Notes on KAM Theory

Zaher Hani

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Abstract

These notes were written loosely to accompany a couple of talks at the UCLA participating analysis seminar. We present a rough introduction to KAM theory avoiding technical details.

1 Motivation

KAM theory (after Kolmogorov, Arnold, and Moser) is concerned with long-time behavior of nearly integrable Hamiltonian systems, namely the persistence of quasi-periodic orbits under small perturbations of a completely integrable system.

The problem originated from the study of celestial mechanics and/or the N-body problem. When two bodies interact under the Newtonian gravitational attraction the motion is well known and explicitly solvable. They oscillate on ellipses with one of the foci being at their common center of mass. However, if we add one more body the problem seizes to be explicitly solvable even if the mass of two of the bodies is very small in comparison to the third as is the case in the solar system.

People have tried to solve this problem by first ignoring the mutual interaction of the two light objects/planets, and then treating this interaction as a perturbation. In fact, by ignoring the mutual interaction we get a completely integrable Hamiltonian system, that is explicitly solvable. After adding the perturbation terms, physicists and astronomers were able to find power series "solutions" to the problem. Unfortunately, proving the convergence of those series was not an easy task and the problem remained open -then- despite a substantial prize from the King of Sweden.

The difficulty resided in the fact that the terms of the series involved coefficients with *small denominators* that could make the series diverge. Dealing with these small denominators is one of the key ideas in KAM theory. To gain some insight to the importance of those small perturbation terms we remind the reader of the forced simple harmonic oscillator. If we excite the oscillator with a frequency that is a multiple of the natural frequency of the oscillator, the motion can become unbounded. This is called *resonance*. In a similar way, the periodic motion of the third planet in the 3-body problem excites the other planets with a frequency equal to its orbital frequency. As a result of this, we expect that if any two orbital frequency are commensurate, then the motion of the system can become unbounded. Even if the two frequencies are not exactly commensurate but almost so, this can lead to convergence problems in the perturbation step.

As an upshot, the mathematical question involved can be formulated as follows. We are given a completely integrable Hamiltonian systems whose orbits (as we will se in the next section) are either periodic or quasi-periodic. After subjecting this system to a small perturbation, the question is: which orbits if any continue to be quasi-periodic? A satisfactory answer to this question was suggested by Kolmogorov in an address to 1954 International Congress of Mathematicians in Amsterdam and is based on the following two ideas:

- Solve a truncated linearized problem that reduces to system into one that is closer to being completely integrable. More precisely, one whose perturbation term is small than the original system. It is in this step that one has to deal with small denominators
- Use a rapidly converging iteration scheme to get rid of the perturbation term. This iteration scheme is nothing but an adaptation of Newton's method. The need for such a rapidly decaying iteration is mandated by the loss of smoothness incurred while dealing with small denominators in the first step.

This scheme was worked out and extended to other contexts by Arnold and Moser in the following ten years or so leading to what is now called the KAM theory.

In what follows we present the mathematical problem involved in the context of nearly integrable Hamiltonian systems along with the statement of the classical KAM theorem.

2 The Classical KAM Theorem

2.1 Completely Integrable Hamiltonian Systems

Recall Hamilton's equations [A]

$$\dot{p} = -\frac{\partial H}{\partial q} \tag{1}$$

$$\dot{q} = \frac{\partial H}{\partial p} \tag{2}$$

where $H(p,q)$ is a smooth function on some open subset of phase space $\mathbf{R}^{\mathbf{n}} \times \mathbf{R}^n$. Recall, the our phase space has a canonical symplectic structure given by

$$\omega = \sum dp_i \wedge dq_i \tag{3}$$

Under this symplectic structure, Hamilton's equation can be regarded as the flow equations of the Hamiltonian vector field X_H given by the symplectic gradient of H. More precisely, $dH = \omega(., X_H)$. The importance of Hamilton's equations is that they are invariant under canonical coordinate transformations. A canonical transformation is a diffeomorphism Φ of the phase space (or an open subset thereof) which preserves the symplectic structure. In other words, $\omega = \Phi^*(\omega)$. This invariance is manifested in the following way. If $\Phi : U \to V$ is a canonical coordinate transformation with U and V being open subsets of $\mathbf{R}^n \times \mathbf{R}^n$ and $\Phi(I, \phi) = (p, q)$, then in the (I, ϕ) coordinates Hamilton's equations have the same form as in (1) with H replace by $\tilde{H}(I, \phi) = H \circ \Phi$. Namely

$$\dot{I} = -\frac{\partial \tilde{H}}{\partial \phi} \tag{4}$$

$$\dot{\phi} = \frac{\partial \tilde{H}}{\partial I} \tag{5}$$

Now if $H(I, \phi) = h(I)$ is independent of ϕ, then these equations can be explicitly solved into $I(t) = I(0) =: I_0$ and $\phi(t) = \frac{dh}{dI}(I_0)t + \phi(0)$. Hence the equations can be completely integrated at least locally and we can get a solution in the (p, q) coordinates by applying the canonical transformation ϕ. Notice that the quantities $I(t) = (I_1(t), \ldots, I_n(t))$ are conserved and are hence *first integrals of the motion.* This can always be done locally in a neighborhood U of any point of the phase space, thanks to Darboux's theorem (or rather its proof)[A].

If O is an open subset of the phase space that is invariant under the Hamiltonian flow, then the Hamiltonian flow is said to be *completely integrable* on O if there are n functions $F_1, \ldots, F_n$ with the following properities:

- $F_1 = H$
- $F_1, \ldots, F_n$ are in involution, i.e. $\{F_i, F_j\} = 0$ for all $1 \leq i, j \leq n$, where $\{.,.\}$ is the Poisson bracket.
- The level sets determined by the map $(p, q) \mapsto (F_1, \ldots, F_n)$ are compact subsets of O
- The functions $F_1, \ldots, F_n$ are independent on each level set (and hence each level set is a smooth, compact, n-manifold of the phase space)

The theorem of Liouville-Arnold-Jost [A] asserts that under the above conditions, there exists a canonical transformation $\Phi : D \times \mathbf{T}^n \to O$ such that $H \circ \Phi(I, \phi) = h(I)$ where D is some open subset of $\mathbf{R}^n$ and $\mathbf{T}^n$ is the standard n-torus $\mathbf{R}^n/(2\pi\mathbf{Z})^n$. These coordinates are called action-angle coordinates. This theorem provides the needed connection between completely integrable systems on $\mathbf{R}^n \times \mathbf{R}^n$ (or any other symplectic manifold) and completely integrable systems on $\mathbf{R}^n \times \mathbf{T}^n$. As a result of this, we will from now on take $D \times \mathbf{T}^n$ as our phase space.

With $H(I, \phi) = h(I)$, the equations of motion become as mentioned above $I(t) = I_0$ and $\phi(t) = \frac{dh}{dI}(I_0)t + \phi(0)$. Hence every solution curve is a straight line which due to the identification in the ϕ coordinates modulo 2π is winding around the torus $\mathcal{T} = \{I_0\} \times \mathbf{T}^n$. In what follows we collect the properties of the orbital behavior of those completely integrable systems:

- Each torus $\{I_0\} \times \mathbf{T}^n$ is invariant under the Hamiltonian flow. The only type of motion it supports is linear flow with frequency $\omega(I_0) = \frac{dh}{dI}(I_0) = (\omega_1(p_0), \ldots, \omega_n(p_0))$. Such tori are called *Kronecker* tori.
- Each torus is clearly *Lagrangian*, i.e. it is a smooth manifold of dimension half of that of the phase space and whose tangent space at each point is null with respect to the symplectic form.
- The whole phase space $D \times \mathbf{T}^n$ is foliated by the above n-parameter family of invariant Kronecker Lagrangian tori

Note that the topological nature of the orbit carried by any invariant torus depends solely on the diophantine nature of the frequency vector $\omega = (\omega_1, \ldots, \omega_n)$ that the torus supports. Here we distinguish to cases:

- the frequencies ω are *nonresonant* or *rationally independent*, i.e.

$$\langle \omega, k \rangle \neq 0 \text{ for all } k \in \mathbf{Z}^n \setminus 0 \tag{6}$$

  In this case, each orbit is dense on the torus and the flow is ergodic.

- the frequencies ω are *resonant* or *rationally dependent*, i.e.

$$\langle \omega, k \rangle = 0 \text{ for some } k \in \mathbf{Z}^n \setminus 0 \tag{7}$$

  A prototype is $\omega = (\omega_1, \ldots, \omega_k, 0, \ldots, 0)$ with $(\omega_1, \ldots, \omega_k)$ nonresonant. In this case, each orbit is dense is a sub-torus $\mathbf{T}^k \subset \mathbf{T}^n$ but not in the entire torus. As a matter of fact, the torus $\mathbf{T}^n$ decomposes into an $n-k$ family of invariant sub-tori. In the special case when $\omega = (\omega_1, 0, \ldots, 0)$, we get an $n-1$ family of periodic orbits of period $2\pi/\omega$.

The importance of the frequency in determining the topological nature of the orbit drives us to define the frequency map

$$h_I : D \to \Omega \; ; \; I \mapsto \omega(I) = h_I(I) \tag{8}$$

where $\Omega \subset \mathbf{R}^n$. For the purposes of KAM theorem (at least the one we will be talking about here), we will assume that the frequency map is *non-degenerate*, i.e.

$$det(h_{II}) = det(\frac{\partial \omega}{\partial I}) \neq 0 \tag{9}$$

As a result of this assumption, Ω is open and the frequency map is a local diffeomorphism. We will later assume that the frequency map is actually a diffeomorphism onto Ω, after which it will be useful to talk about its inverse map (which is incidentally the Legendre transform of h).

2.2 Nearly Integrable Hamiltonian Systems and KAM

The KAM theory is concerned with Hamiltonian systems which are nearly integrable in the sense that

$$H(I, \phi) = h(I) + \epsilon f(I, \phi, \epsilon) \tag{10}$$

where $(I, \phi) \in D \times \mathbf{T}^n$. We remark that we will occasionally think of $H(I, .)$ and $f(I, ., \epsilon)$ as function on $\mathbf{R}^n$ which are 2π-periodic in each of the ϕ component. The case $\epsilon = 0$ was considered above and the question becomes: what happens to the above orbits after a small perturbation (corresponding to small ϵ).

The fist answer to this question is of negative nature and dates back to Poincaré. He observed that the resonant tori are in general *destroyed* after arbitrary small perturbations. The most unstable tori are the periodic orbits (corresponding to the prototype $\omega = (\omega_1, 0, \ldots, 0)$), for which out of each $n-1$-parameter family of periodic orbits on each invariant ω-torus, only finitely many survive the perturbation, while the others behave chaotically. So if the non-degeneracy condition above is assumed a dense set of tori is destroyed after the perturbation. The conclusion that can derived from this is that generic Hamiltonian systems are not integrable (for more on this see [MM]).

Given that a dense set of tori is destroyed, it was a common belief that arbitrarily small perturbations can turn an integrable system into one that is ergodic on each energy surface. This was called the "Boltzmann Ergodicity Hypothesis". There even appeared in the twenties an erroneous proof of this hypothesis by Fermi.

Nonetheless, in 1954 Kolmogorov observed that the converse is in fact true. He proved the persistence of tori that are not only non-resonant but rather *strongly non-resonant* in the following sense: they are of type (α, τ), i.e.

$$\langle \omega, k \rangle \geq \frac{\alpha}{|k|^\tau} \tag{11}$$

where $|k| = |k|_1 = |k_1| + \ldots + |k_n|$.

In what follows we denote Δ^τ_α points of $\mathbf{R}^n$ of type (α, τ) and $\Delta^\tau = \cup_{\alpha>0}\Delta^\tau_\alpha$. In this context we have the following:

Proposition 1.

- *$\Delta^\tau = \emptyset$ for $\tau < n-1$*
- *Δ^{n-1} has Lebesgue measure 0, but Hausdorff dimension n.*
- *Δ^τ has full measure for $\tau > n-1$*

The first statement is easy to prove using the pigeon-whole-principle. The proof of the last statement is left as an exercise to the reader ([P]). The proof of the second is more elaborated and is of no use for us in what follows anyway.

From now on, we fix $\tau > n-1$ and let Ω_α be the set of points in $\Omega \cap \Delta^\tau_\alpha$ that are at a distance greater or equal to α from $\partial\Omega$. This is a Cantor set (closed, perfect, and nowhere dense). Note that if we assume that $\partial\Omega$ is piecewise smooth or has dimension $n-1$ then we have $m(\Omega \setminus \Omega_\alpha) = O(\alpha)$, where m is Lebesgue measure (Ω is assumed to be bounded). We are now in a position to state the classical KAM theorem ([P],[W]). For simplicity, we assume that both h and f are real-analytic functions.

Theorem 1 (The Classical KAM Theorem)**.** *Suppose that $H(I,\phi) = h(I)+\epsilon f(I,\phi,\epsilon)$ is real analytic on $\bar{D}\times\mathbf{T}^n$ and the integrable Hamiltonian system $h(I)$ is non-degenerate in the sense that the frequency map $h_I : D \to \Omega$ is a diffeomorphism. Then there exist γ such that if*

$$\epsilon < \gamma\alpha^2 \tag{12}$$

all Kronecker tori $(\mathbf{T}^n, \omega)$ with $\omega \in \Omega_\alpha$ persist as invariant Lagrangian tori, being only slightly deformed. More precisely, there exist a mapping

$$\Psi : \mathbf{T}^n \times \Omega_\alpha \to D \times \mathbf{T}^n \tag{13}$$

that gives a Lipschitz continuous family of real analytic embeddings of invariant Lagrangian Kronecker tori $\{(\mathbf{T}^n,\omega)\}_{\omega\in\Omega_\alpha}$.

Moreover, the image of Ψ fills $D \times \mathbf{T}^n$ up to a set of measure $O(\alpha)$.

Note that this directly implies that the ergodicity hypothesis mentioned above is false because the invariant tori form an invariant set to the flow that is neither of measure 0 nor of full measure. Note that the set of invariant frequencies is a Cantor set and hence has no interior. In fact, it is physically impossible to tell if an initial position would yield an orbit lying on an invariant torus or not, so from the physical point of view, the statement above is of probabilistic nature saying that with probability $1 - O(\alpha)$ a randomly chosen orbit lies on an invariant torus.

3 Main Ideas of the Proof

In this section we try to explain the main ideas in the proof of the KAM theorem. For full proofs see [W], [P], [B].

Given the perturbed Hamiltonian system $H(I,\phi) = h(I) + \epsilon f(I,\phi,\epsilon)$. The initial plan of the proof is to

- First, find a coordinate transformation Φ such that $H\circ\Phi$ is closer to being completely integrable than H itself. More precisely we would like to write $H_1 = H \circ \Phi = h_1(I) + P_1(I,\phi,\epsilon)$ with $P_1 = O(\epsilon^a)$ for some $a > 1$.
- Use this Hamiltonian H_1 as a new starting point to iterate the above process with the intention of getting rid (in some sense) of the perturbation term in the limit (Newton method argument).

This plan is not without difficulties. The first step above boils down to solving a linearized truncated equation, a fact that is easily done using Fourier coefficients. However, the resulting Fourier coefficients have small divisors (denominators) as we will see below. This forces us to restrict consideration to strongly non-resonant frequencies in a spirit similar to (11). In addition to restricting our frequency domain, there is also an inherent loss in smoothness manifested in the shrinking of the domain of analyticity. This loss of smoothness imposes difficulties in the second step of our plan above and is the reason why usual iteration methods (implicit function theorems) are not enough as they would lead to losing all smoothness after finitely many steps, a fact which would stop the iteration. The way around this difficulty is to use rapidly converging iteration scheme. It turns out that Newton's method does the job, as it provides super-exponential decay. Finally, note that we cannot get rid of the non-integrable while remaining in an open set of action variables, even at the end of the iteration. This is intuitively clear, since if this was indeed true then our perturbed system would be completely integrable which is quite improbable as mentioned while discussing Poincaré's negative result. (In fact, [MM] proves that the set of integrable Hamiltonians is a closed no-where dense subset in the space of smooth Hamiltonians with C^∞ topology).

3.1 Sketch of the Proof:

In what follows we present a very rough sketch of the proof found in [P]. ([W] contains a proof based on the Hamilton-Jacobi method of generating functions).

We are given a perturbed Hamiltonian $H(I,\phi) = h(I) + \epsilon f(I,\phi,\epsilon)$ which is considered to be real analytic. We first extend $H(I,\phi)$ as a complex analytic function onto the domain $D_{r,s} \subset \mathbf{C}^n \times \mathbf{C}^n$.

$$D_{r,s} = \{I \in \mathbf{C}^n : |I - I_0|_\infty < r \ \forall\ I_0 \in D\} \times \{|\Im\phi_j| \le s\} \qquad (14)$$

As mentioned above our aim is to find a *canonical* transformation $\Phi : (I_1,\phi_1) \mapsto (I,\Phi)$ such that $H \circ \Phi(I_1,\phi_1) = h_1(I_1) + P_1(I_1,\phi_1,\epsilon)$ is closer to being completely integrable, in the sense that $P_1 = O(\epsilon^a)$ for some $a > 1$. Notes that since ϵ is supposed to be small, we would expect that this change of variable to be close to the identity in some sense.

Consequently, the first issue we have to deal with is: how to find canonical transformation (that are close to the identity)? Two approaches are possible. The first [W] uses generating functions that solve a Hamilton-Jacobi equation. The second does not require this machinery and is based on the simple observation that transformations provided by Hamiltonian phase flows are canonical! (*Proof:* Let ψ^t be the phase flow after time t of the Hamiltonian vector field $X_F = \nabla_w F$ where $\nabla_w F$ denotes the symplectic gradient of the Hamiltonian function F (the subscript w stands for the symplectic form (3)). Then

$$\frac{d}{dt}(\psi^t)^* w = (\psi^t)^*(L_X w) = 0$$

where L_X is the lie derivative and $L_X w = 0$ since X is a Hamiltonian vector field (uses $L_X w = d \circ i_X w$ since w is closed). The result follows since $\psi^0 = Id$)

We will adopt this second approach here and search for our canonical transformation as the time-1 flow of a Hamiltonian F that is $\Phi = \psi_F^t|_{t=1}$. Since the identity transformation is the time-1 flow of the Hamiltonian $F = 0$ then we would like $F = O(\epsilon)$. Writing ϵF instead of F to emphasize its size we get that Φ is the time-1 flow of the following equations:

$$\dot{I} = -\epsilon\frac{\partial F}{\partial \phi} \qquad (15)$$

$$\dot{\phi} = \epsilon \frac{\partial F}{\partial I} \tag{16}$$

Now using the fact that $d/dt(H \circ \psi^t) = dH(X_F)|_{\psi^t} = w(X_F, X_H)|_{\psi^t} =:$ $\{F, H\} \circ \psi^t$ where $\{F, H\} := w(X_F, X_H)$ is the Poisson Bracket, we get:

$$H \circ \Phi(I, \phi) = (H \circ \psi^t)|_{t=1} = H + \epsilon \int_0^1 \{H, F\} \circ \psi^t dt$$

$$= h(I) + \epsilon\{h, F\} + \epsilon^2 \int_0^1 (1-t)\{\{h, F\}, F\} \circ \psi^t dt + \epsilon f + \epsilon^2 \int_0^1 \{f, F\} \circ \psi^t dt$$

Hence;

$$H_1 := H \circ \Phi = h(I) + \epsilon f + \epsilon\{h, F\} + O(\epsilon^2) = h(I) + \epsilon f_0(I) + \epsilon f'(I, \phi) + \epsilon\{h, F\} + O(\epsilon^2) \tag{17}$$

where $f'(I, \phi) = \sum_{k \neq 0} \hat{f}(I, k) e^{ik\phi}$ and $f_0 = f - f'$ is independent of ϕ. As a result, if F solves:

$$f'(I, \phi) + \{h, F\} = 0 \tag{18}$$

then, at least formally, we have:

$$H_1(I, \phi) = \underbrace{h(I) + \epsilon f_0(I)}_{\text{integrable}} + \underbrace{P_1(I, \phi, \epsilon)}_{O(\epsilon^2)} \tag{19}$$

Now let us make the seemingly unreasonable assumption that $h(I) = cst + \langle \omega, I \rangle$ is affine.(The general case can be reduced to this using normal form reductions which we do not intend to get into here [P] (see also remark 2) below).

Now using the fact that

$$\{h, F\} = \sum_{j=0}^{n} \frac{\partial h}{\partial I_j} \frac{\partial F}{\partial \phi_j}$$

we get the formal solution $F(I, \phi) = \sum_{k \neq 0} \hat{F}(I, k) e^{ik\phi}$ with

$$\hat{F}(I, k) = \frac{\hat{f}(I, k)}{i\langle k, \omega \rangle} \tag{20}$$

These are the infamous *small divisors* with which KAM was developed to deal. We enumerate some important remarks on this issue:

1. First we notice the need to deal with *non-resonant* ω. In fact, in order to prove bounds for our iteration we need ω to be strongly non-resonant as in (11).
2. Notice that since we do not care about terms that are of order ϵ^2 in our analysis, we can always reduce or truncate all expressions that are of order less than or equal to ϵ as long as our reductions or truncations produce error that are second order in ϵ. This is the basic idea behind the normal reductions mentioned above (that allowed us to restrict our attention to affine h).

3. The fact that f is assumed to be analytic implies that the Fourier coefficients $\hat{f}(I,k)$ decay exponentially in k and this is enough to ensure the convergence of the series defining F (Of course, ω is assumed to be non-resonant). Despite this fact,

4. we will need to truncate this series at some $|k| \leq M$. We can do that by choosing M large enough so that the error incurred is second order in ϵ. The reason we do so is as follows. In order to repeat the iteration, we will need ω to belong to some open set and this is impossible if we restrict ω to be strongly non-resonant. However, by truncating the series above, we only need ω to satisfy a finite number of non-resonance conditions which is satisfied in some open neighborhood of Ω_α.

Given all this, we can prove the following bound on F,

$$||F||_{\eta r, s-\sigma} \leq c(\alpha, \sigma, \eta)||f||_{r,s}$$

for some positive σ and small η and hence the following bound on the new Hamiltonian:

$$H_1(I,\phi) = h_1(I) + c(\alpha,\sigma,\eta)\epsilon^2 \tag{21}$$

with H actually defined on the smaller domain $D_{\eta r, s-\sigma}$. The term $c(\alpha,\sigma,\eta)$ is far from being bounded. But luckily enough, we can choose it to be smaller than ϵ^{-1} say $\epsilon^{-1/2}$. This gives:

$$H_1(I,\phi) = h_1(I) + O(\epsilon^{3/2}) \tag{22}$$

Repeating this, while choosing coefficients that close the iteration, we notice that the error term in the Hamiltonian at the n-th step of the iteration is $O(\epsilon^{(3/2)^n})$ (typical decay in Newton iterations) which is small enough to cover up for the inevitable loss of smoothness. In the limit, we get get ridof the perturbation term but we also lose the open set on which the action variables are defined. In fact, the limit frequency-action-angle domain is nothing but $\Omega_\alpha \times \{0\} \times |\Im\phi_j| < s/2$. Nonetheless, this is enough to establish the persistence of all non-resonant tori. See [P] for complete details.

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