NOTES ON SCATTERING FOR NLS

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ABSTRACT. These notes were written loosely to accompany a couple of talks at the UCLA participating analysis seminar. In the first talk, we introduce scattering notions and present some scattering results in the context of nonlinear Schrödinger equation on $\mathbb{R}^d$ (non-periodic case). In the second talk, we will present a non-scattering result for the periodic 2D cubic NLS, which was recently proven in [CKSTT].

CONTENTS

1. INTRODUCTION AND NOTIONS OF SCATTERING

We begin by defining scattering and some notions related to it (cf. [RS]). Scattering occurs in certain dynamical systems which can exhibit two kinds of motion: "free dynamics" and "interaction and/or nonlinear dynamics". Typically, free dynamics occurs in the absence of interaction forces (self or external). Consequently, it is natural to expect that if the interaction terms of the dynamical system become small enough after a certain time, then the motion will become almost linear.

Scattering involves the studying of certain states of interacting systems, especially those that appear to exhibit "free" behavior[1] in the distant past or distant future. This is often a result of the fact that in the distant past or future the interaction terms attenuate to zero, which makes their effect negligible and the motion is hence asymptotically free.

[1]Scattering is also studied in certain dynamical systems which do not have a natural unperturbed free dynamic to compare with the interaction dynamic. In those cases, one can compare the interaction dynamic to certain simple solutions and tries to prove related asymptotic results

Denote by $S(t)$ the nonlinear (or interaction) evolution of the system and S_{lin} that of free evolution. Both $S(t)$ and S_{lin} are defined on the "set of states" Σ of the system, which is the usual phase space in classical mechanics or some Hilbert space in quantum mechanics, or more generally the Cauchy data of some PDE. We say the a nonlinear solution $S(t)u_0$ scatters to a free solution $S_{lin}u_+$ as $t \to +\infty$ if

$$\lim_{t\to+\infty} S(t)u_0 - S_{lin}u_+ = 0 \tag{1.1}$$

where the convergence happens in the relevant space. This should be interpreted as the asymptotic convergence of the interaction solution to a free solution. Similarly, one is interested in systems that scatter to free solutions in infinite past (asymptotically as $t \to -\infty$).

Physically, one can experience scattering by directing particle into a force field (say a Coulomb-type field)(cf. Rutherford Scattering). Initially, the particles are very far away from the field center and hence they move in a uniform motion due to the fact that the effect of the field is negligible. As the particles get closer to the field, the effect of the force leads to some non-linear behavior or non-free interaction. The particle is then repelled[2] from the field which leads to the attenuation of the force the further away it moves from the center. As the force is almost zero, the motion will be almost linear or free.

The basic questions of scattering theory are the following:

1) Existence of Scattering States: Given u_+ in our space of states. $S_{lin}(t)u_+$ is the linear (free) evolution corresponding to u_+. Does their exist a scattering state u_0 such that $\lim_{t\to+\infty} S(t)u_0 - S_{lin}u_+ = 0$? The same question can be asked for $t \to -\infty$

2) Uniqueness of Scattering States: Given u_+, is the scattering state associated to it (if it exists) unique? Note that this question is different from the assertion that $S(t)u_0$ can scatter to at most one linear solution.

Definition 1.1. [3] If questions 1) and 2) are answered affirmatively for any u_+ (respectively u_-) in Σ, we define the *wave operator* Ω_+ (resp. Ω_-):$\Sigma \to \Sigma$ as $\Omega_+ u_+ = u_0$ where $\lim_{t\to+\infty} S(t)u_0 - S_{lin}u_+ = 0$ (resp. $\lim_{t\to-\infty} S(t)u_0 - S_{lin}u_- = 0$).

3) Weak asymptotic completeness[4]: Denote by Σ the space of states. Define

$$\Sigma_{in} = \left\{u_0 \in \Sigma : \exists u_- \text{ with } \lim_{t\to-\infty} S(t)u_0 - S_{lin}u_- = 0\right\} = Image(\Omega_-) \tag{1.2}$$

and

$$\Sigma_{out} = \left\{u_0 \in \Sigma : \exists u_+ with \lim_{t\to+\infty} S(t)u_0 - S_{lin}u_+ = 0\right\} = Image(\Omega_+) \tag{1.3}$$

[2]this can be taken as a heuristic reason why the study of scattering is limited to defocusing/repulsive case for the NLS. see section 2

[3]Please note that here we are parting from the definitions and conventions adopted in [RS]. Unfortunately, there does not seem to be an agreement among authors on a fixed nomenclature.

[4]this is more important in classical mechanics context than in quantum mechanics. We include this definition only for completeness purposes

The system is said to have weak asymptotic completeness if $\Sigma_{in} = \Sigma_{out}$. This basically means that initial states that scatter in infinite past also scatter in infinite future and vice versa.

4) **The S-transformation** (parting from [RS] conventions): The Scattering transformation is defined when an affirmative answer is given to questions 1), 2) and 3). This is defined as

$$S = (\Omega_+)^{-1}\Omega_- : \Sigma \to \Sigma$$

. We will not need the S-transformation in our presentation below so we will limit ourselves to this definition.

5) **Asymptotic completeness**: We will define asymptotic completeness (as we need it in the context of NLS)[5] as the condition that the operator Ω_+ (resp. Ω_-) is bijective.

2. Scattering Theory of NLS in $\mathbb{R}^d$

2.1. **Preliminaries.** We are interested in the nonlinear Schrödinger equation with power type nonlinearity:

$$i\partial_t u + \Delta u = \mu |u|^{p-1} u \tag{2.1}$$

$$u(0) = u_0 \in H^s(\mathbb{R}^d) \tag{2.2}$$

where $\mu = +1$ in the defocusing/repulsive case and $\mu = -1$ in the focusing/attractive case. The initial data is assumed to belong to $H^s(\mathbf{R}^d$. The equation satisfies the following scaling:

$$u(t,x) \longleftrightarrow u_\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) \tag{2.3}$$

This gives the scale-$\dot{H}^s$-invariant regularity $s_c = \frac{d}{2} - \frac{2}{p-1}$. (2.1) and its linear counterpart

$$i\partial_t u + \Delta u = 0 \tag{2.4}$$

are dispersive by nature. Solutions to (2.15) are given by applying the propagator $e^{it\Delta}$ to the initial data. More precisely we have: $S_{lin}(t)u_0 = e^{it\Delta}u_0$ for any initial data $u_0 \in H^s(\mathbb{R}^d)$. One quantitative formulation of the dispersive character of this equation is reflected in the fundamentally important *Strichartz estimates*: For a triple $(q, r, d) \neq (2, \infty, 2)$ that are Schrodinger admissible in the sense that $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ with $2 \leq q, r \leq \infty$ we have the following:

[5]different definitions are provided in [RS] that are needed to deal with so called bound state solutions that do not scatter

$$||e^{it\Delta}u||_{L_t^qL_x^r(I\times\mathbf{R}^d)} \lesssim ||u||_{L^2(\mathbf{R}^d)} \tag{2.5}$$

$$||\int e^{-is\Delta}F(s,x)ds||_{L^2(\mathbf{R}^d)} \lesssim ||F||_{L_t^{q'}L_x^{r'}(I\times\mathbf{R}^d)} \tag{2.6}$$

$$||\int_0^t e^{i(t-s)\Delta}F(s,x)ds||_{L_t^qL_x^r(I\times\mathbf{R}^d)} \lesssim ||F||_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}(I\times\mathbf{R}^d)} \tag{2.7}$$

In the third equation $(\tilde{q},\tilde{r},d)$ is also admissible and as usual q' is the conjugate exponent to q. The three equations are called the homogeneous Strichartz estimate, the dual homogeneous Strichartz estimate, and the retarded Strichartz respectively.

In the particular case of $q=r$ we get the following inequality

$$||e^{it\Delta}\psi||_{L^q_{t,x}} \lesssim C||\psi||_{L^2_x} \text{ where } q=2\frac{(d+2)}{d} \tag{2.8}$$

These estimates are fundamental to establish well-posedness results for the nonlinear equation via the Duhamel formula:

$$u(t)=e^{it\Delta}\phi-i\mu\int_0^t e^{i(t-s)\Delta}|u(s)|^{p-1}u(s)ds \tag{2.9}$$

and Picard Iteration: See [T] for details on well-posedness theory of (2.1).

Recall that solutions respect the following conservation laws:

- Conservation of mass
$$M[u](t):=\int_{\mathbb{R}^d}|u(t,x)|^2dx=M[u](0) \tag{2.10}$$
- Conservation of momentum
$$\overrightarrow{p}[u](t)=\Im\int_{\mathbb{R}^2}\nabla u(t,x)u(\bar{t,x})dx=\overrightarrow{p}[u](0) \tag{2.11}$$
- Conservation of the Hamiltonian
$$E[u](t):=\int_{\mathbb{R}^d}|\nabla u(t,x)|^2+\frac{\mu}{p}|u(t,x)|^pdx=E[u](0) \tag{2.12}$$

These are easily checked formally and can be justified for more general strong solution by usual limiting arguments (see [T]).

The following norms are useful to establish well-posedness results and will come in handy when dealing with scattering. We introduce the *Strichatz space* $S^0(I\times\mathbb{R}^d)$ where $I\subset\mathbb{R}$ is a time interval as the closure of Schwartz functions under the norm

$$||u||_{S^0(I\times\mathbb{R}^d)}:=\sup_{(q,r)\text{ admissible}}||u||_{L_t^qL_x^r(I\times\mathbb{R}^d)} \tag{2.13}$$

where admissibility was defined above for Strichartz estimates[6]. This norm gives a Banach space which has a dual we denote by $N^0(I \times \mathbb{R}^d)$. By construction we have that

$$||F||_{N^0(I\times\mathbb{R}^d)} \leq ||F||_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^d)}$$

whenever (p, r) is an admissible pair. The Strichartz esimates above be combined into

$$||u||_{S^0} \lesssim_d ||u(t_0)||_{L^2} + ||F||_{N^0} \tag{2.14}$$

whenever u satisfies $i\partial_t u + \Delta u = F$. Similarly we define those norms for higher regularities. For instance,

$$||u||_{S^1} = ||u||_{S^0} + ||\nabla u||_{S^0}$$

and

$$||u||_{N^1} = ||u||_{N^0} + ||\nabla u||_{N^0}$$

.

2.2. **Scattering Theory.** In the context of NLS[7] (2.1), the nonlinear evolution will be compared to free one exhibited by the linear equation:

$$i\partial_t u + \Delta u = 0 \tag{2.15}$$

Solutions to this equation are given by applying the propagator $e^{it\Delta}$ to the initial data. More precisely we have: $S_{lin}(t)u_0 = e^{it\Delta}u_0$ for any initial data $u_0 \in H^s(\mathbb{R}^d)$. In the spirit of section 1, we denote the solution $u(t, x)$ to (2.1) with initial data u_0 by $S(t)u_0$. The nonlinear solution scatters to a linear one if there exists u_+ such that $S(t)u_0 - S_{lin}(t)u_+ \to 0$ as $t \to +\infty$ is some relevant space (we will limit ourselves here to convergence in H^s).

As mentioned in section 1, scattering occurs when the nonlinear/ interaction factors vanish asymptotically. The same is true in the context of NLS. If we manage to prove in some way that the nonlinearity $|u|^{p-1}u \to 0$ in some relevant sense as $t \to \infty$ then we expect the evolution to become close to linear asymptotically. Notice that if u decays in some norm, then typically $|u|^{p-1}u$ decays even faster making the nonlinear terms in (2.1) much less significant in comparison to the linear term Δu responsible for the linear evolution. For this reason, it is usually easier to prove scattering when the power of the nonlinearity is not too small and not too big [8]. For instance, it can be shown that scattering cannot occur in the *long-range* case $p < 1 + \frac{4}{d}$. This can be seen heuristically by looking at solutions of the form:

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[6]We ignore the technicality in $d = 2$ resulting from the fact that the range of exponents is not compact

[7]It is mentioned in [KV] that scattering for NLS implies complete integrability (NLS is Hamiltonian). We will not pursue this interesting observation attributed by [KV] to Mozer in this talk

[8]Recall that local well-posedness arguments require the exponent not to be very high as this will exacerbate the large values of the solution and thus make the perturbation argument harder to close

$$u(t,x) = \frac{1}{(it)^{d/2}}\alpha \exp(\frac{i|x|^2}{2t} + \frac{i\mu|\alpha|^{p-1}}{\eta t^\eta}) \tag{2.16}$$

where $\eta := \frac{d}{2}(p - [1+4/d]) + 1$. These exhibit non-linear behavior in their phases (second term in the argument of the exponential). This non-linear behavior becomes less significant as time increases if $\eta > 0$ which is the case $p > 1 + \frac{4}{d}$ however this is not true if $p \leq 1 + \frac{4}{d}$. This is an indication that very small exponents are usually not enough to force scattering.

We will discuss scattering theory for NLS in the Sobolev spaces H^s. We shall only consider scattering from $t = 0$ to $t = +\infty$ and vice versa, i.e. proving results for the operator Ω_+. Results for operator Ω_- are similar and can be recovered by the symmetries of the equation.

Before discussing if the solution scatters or not we must make sure that we are considering an equation that is globally well-posed. In this case we have the Duhamel formula for all t:

$$u(t) = e^{it\Delta}u_0 - i\mu \int_0^t e^{i(t-s)\Delta}|u(s)|^{p-1}u(s)ds \tag{2.17}$$

The fact that the nonlinear solution scatters to a linear one can be written as:

$$||u(t) - e^{it\Delta}u_+||_{H^s(\mathbb{R}^d)} \to 0$$

as $t \to \infty$. This can be recast by the unitarity of the Schrödinger propagator as:

$$||e^{-t\Delta}u(t) - u_+||_{H^s} \to 0.$$

But

$$e^{-t\Delta}u(t) = u_0 - i\mu \int_0^t e^{-is)\Delta}|u(s)|^{p-1}u(s)ds$$

and so u scatters to a linear solution if and only if the integral

$$\int_0^\infty e^{-is)\Delta}|u(s)|^{p-1}u(s)ds$$

is conditionally convergent in H^s and in this case the asymptotic state is given by

$$u_+ = u_0 - i\mu \int_0^\infty e^{-is)\Delta}|u(s)|^{p-1}u(s)ds \tag{2.18}$$

Using this equation to eliminate u_0 in (2.17) we get

$$u(t) = e^{it\Delta}u_+ + i\mu \int_t^\infty e^{i(t-s)\Delta}|u(s)|^{p-1}u(s)ds \tag{2.19}$$

which can be understood as the Duhamel formula dealing with initial conditions at $t = +\infty$!

For completeness,we repeat some definitions from section 1. If for every $u_+ \in H^s$ there exists a unique $u_0 \in H^s$ such that $S(t)u_0$ scatters to $S_{lin}(t)u_+$, then we can define the *wave operator* $\Omega_+ : H^s \to H^s$ by $\Omega_+ u_+ = u_0$. If Ω_+ is bijective then we have *asymptotic completeness.*

Usually, proving the existence and injectivity[9] of the wave operator is easy using the Duhamel formula (2.19) and similar arguments to those used in proving well-posedness. However proving the surjectivity of the operator Ω_+ is harder and usually requires decay estimates of the solution (and hence faster decay of the non-linearity). This necessitates that we deal with defocusing equations only (see footnote 2) since soliton solutions of focusing equations clearly do not scatter[10].

We illustrate the above by sketching a simple example. More complete details can be found in [T] and [C]. We consider the cubic ($p = 3$) defocusing ($\mu = +1$) 3D ($d = 3$) nonlinear Schrödinger equation in $H^1(\mathbb{R}^3)$. This equation is $\dot{H}^1$ subcritical ($s_c = 1/2$) and is globally well-posed in H^1 (subcritical local well-posedness as a result of Picard iteration + energy conservation).

Theorem 2.3. *The wave operator for the 3D cubic NLS in* $\mathbb{R}^3$ *exists and is continuous and injective.*

Idea behind the proof: The problem is split in two. First we flow from $t = \infty$ to some large time T and then we flow from T to 0 using global well-posedness. The reason why we do this here is that despite the fact that the equation is subcritical we cannot use Hölder in time to get a power of T as in the proof of local well-posedness for subcritical equations. Since we are solving the Duhamel problem at $+\infty$ our arguments should be as scale-invariant as possible. For this fix $u_+ \in H^1$ with $||u_+||_{H^1} < A$ then by Strichartz estimates we have

$$||e^{it\Delta}u_+||S^1(\mathbb{R} \times \mathbb{R}^3) \lesssim_A 1$$

Since our argument will be scale-invariant we have to make some norm of the iterates small. This is usually established by using monotone convergence to get a large T depending on the profile of u_+ such that $||e^{it\Delta}u_+||S^1([\mathbb{T}, \infty) \times \mathbb{R}^3) \leq \epsilon$ for some epsilon to be specified later. Note the use of the fact that the linear equation is dispersive. But this cannot be done as the S^1 norm contains norms of the form $L_t^\infty L_x^r$ for which we cannot apply monotone convergence. No problem. We will iterate keeping the following norm small:

$$||u||_{S_0} := ||u||_{L^5_{t,x}} + ||u||_{L_t^{10/3} W_x^{1,10/3}}$$

Note that the norm $||u||_{L^5_{t,x}}$ is not admissible but is controlled by an admissible one namely $||e^{it\Delta}u_+||_{L_t^5 W_x^{1,30/11}}$ courtesy of non-endpoint Sobolev embedding. This gives

$$||e^{it\Delta}u_+||_{S_0} \lesssim ||e^{it\Delta}u_+||_{L_t^5 W_x^{1,30/11}} + ||u||_{L_t^{10/3} W_x^{1,10/3}} \lesssim ||e^{it\Delta}u_+||S^1 \lesssim_A 1$$

[9]Note the the injectivity of the wave operator is easy to establish but is different from uniqueness of well-posed solutions to NLS which only says that $S(t)u_0$ can scatter to at most one linear solution

[10]these are solutions of the form $u(t, x) = e^{it\tau}Q(x)$ where $Q(x)$ solves the ground state equation $\Delta Q - 2\mu|Q|^{p-1}Q = 2\tau Q$ and $\tau > 0$. If the exponent is energy-subcritical i.e. $s_c(p) < 1$ then positive radial Schwartz solutions Q exist

Now perform the usual iteration in S^1 keeping the solution small in S_0 norm and the nonlinearity in $L_t^{10/7}W_x^{1,10/7}$ (these are the conjugate exponents of an admissible pair).

This closes the iteration and gives a strong solution in $S^1([T,\infty)\times\mathbb{R}^3)$. Gluing it to the strong solution on $[0,T]$ gives a strong solution on $[0,\infty)$. Continuity is established by concatinating the two continuity results from 0 to T and then from T to ∞. Unconditional uniqueness can be established by Strichartz estimates and a continuity argument.

Note that flowing on $[T,\infty)$ was easier than flowing on $[0,T]$ and did not require energy conservation or knowledge of the sign of the nonlinearity. This is very similar to small data global well-posedness results that are free by product of (critical) local well-posedness perturbation theory. This is because for large times the asymptotic (linear) state is so dispersed (thanks to our choice of T) and this keeps the solution small on the interval $[T,\infty)$.

Now we turn to proving the surjectivity of the wave operators and hence asymptotic completeness. As mentioned before, this heavily relies on proving decay estimates for the solution. This decay is responsible for attenuation of the non-linearity and its effects and thus the asymptotically linear behavior. Asymptotic completeness will follow from the following two propositions and the interaction Morowetz estimate (see [T] section 3.5).

Proposition 2.4. *(Strong space-time bound implies scattering) Consider the cubic 3D defocusing NLS and suppose that the solution satisfies the space-time bound*

$$||u||_{S^1(\mathbb{R}\times\mathbb{R}^d)} \lesssim 1 \tag{2.20}$$

That is we assume that the non-linear solution satisfies the same bounds as the linear one. Then $u(t)$ scatters to some linear solution as $t\to\infty$. If this bound is satisfied by all solutions, then Ω_+ is a homeomorphism of $H^1(\mathbb{R}^3)$

Proof: Recall that the solution scatters to a linear one if and only if the integral

$$\int_0^\infty e^{-it\Delta}|u(t)|^2u(t)dt$$

is conditionally convergent in H^1. This is equivalent to it being Cauchy:

$$||\int_I e^{-it\Delta}|u(t)|^2u(t)dt||_{H_x^1(\mathbb{R}^3)} \lesssim |||u|^2u||_{N^1(I\times\mathbb{R}^3)} \leq \sum_{k=0}^1 ||\nabla^k(|u|^2u)||_{L_{t,x}^{10/7}(I\times\mathbb{R}^3)}$$

$$\lesssim \sum_{k=0}^1 |||\nabla^k u||u|^2||_{L_{t,x}^{10/7}(I\times\mathbb{R}^3)} \lesssim ||\nabla u||_{L_t^{10/3}L_x^{10/3}(I\times\mathbb{R}^3)}||u^2||_{L_{t,x}^{5/2}(I\times\mathbb{R}^3)}$$

$$\lesssim ||u||_{L_t^{10/3}W_x^{1,10/3}(I\times\mathbb{R}^3)}||u||^2_{L_{t,x}^5(I\times\mathbb{R}^3)} \lesssim ||u||_{L_t^{10/3}W_x^{1,10/3}(I\times\mathbb{R}^3)}||u||^2_{L_t^5W_x^{1,30/11}(I\times\mathbb{R}^3)}$$

Since these latter two norms are admissible, they are controlled by the S^1 and hence they are bounded themselves on $\mathbb{R}\times\mathbb{R}^3$ and this is enough to prove Cauchyness.

As for continuity we split the time line in two pieces. We need to show that the map $u_0 \mapsto u_+$ is continuous. The first piece $[0, T]$ and the other $[T, \infty]$ where T is to be specified later. Continuous dependence of $u(T)$ on $u(0)$ is a direct result of global well-posedness. To establish the continuity of $u(T) \mapsto u_+$ we apply a scale-invariant perturbation argument like that in proving the existence of the wave operator. T is chosen similarly so that $||e^{it\Delta}u_0||_{S_0}$ is small enough. This gives that the solution maps $u(T) \in H^1(\mathbb{R}^3) \mapsto u(t) \in S^1([T, \infty) \times \mathbb{R}^3)$ is Lipschitz which gives the continuous dependence of u_+ on $u(T)$. One may also try using the formula:

$$u_+ = u_0 - i \int_0^\infty e^{-it\Delta}|u(t)|^2 u(t) dt$$

□

Proposition 2.5. *(Weak space-time bound implies strong space-time bound) Suppose that*

$$||u||_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^3)} \lesssim_{||u_0||_{H^1}} 1 \tag{2.21}$$

for all H^1 well-posed solutions and for some fixed $10/3 \leq q \leq 10$. Then (2.20) holds.

This says that we can essentially bootstrap this weaker estimate into the stronger one thanks to Strichartz estimates and conservation of energy. The room in the exponent range is due to sub-criticality of the equation. In the critical case, we can only hope to control the relevant Strichartz norm which is scale invariant by a scale-invariant norm and the exponent range will not be open.

Proof. Let u be an H^1 well-posed solution, again we split the time axis into a finite number of intervals such that on each interval I we have the estimate $||u||_{L^q_{t,x}(I\times\mathbb{R}^3)} \leq \epsilon$ where ϵ is to be specified later. Now by Sobolev embedding we have that for all $10/3 \leq r \leq 10$ we have $||u||_{L^r_{t,x}(I\times\mathbb{R}^3)} \lesssim ||u||_{S^1}$ and hence by interpolating these two estimates we get that $||u||_{L^5_{t,x}(I\times\mathbb{R}^3)} \lesssim \epsilon^\alpha ||u||_{S^1(I\times\mathbb{R}^3)}$ for some $\alpha > 0$. By the same estimates as in the proof of the previous proposition (and Strichartz's estimates and boundedness of H^1 norm thanks to energy conservation) we get that

$$||u||_{S^1(I\times\mathbb{R}^3)} \lesssim ||u(0)||_{H^1} + |||u|^2u||_{N^1(I\times\mathbb{R}^3)} \lesssim ||u(0)||_{H^1} + \epsilon^{2\alpha}||u||^{3-2\alpha}_{S^1(I\times\mathbb{R}^3)}$$

Choosing ϵ small enough depending only on $||u(0)||_{H^1}$ gives the result by a continuity argument. □

Finally, note that this weak space-time bound can be established in the radial case by combining Morawetz estimate

$$\int_{\mathbb{R}}\int_{\mathbb{R}^3} \frac{|u(t,x)|^4}{|x|} dx dt \lesssim_{||u_0||_{H^1}} 1$$

with radial Sobolev embedding which says that $|||x|^s u||_{L^\infty_{t,x}} \lesssim ||u||_{H^1}$ for all $\frac{d-2}{2} \leq s \leq \frac{d-1}{2}$ to get the decay estimate. In the non-radial case, one can use interaction Morawetz estimate discovered by Colliander, Keel, Staffilani, Takoaka, and Tao which directly gives the decay estimate. See [T] for details.

3. Cubic NLS on $\mathbb{T}^2$ Cannot Scatter [CKSTT]

In this section we consider the cubic defocusing NLS on the torus $\mathbb{T}^2$ given by:

$$-i\partial_t u + \Delta u = |u|^2 u \tag{3.1}$$

$$u(0) = \phi \tag{3.2}$$

where $\phi \in H^1(\mathbb{T}^2)$. This equation is globally well-posed in H^1 thanks to the conservation of the Hamiltonian (subcritical local well-posedness was established for $s > 0$ in [B]).

We consider the scattering question in $H^1(\mathbb{T})$. This equation has a family of single frequency solutions:

$$u(t,x) = Ae^{i\kappa}e^{i(Nx+|N|^2t+A^2t)} \tag{3.3}$$

where $A > 0$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. These solutions are important in establishing some ill-posedness and instability results for this equation in negative Sobolev spaces. Note that these solutions are not exactly linear due to the phase factor e^{iA^2t} caused by the nonlinearity. Clearly do not scatter to any linear solution whose form is:

$$u_{lin}(t) = e^{-it\Delta}\phi = \sum_{n\in\mathbb{Z}^2} \hat{\phi}(n)e^{i(nx+|n|^2t)} \tag{3.4}$$

The next question is: "Does there exist any non-zero scattering solution?". The answer of this question is negative. In fact, it is shown that the only solutions that are asymptotically linear modula a phase factor are those given in (3.3).

Theorem 3.1. *(Theorem 5.1 of* [CKSTT]*) Suppose that $u(t,x)$ is a strong H^1 solution to (3.1) that scatters modulo phase in H^1, i.e.*

$$||u(t) - e^{i\theta(t)}e^{-it\Delta}u_+||_{H^1(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty \tag{3.5}$$

Then $u(t,x)$ is a single frequency solution (3.3).

The proof is based on the following compactness property of the linear solution group:

Lemma 3.2. *Let $u_+ \in H^1(\mathbb{T}^2)$. The set $\{e^{i\theta}e^{-it\Delta}u_+ : \theta \in \mathbb{R}/(2\pi\mathbb{Z}), t \in \mathbb{R}\}$ is compact in $H^1(\mathbb{T}^2)$.*

Proof. This set is the image of continuous map from a compact domain, namely $(\theta, t) \in \mathbb{R}/(2\pi\mathbb{Z})\times\mathbb{R}/(2\pi\mathbb{Z}) \mapsto e^{i\theta}e^{-it\Delta}u_+ \in H^1(\mathbb{T}^2)$. The main point here is that $e^{-i(t+2\pi)\Delta}u_+ = e^{-it\Delta}u_+$. □

We will also need the following two lemmas in the proof:

Lemma 3.3. *If $u \in H^1(\mathbb{T}^2)$ then $|u| \in H^1(\mathbb{T}^2)$ and $|||u|||_{H^1(\mathbb{T}^2)} \le ||u||_{H^1(\mathbb{T}^2)}$.*

Proof. It is clearly enough to prove that for $u = R + iI$ the distributional derivative of $|u|$ is given by

$$\nabla|u| = \begin{cases} \frac{R\nabla R + I\nabla I}{|u|} & \text{if } |u| \neq 0 \\ 0 & \text{if } |u| = 0 \end{cases}$$

A limiting argument shows that it is enough to show this for smooth u. For this we consider the function $G_\epsilon(R, I) = \sqrt{R^2 + I^2 + \epsilon^2} - \epsilon$, then we have that G_ϵ is smooth and converges to $|u|$ in the sense of distributions. For any test function ψ

$$\int |u|\nabla\psi = \lim_{\epsilon\to 0}\int_{|u|>0} G_\epsilon(R, I)\nabla\psi = -\lim_{\epsilon\to 0}\int_{|u|>0} \nabla G_\epsilon(R, I)\psi = -\lim_{\epsilon\to 0}\int_{|u|>0} \frac{R\nabla R + I\nabla I}{\sqrt{|u|^2 + \epsilon^2}}\psi$$
$$= \int_{|u|>0} \frac{R\nabla R + I\nabla I}{|u|}\psi$$

where the last integral follows by dominated convergence. □

We will also need the fact that $H^1(\mathbb{T}^2)$ contains no step functions:

Lemma 3.4. *Suppose that $u \in H^1(\mathbb{T}^2)$ is a function that takes at most two values. Then u is constant.*

Proof. Without any loss of generality u takes only the values 0 and 1. Then $u^2 = u$. Deriving this identity we get that $2u\nabla u = \nabla u$ which implies that $(1 - 2u)\nabla u = 0$. But $1 - 2u \neq 0$ as u takes only the values 0 and 1. So $\nabla u = 0$ which means that u is a constant. The only thing that needs justifying is the gradient of u^2. This can be done by a limiting argument thanks to the fact that u is bounded. □

Proof of Theorem 3.1: Let $u(t, x)$, $\theta(t)$, and u_+ be as in the statement of the theorem. (3.5) and lemma 3.2 implies that the set $\{u(t, x)\}$ is totally bounded in H^1 and hence there exists a convergent subsequence $\{u(t_n, x)\}$ that converges to v_0. Also by lemma 3.2 we can assume that $e^{-it_n\Delta}u_+$ also converges in H^1 to v_+.

Now let $v^{(n)}(t) = u(t + t_n)$. Then $v^{(n)}(0) = u(t_n)$ which converge in H^1 to v_0. So if $v(t)$ is the solution of (3.1) with initial data v_0, global well-posedness implies that $v^{(n)}(t) \to v(t)$ uniformly on compact time intervals. But

$$||u(t + t_n) - e^{i\theta(t+t_n)}e^{-i(t+t_n)\Delta}u_+||_{H^1(\mathbb{T}^2)} \to 0 \text{ as } n \to \infty$$

and hence

$$||v(t) - e^{i\theta(t+t_n)}e^{-it\Delta}v_+||_{H^1(\mathbb{T}^2)} \to 0 \text{ as } n \to \infty$$

and this gives that:

$$v(t) = e^{i\alpha(t)}e^{-it\Delta}v_+ \tag{3.6}$$

Multiplying by $e^{it\Delta}\bar{v_+}$ and integrating we get that:

$$e^{i\alpha(t)} = \frac{1}{||v_+||_{L^2}^2}\int v(t,x)e^{it\Delta}\bar{v_+}dx$$

$v(t)$ is continuously differentiable in H^{-1} and $e^{it\Delta}v_+$ is continuous in H^1 and vice-versa. Hence $e^{i\alpha(t)}$ is C^1 in time and we get that α is continuously differentiable in time. (This can be made rigorous in the following way. Suppose the w is a C^1 function satisfying $w\bar{w} = 1$. Deriving this we get that $w'\bar{w}$ is pure imaginary. So if F is such that $iF'(t) = w'\bar{w}$, then F is real $w = ce^{iF}$).

Now applying $(-i\partial_t + \Delta)$ to both sides of (3.6) we get that:

$$|v(t,x)|^2 v(t,x) = \alpha'(t)e^{i\alpha(t)}e^{-it\Delta}v_+ = \alpha'(t)v(t,x)$$

Which gives that $v(t,x)$ can only take the values 0 and $\alpha'(t)$ (actually integrating both sides after canceling $v(t,x)$ when it's not 0, shows that $\alpha' = M^2$ where M is the conserved mass). Hence we get that $|v(t,x)| = cst$. But $|v(t,x)| = |e^{-it\Delta}v_+|$. Since $|e^{-it\Delta}v_+| \in H^1(\mathbb{T}^2)$ we get after a similar justification as above[11] that $e^{-it\Delta}v_+ = Ae^{iF(t,x)}$ for some $A > 0$ and $F \in H^1_{loc}(\mathbb{R}^2)$ real, with periodic gradient.

Applying $-i\partial_t + \Delta$ to both sides we get:

$$0 = Ae^{iF}(\partial_t F + i\Delta F + |\nabla F|^2)$$

in the sense of distributions. Taking imaginary parts we get that F is harmonic on $\mathbb{R}^2$ and ∇F is periodic. Therefore F is linear for each fixed t. In particular we get that

$$v_+ = Ae^{iF(0,x)} = Ae^{n.x+\beta}$$

which in turn gives that $u_+ = Ae^{i\gamma}e^{inx}$. Applying the phase rotation and Galilean symmetries of (3.1) to (3.5) we may assume that $\gamma = n = 0$ and hence we get that

$$||u(t) - e^{i\theta(t)}e^{it\Delta}A||_{H^1(\mathbb{T}^2)} \to 0 \text{ as } t \to \infty$$

which gives that

$$\int |u(t,x)|^2 dx = A^2$$

and hence

$$\int |u(t,x)|^4 dx \geq (\int |u(t,x)|^2)^2 = A^4$$

[11]In fact, differentiating the identity $f\overline{f} = 1$ we see that $\partial_j f\overline{f}$ is imaginary for $j = 1, 2$. This in turn implies that $\Im(\nabla f\overline{f})$ is curl-free, and also lies in $L^2(\mathbb{T}^2)$, so by Hodge theory we can find a function $F : \mathbb{R}^2 \to \mathbb{R}$ which is locally in H^1 such that $\nabla f\overline{f} = i\nabla F$. This implies that $\nabla(fe^{-iF}) = 0$, and thus f is a constant multiple of e^{iF}

by Hölder. But this means that the potential energy consumes all the conserved Hamiltonian

$$H[u](t) = \int_{\mathbb{T}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}|u(t,x)|^4 dx = H[u_+] = A^4/4$$

Hence we get

$$\int_{\mathbb{T}^2} \frac{1}{2}|\nabla u|^2 dx = 0$$

for all t. Thus $u(0)$ is constant, say $u(0) = \eta$ and consequently $u(t) = \eta e^{i|\eta|^2 t}$ by uniqueness. $\square$

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