, 0) \in X(\mathbb{T})$ under the isomorphism $X(\mathbb{T}) \cong \mathbb{Z}^3$ defined in the previous section.
Fix a Frobenius element $Frob_v$ in $D_v$. As we explain above, the eigenvalues of the matrix $\rho_{v,p}(Frob_v)$ are all distinct and lie in $\mathbb{Z}_p^\times$. So we can modify the basis of $T_p A(\mathbb{Q})$ if necessary and assume that the matrix $\rho_{v,p}(Frob_v)$ is diagonal. As $\rho_{v,p}(Frob_v)$ generates a maximal torus in $G^G_{p,\mathbb{Q}_p}$, by the explicit calculation of the conjugation on $\rho_{v,p}(I_v)$ by the matrix $\rho_{v,p}(Frob_v)$, we see that the entries $b_{i,5-i}: I_v \to \mathbb{Z}_p$, $i = 1, 2, 3, 4$ of $B$ give the weight $(2, 0, 0) \in X(T)$, and no entry of $B$ gives the weight $(-2, 0, 0)$. Hence $b_{i,j} = 0 : I_v \to \mathbb{Z}_p$ if $i + j \neq 5$ and the set $\{b_{4,\sigma}(\sigma), b_{23}(\sigma), b_{32}(\sigma), b_{41}(\sigma)\} |_{\sigma \in I_v}$ spans a $1$-dimensional $\mathbb{Q}_p$ vector space inside $G^G_{p,\mathbb{Q}_p}$.

Now from the discussion in section 1, we see that the Serre-Tate coordinates $q(A, Frob_v) : \text{Sym}(T_p A_v(k) \otimes \mathbb{Z}_p) \to \hat{G}_m(W(k))$ satisfies the following properties: there exists $(\lambda_1, \lambda_2) \in \mathbb{Z}_p^2 \setminus \{(0, 0)\}$, such that

$$q(u_1 \otimes u_4)^{\lambda_1} = q(u_4 \otimes u_1)^{\lambda_1} = q(u_2 \otimes u_3)^{\lambda_2} = q(u_3 \otimes u_2)^{\lambda_2},$$

and

$$q(u_i \otimes u_j) = 0 \text{ for } i + j \neq 5.$$

Let $U_{W(k)}$ be the formal torus $\text{Hom}_{\mathbb{Z}_p}(\text{Sym}^2(T_p A_v(k)), \hat{G}_m)$. The $\mathbb{Z}_p$-basis $\{u_i \otimes u_j | 1 \leq i \leq j \leq 4\}$ of $\text{Sym}^2(T_p A_v(k))$. Under this basis, we have ten coordinates $t_{ij}, 1 \leq i \leq j \leq 4$ on $U_{W(k)}$. Set $T_{ij} = t_{ij} - 1$, for $1 \leq i \leq j \leq 4$, and then we have an isomorphism of formal tori over $W(k)$:

$$\mathcal{U} \to \text{Spf}(W(k)[[T_{ij}]])_{1 \leq i \leq j \leq 4}.$$

Now we define a rank one formal subtorus $\mathcal{U}_{W(k)}$ of $U_{W(k)}$, such that $\mathcal{U}$ corresponds to the formal torus $\text{Spf}(W(k)[[T_{ij}]])_{1 \leq i \leq j \leq 4}/(T_{11}, T_{22}, T_{33}, T_{44}, T_{12}, T_{13}, T_{24}, T_{34}, (1 + T_{14})^{\lambda_1} - (1 + T_{23})^{\lambda_2})$ under the above isomorphism. From the discussion in this section, we see that the abelian variety $A_{Frob}$ sits on the subtorus $\mathcal{U}_{W(k)}$.

In general, we do not assume that all the eigenvalues of $\rho_{I}(Frob_v)$ are in $\mathbb{Z}_p$. Then we can choose a symplectic basis $\{v_0^1, v_0^2, v_0^3, v_0^4, v_1^t, v_2^t, v_3^t, v_4^t\}$ of the $p$-adic Tate module $T_p A(\mathbb{Q})$ such that the local Galois representation is of the shape:

$$\rho_{v,p} : D_v = \text{Gal}(F_v/F_v) \to \text{GSp}_8(\mathbb{Z}_p),$$

$$D_v \ni \sigma \mapsto \begin{pmatrix} T_1(\sigma) & B(\sigma) \\ 0 & T_2(\sigma) \end{pmatrix},$$

where $\chi_p$ is again the $p$-adic cyclotomic character, $B : D_v \to M_{4 \times 4}(\mathbb{Z}_p)$ is a map valued in $4 \times 4$ symmetric matrices, and $A(\cdot)$ (resp. $A^{-1}(\cdot)$): $D_v \to \text{GL}_4(\mathbb{Z}_p)$ is an unramified homomorphism which sends any Frobenius $Frob_v \in D_v$ to a matrix $A \in \text{GL}_4(\mathbb{Z}_p)$ (resp. $A^{-1} \in \text{GL}_4(\mathbb{Z}_p)$). From the discussion in section 5, there exists a Galois extension $M/Q_p$ with degree at most 4, such that all the eigenvalues of $\rho_{I}(Frob_v)$ are in $M$. Then we can find a matrix $W \in \text{GL}_4(O_M)$ such that

$$W A W^{-1} = \begin{pmatrix} \psi_{(-1,1,1)}(Frob_v) & 0 & 0 & 0 \\ 0 & \psi_{(-1,1,-1)}(Frob_v) & 0 & 0 \\ 0 & 0 & \psi_{(-1,-1,1)}(Frob_v) & 0 \\ 0 & 0 & 0 & \psi_{(-1,-1,-1)}(Frob_v) \end{pmatrix}.$$

Then we consider the conjugation of the Galois representation

$$\rho'_{v,p} = \begin{pmatrix} W & 0 \\ 0 & (W^t)^{-1} \end{pmatrix} \rho_{v,p} \begin{pmatrix} W^{-1} & 0 \\ 0 & W^t \end{pmatrix} : D_v \to \text{GSp}_8(O_M),$$

$$D_v \ni \sigma \mapsto \begin{pmatrix} T'_1(\sigma) & B'(\sigma) \\ 0 & T'_2(\sigma) \end{pmatrix},$$

where $T'_2 : D_v \to \text{GL}_4(O_M)$ is an unramified homomorphism sending (any) Frobenius element to the matrix $W A W^{-1}$, and $T'_1 = \chi_p \cdot (T'_2)^{-1}$, and $B' = (b'_{ij})_{1 \leq i \leq j \leq 24} : D_v \to M_{4 \times 4}(O_M)$ is a map.

Take another conjugation if necessary, we can assume that $\rho'_{v,p}(Frob_v)$ is diagonal for some Frobenius element $Frob_v \in D_v$. As $\rho'_{v,p}(Frob_v)$ generates a maximal torus of $G_{p,\mathbb{Q}_p}$, we can again apply Noot’s results to conclude that $b'_{ij} = 0$ if $i + j \neq 5$ and the set $\{b'_{41}(\sigma), b'_{23}(\sigma), b'_{32}(\sigma), b'_{41}(\sigma)\} |_{\sigma \in I_v}$ spans a
1-dimensional $M$-vector space inside $M^4$. For each pair $1 \leq i, j \leq 4$, the map $b_{ij}: I_v \to O_M$ is an $O_M$-linear combination of the maps $b_{kl}: I_v \to \mathbb{Z}_p$, $1 \leq k, l \leq 4$. From Theorem 5 and Remark 2.1, the entries $b_{kl}$'s determines the Serre-Tate coordinates of $A_{W(k)}$. Hence the above restrictive conditions on the entries $b_{kl}$'s can be translated to the restrictive conditions on the Serre-Tate coordinates of $A_{W(k)}$. It may not be obvious from this observation that we get a rank 1 formal subtorus of $U_{W(k)}$, but we will use this observation in next section.

To see that the above restrictive conditions define a rank 1 formal subtorus of $U_{W(k)}$, we use our special choice of the finite place $v$ at the end of section 5. Replacing the number field by a finite extension if necessary, we can assume that the representation $\rho_v: D_v \to G_{p}(Q_p)$ can be lifted to the semisimple group $\tilde{\rho}_{v,p}: D_v \to \tilde{G}_p(Q_p)$. Consider the element $\tilde{\rho}_{v,p}(Frob_v) \in \tilde{G}_p(Q_p) \subseteq G_p$. By our assumption on the algebraic group $G_p$, we know that over $\bar{Q}_p$, we have an isomorphism:

$$\tilde{G}_p/\bar{Q}_p \cong G_m/\bar{Q}_p \times (SL_2, \bar{Q}_p)^3.$$ 

By our choice of the place $v$, the projection of $\tilde{\rho}_{v,p}(Frob_v)$ to the first factor of $(SL_2, \bar{Q}_p)^3$ actually sits inside $SL_2(Q_p)$ and hence generates a torus over $Q_p$. On the other hand, from the previous discussion, the conjugation action of the maximal torus $T_{\bar{Q}_p}$ generated by $\rho_v(Frob_v)$ on the group $\rho(I_v)$ can only gives the root $(2, 0, 0) \in X(T)$. From the general theory of reduction groups (see [18]), the Lie algebra of $G_p$ on which the maximal torus $T$ acts through the root $(2, 0, 0) \in X(T)$ has dimension 1 over $\bar{Q}_p$. This mean that the set $\{b_{ij}(\sigma)\}_{i, j}$ determines the Serre-Tate coordinates of $\bar{Q}_p$. Again from Theorem 5 and Remark 2.1, we see that the above conditions define a rank 1 formal subtorus $\mathfrak{3}$ of $U_{W(k)}$.

7. Proof of the main result

In this section, we prove the main result Theorem 7 in this paper. It is enough to prove the following:

**Theorem 10.** Let $F$ be a number field. If $A_{W(k)}$ is an abelian variety with Galois representation of Mumford’s type, then $A_{W(k)}$ come from a Shimura curve constructed by Mumford in [11]. In particular, the Mumford-Tate conjecture holds for $A_{W(k)}$.

Replacing $F$ by a finite extension if necessary, we can assume that $A_{W(k)}$ has a principal polarization $\lambda: A \to A^t$ and the algebraic groups $G_{l}(Q_l)$ are connected for all primes $l$. For each integer $N \geq 3$, we choose a symplectic level $N$ structure $\eta_N$ of $A_{W(k)}$. Then the triple $(A_{W(k)}, \lambda, \eta_N)$ gives an $F$-valued point $x$ on the Siegel moduli space $A_{1,N}$.

Now recall that we choose a finite place $v$ of $F$ in section 5 at which the abelian variety $A_{W(k)}$ has good ordinary reduction. Let $k_v$ be the residue field of $F$ at $v$ with characteristic $p = p_v$ and fix an algebraic closure $\bar{k}_v$ of $k_v$. The reduction $A_{1,N} \to A_{1,N}$ at $v$ gives a closed point $x_v \in A_{1,N}(k)$. As explained in section 3, the formal completion $U_{W(k)}$ of $A_{1,N}$ along the closed point $x_v$ has a formal group structure and is isomorphic to

$$\text{Hom}_{\mathbb{Z}_p}(\text{Sym}(T_pA_{W(k)}) \otimes_{\mathbb{Z}_p} T_pA_{W(k)}, \hat{G}_m, U_{W(k)}).$$

In the previous section, we determine a rank 1 formal subtorus $\mathfrak{3}$ of $U_{W(k)}$ on which the point $x$ lies.

In section 6, we fix a basis $\{v_1^i, \ldots, v_4^i\}$ of $T_pA(Q_p)$ such that $\{v_1^i, \ldots, v_4^i\}$ is a basis of $T_pA(Q_p)$, and $\{v_1^i, \ldots, v_4^i\}$ corresponds to a basis $\{u_1, \ldots, u_4\}$ of $T_pA_{W(k)}$ under the reduction map. Moreover, this basis is symplectic in the sense that under the Weil pairing $E_{p,v}: T_pA(Q_p) \times T_pA(Q_p) \to T_pA_{W(k)}(Q_p)$ induced from the polarization $\lambda$, we have $E_{p,v}(v_i^1, v_j^1) = \zeta_p^i$ if $i = j$ and $E_{p,v}(v_i^1, v_j^1) = 1$ if $i \neq j$, where $\zeta_p^i$ is a fixed basis of $T_pA_{W(k)}(Q_p)$.

The basis $\{v_1^i, \ldots, v_4^i\}$ gives a full level structure at $p$ of the abelian variety $A_{W(k)} \eta_{p,v}: L_p \to T_pA(Q_p)$ such that $\eta_{p,v}(e_i) = v_i^1$, for $1 \leq i \leq 4$, and $\eta_{p,v}(e_i) = v_i^1$, for $5 \leq i \leq 8$. The level structure $\eta_{p,v}$ induces an isomorphism $W_p \cong L_p \otimes_{\mathbb{Z}_p} \bar{Q}_p \to T_pA(Q_p) \otimes_{\mathbb{Z}_p} \bar{Q}_p = V_p$, which gives an isomorphism of algebraic groups: $\text{Aut}_{\bar{Q}_p}(W_p) \cong \text{Aut}_{\bar{Q}_p}(V_p)$. As $G_{p/\bar{Q}_p}$ is an algebraic subgroup of $\text{Aut}_{\bar{Q}_p}(V_p)$, we can regard $G_{p/\bar{Q}_p}$ as a subgroup of $\text{Aut}_{\bar{Q}_p}(W_p)$ under the above isomorphism.

Now let $(A_{can}(W(k)), \lambda_{can}, \eta_{N,N,can})$ be the canonical lifting of $x_v$, which corresponds to the identity element in the group $U(W(k))$. From [8] Lemma 2.8, the abelian variety $A_{can}$ has complex multiplication and hence is defined over some number field $F_1$. 


Fix a complex embedding \( i : F_1 \hookrightarrow \mathbb{C} \) and set \( A_{\text{can}}/\mathbb{C} = A_{can} \). Let \( H_1(A_{\text{can}}/\mathbb{C}, \mathbb{Q}) = V_{\text{can}} \) be the first rational homology group of \( A_{\text{can}}/\mathbb{C} \) and let \( \text{MT}(A_{\text{can}})/\mathbb{Q} \hookrightarrow \text{Aut}_{\mathbb{Q}}(V_{\text{can}}) \) be the Mumford-Tate group of \( A_{\text{can}} \). On the other hand, fix an algebraic closure \( F_1 \) of \( F_1 \). Let \( T_p A_{\text{can}}(F_1) \) be the \( p \)-adic Tate module of \( A_{\text{can}} \) and set \( V_{\text{can}, p} = T_p A_{\text{can}}(F_1) \otimes_{\mathbb{Q}} \mathbb{Q}_p \). By comparison theorem, we have an isomorphism \( V_{\text{can}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to V_{\text{can}, p} \), which induces an isomorphism of algebraic groups: \( \text{Aut}_{\mathbb{Q}}(V_{\text{can}}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \text{Aut}_{\mathbb{Q}_p}(V_{\text{can}, p}) \).

Now we give a full level structure at \( p \) of \( A_{\text{can}}/F_1 \). Recall that \( A_{\text{can}}/W(k) \) is the canonical lifting of the ordinary abelian variety \( A_{v'/k} \). Then connected-\( \acute{e} \)tale exact sequence of Barsotti-Tate groups

\[
0 \to \hat{A}_{\text{can}} \to A_{\text{can}}[p^{\infty}] \to T_p A_{v}(k) \times_{\mathbb{Z}} (\mathbb{Q}_p/\mathbb{Z}_p) \to 0
\]
splits over \( W(k) \). As we have an inclusion from \( F_1 \) to the fractional field of \( W(k) \), it induces a finite place \( v_1 \) of \( F_1 \) over \( p \). Let \( I \subset \text{Gal}(\overline{F_1}/F_1) \) be the inertia group at \( v_1 \). The above splitting exact sequence of Barsotti-Tate groups gives a splitting of the exact sequence of the \( p \)-adic Tate modules as \( I \)-modules:

\[
0 \to \hat{A}_{\text{can}}(\mathbb{Q}_p) \to T_p A_{\text{can}}(\mathbb{Q}_p) \to T_p A_{v}(k) \to 0.
\]

Under the Weil pairing on \( T_p A_{\text{can}}(\mathbb{Q}_p) \) induced from the polarization \( \lambda_{\text{can}} \) of \( A_{\text{can}} \), we can choose a symplectic basis \( \{v_1^{\text{can}}, \ldots, v_4^{\text{can}}\} \) of \( T_p A_{\text{can}}(\mathbb{Q}_p) \) such that \( \{v_1^{\text{can}}, \ldots, v_4^{\text{can}}\} \) is a basis of \( T_p \hat{A}_{\text{can}}(\mathbb{Q}_p) \), and \( \{v_1^{\text{can}}, \ldots, v_4^{\text{can}}\} \) corresponds to the basis \( \{u_1, \ldots, u_4\} \) of \( T_p A_{v}(k) \) under the splitting of the above exact sequence. This symplectic basis allows us to endow \( A_{\text{can}}/F_1 \) with a full level structure at \( p \) \( \eta_{\text{can}, p^\infty} : L_p \to T_p A_{\text{can}}(\mathbb{Q}_p) \) such that \( \eta_{\text{can}, p^\infty}(e_i) = v_i^{\text{can}} \) for \( 1 \leq i \leq 4 \), and \( \eta_{\text{can}, p^\infty}(e_i) = v_i^{\text{can}} \) for \( 5 \leq i \leq 8 \). By inverting \( p \), the level structure \( \eta_{\text{can}, p^\infty} \) gives an isomorphism \( W_p = L_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to T_p A_{\text{can}}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_{\text{can}, p} \).

Then we define an embedding of algebraic groups over \( \mathbb{Q}_p \):

\[
i\text{can} : \text{MT}(A_{\text{can}}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \text{Aut}_{\mathbb{Q}}(V_{\text{can}}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \text{Aut}_{\mathbb{Q}_p}(V_{\text{can}, p}) \to \text{Aut}_{\mathbb{Q}_p}(W_p).
\]

Similarly, for each \( p \)-th power root of unity \( \zeta \in \mathbb{Q}_p \), let \( x_\zeta = (A_{\text{can}}/\mathbb{Q}, \lambda_\zeta, \eta_{\text{can}, \zeta}) \in \mathcal{S}(\zeta) \) be any nontrivial torsion point on the rank 1 formal torus \( \mathcal{S} \) where \( R \) is a finite flat \( W(k) \)-algebra. From [3] Lemma 2.8 and Definition 2.9, the abelian scheme \( A_{\zeta}/R \) is a quasi-canonical lifting of \( A_{v'/k} \) and has complex multiplication. In particular \( A_{\zeta} \) is defined over some number field \( F'_1 \).

As before, let \( V_{\zeta} = H_1(A_{\zeta}/\mathbb{C}, \mathbb{Q}) \) be the first rational homology group of \( A_{\zeta} \) and let \( \text{MT}(A_{\zeta})/\mathbb{Q} \) \( \hookrightarrow \text{Aut}_{\mathbb{Q}}(V_{\zeta}) \) be its Mumford-Tate group. As \( A_{\zeta}/R \) is a lifting of the ordinary abelian variety \( A_{v'/k} \), we can choose a symplectic basis \( \{v_1^{\zeta}, \ldots, v_4^{\zeta}\} \) with respect to the Weil pairing induced from the polarization \( \lambda_\zeta \) such that \( \{v_1^{\zeta}, \ldots, v_4^{\zeta}\} \) corresponds to the basis \( \{u_1, \ldots, u_4\} \) of \( T_p A_{v}(k) \) under the reduction map. This basis gives a full level structure at \( p \) of \( A_{\zeta} \) which induces an isomorphism \( W_p \to T_p A_{\zeta}(\mathbb{Q}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = V_{\zeta, p} \).

Similarly, we define an embedding of algebraic groups over \( \mathbb{Q}_p \):

\[
i_{\zeta} : \text{MT}(A_{\zeta}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \text{Aut}_{\mathbb{Q}}(V_{\zeta}) \times_{\mathbb{Q}} \mathbb{Q}_p \to \text{Aut}_{\mathbb{Q}_p}(V_{\zeta, p}) \to \text{Aut}_{\mathbb{Q}_p}(W_p).
\]

From the above construction we have:

**Lemma 7.1.** The embeddings \( i_{\text{can}} \) and \( i_{\zeta} \)’s factor through \( G_{p}/\mathbb{Q}_p \).

**Proof.** As the canonical lifting can also be regarded as a quasi-canonical lifting, we only prove that \( i_{\zeta} \) factors through \( G_{p}/\mathbb{Q}_p \). Recall that the quasi-canonical lifting \( A_{\zeta}/R \) has complex multiplication and hence is defined over some number field \( F' \). Fix an algebraic closure \( F' \) of \( F' \). Under the symplectic basis \( \{v_1^{\zeta}, \ldots, v_4^{\zeta}, v_1^{\zeta}, \ldots, v_4^{\zeta}\} \) of \( T_p A_{\zeta}(F'') \), we can consider the Galois representation \( \rho_\zeta : \text{Gal}(F''/F') \to \text{GSp}_{4}(\mathbb{Z}_p) \). Let \( G_{\zeta}/\mathbb{Q}_p \) be the Zariski closure of the image of \( \rho_\zeta \) inside \( \text{GSp}_{4}(\mathbb{Q}_p) \). As the Mumford-Tate conjecture is known to be true for abelian varieties with complex multiplication, from the construction of the embedding \( i_{\zeta} \), the image of \( i_{\zeta} \) is nothing but \( G_{\zeta}(\mathbb{Q}_p) \).

As we have an embedding from \( F'' \) to the quotient field of \( R \), it induces a \( p \)-adic place \( v' \) of \( F' \). Let \( D_{v'} \subseteq \text{Gal}(F''/F') \) (resp. \( I_{v'} \subseteq \text{Gal}(F''/F') \)) be the decomposition group (resp. inertia group) at \( v' \). The
local Galois representation \( \rho_{\zeta, \nu'} = \rho_{\zeta, D_v'} \) is of the shape:

\[
\rho_{\zeta, \nu'} : D_v' \to \text{GSpg}(\mathbb{Z}_p), \\
D_v' \ni \sigma \mapsto \begin{pmatrix} T_1(\sigma) & B_2(\sigma) \\ 0 & T_2(\sigma) \end{pmatrix},
\]

where \( T_1, T_2 \) have the same meaning as the previous section, and \( B_1(\sigma) \in M_{4 \times 4}(\mathbb{Z}_p) \) is a matrix depending on \( \sigma \). Since the quasi-canonical lifting \( A_\zeta \) comes from the rank 1 formal subtorus \( \mathfrak{g}/W(k) \), we see that if we consider the conjugation of \( \rho_{\zeta, \nu'} \):

\[
\rho_{\zeta, \nu'} = \begin{pmatrix} W & 0 \\ 0 & (W^*)^{-1} \end{pmatrix} \rho_{\zeta, \nu'} \begin{pmatrix} W^{-1} & 0 \\ 0 & W^* \end{pmatrix} : D_v \to \text{GSpg}(O_M),
\]

\[
D_v' \ni \sigma \mapsto \begin{pmatrix} T_1'(\sigma) & B'_2(\sigma) \\ 0 & T_2'(\sigma) \end{pmatrix},
\]

then we have \( B'_{\zeta, ij} = 0 \) if \( i + j \neq 5 \) and the set \{\( B'_{\zeta, 14}(\sigma), B'_{\zeta, 23}(\sigma), B'_{\zeta, 32}(\sigma), B'_{\zeta, 41}(\sigma) | \sigma \in I_v \}\} lies in the same 1-dimensional vector space in \( M^4 \) as in the previous section. Here \( B'_{\zeta} = (b'_{\zeta, ij})_{1 \leq i, j \leq 4} \) are the entries of the matrix \( B'_{\zeta} \). In particular, we see that the local Galois representation \( \rho_{\zeta, \nu'} \) factors through \( G_p(\mathbb{Q}_p) \).

On the other hand, from the analysis in Section 5, the special fiber \( A_{v/k} \) is either a product of an elliptic curve and a simple abelian threefold, or a simple abelian fourfold. From the isogeny type of \( A_{v/k} \) in Section 5, the Mumford-Tate group of \( A_\zeta \) is contained in the torus \( \mathbb{G}_m/\mathbb{Q} \times T'_L/\mathbb{Q} \) which is a rank 4 torus (here recall that \( T'_L = (\sigma) \) is of the shape:

\[
\text{on the Shimura variety } S = \text{Sh}(\mathbb{Q}) \), we have \( \text{the inclusions } \rho_{\zeta, \nu'}(\text{Frob}_{\nu'}) \) for any Frobenius element in \( D_v' \).

Combining the above facts together, we see that the embedding \( i_\zeta \) factors through \( G_p(\mathbb{Q}_p) \).

We fix a compatible sequence \( (\zeta_n)_{n \geq 1} \) of \( p \)-th power roots of unity in the sense that \( \zeta_n \) is a primitive \( p^n \)-th root of unity and \( \zeta_n = \zeta_{n-1} \) for each \( n \).

As the above construction is valid for any integer \( N \) prime to \( p \), we have \( \mathbb{Q}_p \)-valued point \( x_{\zeta} = \lim_{N \to \infty} (x_{\zeta} \in \text{Sh}(\mathbb{Q})) \) (corresponding to the canonical lifting of \( x_v \)) and \( x_{\zeta} = \lim_{N \to \infty} (x_{\zeta} = \text{Sh}(\mathbb{Q})) \) (corresponding to the quasi-canonical liftings of \( x_v \)). As the abelian variety \( A_{v/k} \) is the canonical lifting of the ordinary abelian variety \( A_{v/k} \), it has complex multiplication by a CM-algebra \( M = \text{End}^0(A_{v/k}) = \text{End}(A_{v/k}) \). From the reciprocity law at special points (section 7.2.2), we have an embedding of groups: \( : T_M = \text{Res}_{\mathbb{Q}/\mathbb{Q}} G_m(\mathbb{Q}) \to \text{GSpg}(A(\infty)) \) which acts on the Shimura variety \( \mathbb{Q}_p \) which stabilizes the point \( x_{\zeta} \) and acts transitively on the set \( \{x_{\zeta} | n \geq 1\} \).

As \( T_M \) is a \( \mathbb{Q}_p \)-torus, the closure of \( \{x_{\zeta} | n \geq 1\} \) in \( \text{Sh}(\mathbb{C}) \) under the complex topology is contained in a set \( \Omega \) homeomorphic to \( (S^1)^n \) for some \( n \geq 1 \), where \( S^1 = \{ z \in \mathbb{C} | |z| = 1 \} \) is the unit circle in the complex plane. As the set \( \{x_{\zeta} | n \geq 1\} \) is countable, we can find a simply connected open subset \( \Omega' \subseteq \Omega \) containing \( \{x_{\zeta} | n \geq 1\} \).

Now let \( f : A \to \text{Sh}(\mathbb{Q}) \) be the universal abelian scheme, the restriction of the local system \( R^1 f_* \mathbb{Q} \) to \( \Omega' \) is constant. Hence we can identify all the cohomology groups \( H^1(A_{\zeta}, \mathbb{Q}) \)'s with the 8-dimensional \( \mathbb{Q} \)-vector space \( W \).

**Definition 7.2.** Define an algebraic group \( G_{\mathbb{Q}} \) to be the smallest algebraic subgroup of \( \text{Aut}_{\mathbb{Q}}(W) \) with the property that the embeddings \( i_{\zeta} \)'s factor through \( G_{\mathbb{Q}_p}(\mathbb{Q}_p) \) for all \( n \geq 1 \).

From the above definition, the algebraic group \( G_{\mathbb{Q}} \) is an algebraic subgroup of \( \text{GSpg}(W, \psi) \).

As the algebraic group is defined over \( \mathbb{Q} \), for any field automorphism \( \sigma : \mathbb{Q}_p \to \mathbb{Q}_p \) (which of course fix \( \mathbb{Q} \) pointwise), we have the inclusions \( \tau(i_{\zeta} \text{MT}(A_{\zeta})(\mathbb{Q}_p))) \subseteq G_{p}(\mathbb{Q}_p) \) and \( \tau(i_{\zeta} \text{MT}(A_{\zeta})(\mathbb{Q}_p))) \subseteq G_{p}(\mathbb{Q}_p) \).

From our construction, the algebraic group \( \text{MT}(A_{\zeta}) \times_{\mathbb{Q}} G_{\mathbb{Q}_p} \) gives a maximal torus of \( G_{\mathbb{Q}_p}(\mathbb{Q}_p) \) under the embedding \( i_{\zeta} \). On the other hand, the group generated by \( i_{\zeta} \text{MT}(A_{\zeta}) \times_{\mathbb{Q}} G_{\mathbb{Q}_p} \) and \( i_{\zeta} \text{MT}(A_{\zeta}) \times_{\mathbb{Q}} G_{\mathbb{Q}_p} \)
contains a unipotent such that the action of $i_{can}(\text{MT}(A_{can})) \times_{\mathbb{Q}} \mathbb{Q}_p$ on this unipotent by conjugation corresponds to the root $(2, 0, 0) \in X(T'_L)$ ($X(T'_L)$ is the character group of the torus $T'_L$). From the analysis in Section 5, the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts transitively on the set $\{(\pm 2, 0, 0), (0, \pm 2, 0), (0, 0, \pm 2)\} \subseteq X(T'_L)$. It follows that the groups $\tau(i_{can}(\text{MT}(A_{can}))(\mathbb{Q}_p)) \subseteq G_p(\mathbb{Q}_p)$ and $\tau(i_{\epsilon}(\text{MT}(A_{\epsilon}))(\mathbb{Q}_p)) \subseteq G_p(\mathbb{Q}_p)$ for all $\tau$ generate the group $G_p(\mathbb{Q}_p)$ and hence we have:

**Lemma 7.3.** We have the equality

$$G \times_{\mathbb{Q}} \mathbb{Q}_p = G_p$$

over $\mathbb{Q}_p$. In other words, $G/\mathbb{Q}$ is a $\mathbb{Q}$-form of the algebraic group $G_p/\mathbb{Q}_p$.

Now we can give a proof of Theorem 7.4.

**Proof.** Replacing the algebraic group $G/\mathbb{Q}$ by the semisimple group $G/\mathbb{Q}$ if necessary, we can assume that $G/\mathbb{Q}$ is semisimple. Since the Lie algebra of $G_p$ is isomorphic to $\mathfrak{c} \oplus \mathfrak{sl}_3$ (where $\mathfrak{c}$ is the one dimensional center) over an algebraic closure of $\mathbb{Q}_p$, we have an isomorphism:

$$G/\mathbb{Q} \cong G_{m/\mathbb{Q}} \times \text{SL}_2/\mathbb{Q} \times \text{SO}_4/\mathbb{Q}.$$  

On the other hand, the morphism $G/\mathbb{Q} \hookrightarrow \text{GSp}_{8/\mathbb{Q}}$ gives faithful symplectic representation of $G/\mathbb{Q}$, hence $i = 1$ or $i = 3$.

Now consider the homomorphism $h_{can} : S = \text{Res}_{C/\mathbb{R}} \rightarrow \text{GSp}_8(\mathbb{R})$ (resp. $h_{\epsilon} : S = \text{Res}_{C/\mathbb{R}} \rightarrow \text{GSp}_8(\mathbb{R})$) which defines the complex structure of the abelian variety $A_{can}$ (resp. $A_{\epsilon}$). These homomorphisms factor through $G(\mathbb{R})$ by our construction. Let $X$ be the $G(\mathbb{R})$-conjugacy class of $h_{can}$. Then the pair $(G/\mathbb{Q}, X)$ is a Shimura datum. From [14] Lemma, for the fixed prime $p$, we can find a integer $\alpha$ prime to $p$, and a Shimura variety $Sh_G$ coming from the Shimura datum $(G/\mathbb{Q}, X)$ by adding a sufficient deep level structure, such that there is a closed immersion $Sh_G \hookrightarrow A_{1,n}$. The abelian varieties $A_{can}$ and $A_{\epsilon}$'s certainly lie on the Shimura variety $Sh_G$ by our construction. Their special fiber $A_{\epsilon}/k$ gives a closed ordinary point $x_{\epsilon}$ of $Sh_G$. From [14] Theorem 3.7 or [8] Theorem 4.2, the formal completion of $Sh_G$ along the closed point $x_{\epsilon}$ is a union of formal tori. As the canonical lifting $A_{can}$ and the quasi-canonical liftings $A_{\epsilon}$'s are dense in the rank 1 formal torus $\mathfrak{F}_{W(k)}$. $\mathfrak{F}_{W(k)}$ is contained in the formal completion of $Sh_G$ along the point $x_{\epsilon}$. As the abelian variety $A_{1,F}$ sits on the formal torus $\mathfrak{F}_{W(k)}$, it is a point of the Shimura variety $Sh_G$. Since the absolute endomorphism algebra of $A_{1,F}$ is $\mathbb{Z}$, we see that $i = 1$ and thus

$$G/\mathbb{Q} \cong G_{m/\mathbb{Q}} \times \text{SL}_2/\mathbb{Q} \times \text{SO}_4/\mathbb{Q}.$$  

This shows that $A_{1,F}$ arises from a Shimura curve constructed by Mumford in [11], which is exactly what we want to prove. □

**Remark 7.4.** From the above proof, we see that after constructing the reductive group $G/\mathbb{Q}$, the quasi-canonical lifting $A_{\epsilon}$'s give points on the Shimura variety $Sh_G$ after choosing a suitable level structure. In particular, we have an embedding $i_{\epsilon} : \text{MT}(A_{\epsilon}) \hookrightarrow G/\mathbb{Q}$ for each $\epsilon$, such that $i_{\epsilon}(\text{MT}(A_{\epsilon}))$ is a maximal torus of $G/\mathbb{Q}$, and $G/\mathbb{Q}$ is the smallest algebraic subgroup of $\text{GSp}_{8/\mathbb{Q}}$ containing all these tori.

However, before we construct the group $G/\mathbb{Q}$, it might be difficult to find an appropriate embeddings $i_{\epsilon}$ : $\text{MT}(A_{\epsilon}) \hookrightarrow \text{GSp}_{8/\mathbb{Q}}$ which factors through $G/\mathbb{Q}$. In fact, if we add a suitable level structure $\eta_{N,\epsilon} : L/NL \rightarrow A_{\epsilon}[N](\mathbb{C})$ on $A_{\epsilon}$, the triple $(A_{\epsilon}/\mathbb{C}, \lambda_{\epsilon}, \eta_{N,\epsilon})$ gives a point on the Siegel moduli space $A_{1,N}(\mathbb{C}) = \mathcal{H}_4/\Gamma(N)$, where $\Gamma(N) = \Gamma(N) \cap \text{GSp}_8(\mathbb{Q})$. In this setting the embedding $\text{MT}(A_{\epsilon}) \hookrightarrow \text{GSp}_{8/\mathbb{Q}}$ is determined up to conjugation in $\Gamma(N)$ as the isomorphism $H_1(A_{\epsilon}/\mathbb{C}, \mathbb{Z}) \rightarrow L$ is so. Of course not all of these conjugations factor through the group $G/\mathbb{Q}$, but it is difficult to tell which embeddings have this property as we have not constructed the group $G/\mathbb{Q}$ yet. So we consider a base change. After giving a full level structure $\eta_{p,\epsilon,\zeta}$ at $p$ on $A_{\epsilon}$, it allows us to give an embedding $i_{\epsilon} : \text{MT}(A_{\epsilon}) \times_{\mathbb{Q}} \mathbb{Q}_p \hookrightarrow G_p$. To satisfy the last condition, we cannot choose an arbitrary level structure. In fact, if $\eta_{p,\epsilon,\zeta} : L_p \rightarrow T_p A_{\epsilon}(\mathbb{Q}_p)$ is such a level structure constructed in the proof of [7.1], all the other level structures satisfying the last condition are $\eta \circ g$, where $g \in G_p(\mathbb{Q}_p) \cap \text{GSp}_8(\mathbb{Z}_p)$. As we see in the proof of [7.1] the determination of the Serre-Tate coordinates of $A_{1,F}$ and the rank 1 formal torus $\mathfrak{F}_{W(k)}$ is crucial to find a desired level structure $\eta_{p,\epsilon,\zeta}$. 


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