DISCRETE AND CONTINUOUS PARAMETRICES FOR VARIABLE COEFFICIENT WAVE AND SCHRÖDINGER EQUATIONS

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ABSTRACT. These notes were loosely written after reading H. Smith's: "A Parmetrix Construction for the Wave Equation with $C^{1,1}$ Coefficients" [Smith] and D. Tataru's "Phase space transforms and microlacal analysis" [T] . In [Smith], the author introduces a new parametrix construction for the wave equation under the assumption that the coefficients of the principal term are $C^{1,1}$ in the space variable and $C^{0,1}$ in the time variable. He then uses the parametrix to prove Strichartz and Pecher estimates in dimensions 2 and 3.

1. INTRODUCTION

We consider the equation:

$$\begin{aligned} \partial_t^2 u(t,x) - A(t,x,\partial_x)u(t,x) &= F(t,x) \\ u(t=0,x) &= f(x) \\ v(t=0,x) &= g(x) \end{aligned} \tag{1.1}$$

where $(t,x) \in [-T,T] \times \mathbb{R}^n$ and $A(t,x,\partial_x) = a^{ij}(t,x)\partial_i\partial_j$ satisfy the following conditions:

(i) There exists $c > 0$ such that:

$$c|\xi|^2 \leqslant a^{ij}(t,x)\xi_i\xi_j \leqslant c^{-1}|\xi|^2 \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and $(t,x) \in [-T,T] \times \mathbb{R}^n$.

(ii) A Lipschitz condition in t:

$$|a^{ij}(t,x) - a^{ij}(t',x)| \leqslant C|t-t'|$$

(iii) A Lipschitz condition for $\nabla_x a^{ij}$:

$$|\nabla_x a^{ij}(t,x) - \nabla_x a^{ij}(t,x')| \leqslant C|x-x'|$$

Smith mentions that these regularity assumptions are the least needed to prove well-posedness of the Hamiltonian flow and that the Strichartz estimates are known to fail under less regularity assumptions on the space derivative by a result of him and Sogge.

We say that $u(t,x) \in C_t H^{\alpha+1} \cap C_t^1 H^\alpha([-T,T]\times\mathbb{R}^n)$ with $\alpha > -1$ is a weak solution of (1.1) if u solves the equation weakly and satisfies the initial conditions in the vector-valued sense at $t = 0$. Notice that since $\alpha \geqslant -1$, $u \in L^2_{t,x}$ and the left hand side of the equation makes sense weakly (otherwise u is a distribution rather than a function, there is no way to define its action on the function $\partial_t^2 - \partial_i\partial_j a^{ij}(t,x)\phi$ where ϕ is a test function).

For $-1 \leqslant \alpha \leqslant 2$, $f \in H^{\alpha+1}$, $g \in H^{\alpha}$, and $F \in L^1_t H^{\alpha}$, we establish the existence of a unique solution to the wave equation. This is achieved by constructing an operators $s(t,s)$ for $-T \leqslant s,t \leqslant T$ that are essentially Fourier integral operators of order -1 that satisfy:

$$\mathbf{s(t,s)}|_{\mathbf{t=s}} = \mathbf{0} \qquad \partial_{\mathbf{t}}\mathbf{s(t,s)}|_{\mathbf{t=s}} = \mathbf{Id}$$

These should be compared to $\frac{sin(t-s)|\nabla|}{|\nabla|}$ corresponding to the fundamental solution for the constant coefficient wave equation. Using the operators $\mathbf{s(t,s)}$, we set:

$$u(t,x) = \int_0^t \mathbf{s(t,s)G(s,x)ds}$$

and obtain that:

$$(\partial_t^2 - A(t,x,\partial_x))u(t,x) = G(t,x) + \int_0^t (\partial_t^2 - A(t,x,\partial)\mathbf{s(t,s)G(s,x)ds}$$

The left-hand side is a Volterra-type equation that will be solved for G by iteration.

The operators $s(t,s)$ are constructed as matrices acting on a frame of functions on $L^2(\mathbb{R}^n)$ which are sufficiently localized in phase space so that the action of the wave propagator on each of them can be well-approximated by following the Hamiltonian (bicharacteristic) flow to the center of the wave packet.

The rough coefficients a^{ij} are regularized by first projecting each of them onto space frequencies $\leqslant 2^{k/2}$ and then convolving them in time with a compactly supported smooth function dilated by a factor of 2^{-k} (which has the essential effect of multiplying on the frequency side with a function localized at temporal frequencies $\lesssim 2^k$). As a result of this, we get the coefficients a^k_{ij} that satisfy the following:

(i)
$$|\partial_t^m \partial^\beta a^k_{ij}(t,x)| \lesssim C_{m,\beta} 2^{\frac{k}{2}(2m+|\beta|-2)} \quad , 2m + |\beta| > 2$$
This follows from the assumptions on the coefficients and the fact that if $f \in C^{0,1}$ then $\partial P_{\leqslant N} f$ is bounded in L^∞ independent of N ($P \leqslant Nf$ is differentiable and every difference quotient is bounded).

(ii)
$$|a_{ij}(t,x) - a^k_{ij}(t,x)| \lesssim 2^{-k}$$
$$|\nabla_x a_{ij}(t,x) - \nabla_x a^k_{ij}(t,x)| \lesssim 2^{-k/2}$$

These two fact can be seen as follows: Let us start with the second one as it is easier: Suppose that f is a Lipschitz function on $\mathbb{R}^n$ (here $f = \nabla_x a_{ij}$) then

$$|f(x) - P_{\leqslant N} f| = |\int_{R^n} k_N(y)[f(x-y) - f(x)]dy| \lesssim C \int_{R^n} \int_{\mathbb{R}^n} |y| k_N(y) dy \lesssim \frac{1}{N}$$

since $k_N(y) = N^d k_0(Ny)$. We now turn to the first estimate and assume that f is a function on $\mathbb{R}^n$ satisfying $\nabla_x f \in C^{0,1}$, then

$$
\begin{aligned}
|f(x) - P_{\leqslant N} f| &= |\int_{R^n} k_N(y)[f(x-y) - f(x)]dy| = |\int_{R^n}\int_0^1 k_N(y)y.\nabla f(x+sy)dsdy| \\
&= |\int_{R^n}\int_0^1 k_N(y)y.[\nabla f(x+sy) - \nabla f(x)]dsdy| \\
&\lesssim \int_{\mathbb{R}^n} k_N(y)|y|^2 dy \lesssim \frac{1}{N^2}
\end{aligned}
$$

where we used the fact that $\int_{\mathbb{R}}^n k_N(y)y_i dy = 0$ because $\hat{k}$ is constant near the origin.

With these regularizations made: $\mathbf{s}(\mathbf{t},\mathbf{s})$ will be given a sum involving operators $\mathbf{s_k}(\mathbf{t},\mathbf{s})$ which are localized to frequencies $|\xi| \sim 2^k$ and satisfy:

$$
\sum_{k=1}^{\infty} (\partial_t^2 - A_k(t,x,\partial))\mathbf{s_k}(\mathbf{t},\mathbf{s})
$$

is a FIO of order 0.

This regularization seems at the balance of making $\mathbf{s}$ an FIO with symbol $S_{\frac{1}{2}\frac{1}{2}}$ which is the largest class for which the standard theory of FIO goes through and the requirement of making the error term make by replacing A by A_k bounded on H^α.

2. The frame of functions

The parametrix construction is based on a "frame" of coherent wave packets that are well-localized in phase space, i.e. in space and in frequency. Every function in L^2 can be written(though not uniquely) as a convergent sum of such wave packets. The advantage of this frame is that the action of wave operator is given essentially by a rigid motion (in phase space applied to the wave packet): namely move it by a flow very close to the bicharacteristic flow of the symbol $\sqrt{A(t,x,\xi)}$(recall we want to first find a parametrix for $e^{it|nabla|}$).

The frame of coherent wave packets starts with a partition of unity in frequency space exactly the same as that used in Chapter 9 of Stein in the proof of L^p boundedness of Fourier integral operators. It is often referred to as the "dyadic-parabolic decomposition" or second dyadic decomposition as Stein refers to it. Here one has a partition of unity of $\mathbb{R}^n$ given by:

$$
1 = |h_0(\xi)|^2 + \sum_{k=1}^{\infty}\sum_{\omega} |h_k^\omega(\xi)|^2 \tag{2.1}
$$

where ω runs over a $2^{-k/2}$ separated set on the sphere S^{d-1} (and hence its cardinality is $\sim 2^{k(d-1)/2}$). Also for each k, $\sum_\omega |h_k^\omega(\xi)|^2$ can be taken to be $|\phi(\frac{|\xi|}{2^k})|^2$ with $|h_0(\xi)|^2 + \sum_{k=1}^{\infty} |\phi(\frac{|\xi|}{2^k})|^2 = 1$. (Essentially one starts with this latter partition of unity of $\mathbb{R}^n$ and a partition of unity of S^{d-1} where each piece is supported in a $\sim 2^{-k/2}$ neighborhood of ω with finite overlap) and then multiplies this partition of unity of S^{d-1} with each piece of the partition of unity of $\mathbb{R}^d$. As a result, it is easy make sure that each $h_k^\omega(\xi)$ is supported in the set:

$$
2^{k-1/2} \leqslant |\xi| \leqslant 2^{k+1/2} \qquad \left|\frac{\xi}{|\xi| - \omega}\right| \leqslant 2^{-k/2}
$$

and h_k^ω can be made to satisfy the following bounds:

$$\left|\langle\omega,\partial_\xi\rangle^j\partial_\xi^\alpha h_k^\omega(\xi)\right| \lesssim_{j,\alpha} 2^{-k(j+|\alpha|/2)} \tag{2.2}$$

(remember that $2^{-k}\leqslant 2^{-k/2}$:)) Notice that essentially each h_k^ω is supported in a rectangular box with one long side along ω of length 2^k and perpendicular sidelengths of $2^{k/2}$. By the uncertainty principle, we expect that the inverse Fourier transform of h_k^ω to vary very little in the dual rectangular boxes: these are boxes of length 2^{-k} in the direction of ω and $2^{-k/2}$ in the perpendicular directions. In fact, it can be easily seen by the (2.2) that the inverse Fourier transform of h_k^ω is very concetrated on a dual rectangular box centered near the origin and decays sharply as $2^k|\langle\omega,x\rangle|+2^{k/2}|\langle\omega^\perp,x\rangle|$ becomes large (where we have denoted $\langle\omega^\perp,x\rangle$ the projection of x on $\omega^\perp$)

As a result, in order to capture the behaviour of any function supported on the support of ω one has to essentially know its behaviour on a lattice of points that is seperated by 2^{-k} in the direction of ω and $2^{-k/2}$ in orthogonal directions. We call this lattice Ξ_k^ω we denote by Γ the set of triples $\{(x,\omega,k): x\in\Xi_k^\omega\}$. We set:

$$\widehat{\phi_\gamma}(\xi)=(2\pi)^{-n/2}2^{-\frac{k(n+1)}{4}}e^{-i\langle x,\xi\rangle}h_k^\omega(\xi) \tag{2.3}$$

This is L^2 normalized and has mean zero for $k>0$. Also notice the following bounds:

$$\left|\langle\omega,\partial_y\rangle^j\langle\omega^\perp,\partial_y\rangle^\alpha\phi_\gamma(y)\right| \lesssim_{j,\alpha} 2^{k\left(-\frac{(n+1)}{4}+j+\frac{|\alpha|}{2}\right)}(1+2^k|\langle\omega,x-y\rangle|+2^k|x-y|^2)^{-N} \tag{2.4}$$

So ϕ_γ is sharply localized in the boxes determined by γ and is L^2 localized.

The set of function ϕ_γ with $\gamma\in\Gamma$ is a "frame" for $L^2(\mathbb{R}^n)$. Following the books "Wavelets and operators" of Meyer, a frame for a Hilbert space consists first of a bounded surjective mapping from $l^2(\Gamma)$ to H where we denote by $\delta_\gamma\in l^2(\gamma)$ the element that is 1 for γ and 0 otherwise (I think I might be missing part of the definition here though). For us, the exact definition is irrelevant. We have:

If

$$c(\gamma)=\int_{\mathbb{R}^n}\overline{\phi_\gamma(y)}f(y)dy$$

then

$$f(y)=\sum_\gamma c(\gamma)\phi_\gamma(y) \qquad \int_{\mathbb{R}^n}|f(y)|^2dy=\sum_\gamma|c(\gamma)|^2$$

The first property follows by working in Fourier space and using Poisson summation. The second follows trivially by writing $\int_{\mathbb{R}^n}|f(y)|^2dy=\sum_\gamma\int_{R^n}\overline{c(\gamma)\phi_\gamma(y)}f(y)dy=\sum_\gamma|c(\gamma)|^2$.

The frame ϕ_γ is not orthogonal (not even independent as a spanning set) but it satisfies the following property (which is the boundedness statement in Meyer's definition):

$$\int_{\mathbb{R}^n}\left|\sum_\gamma d(\gamma)\phi_\gamma(y)\right|^2dy\lesssim\sum_\gamma|d(\gamma)|^2$$

(I verified this using the decay poperties of the Fourier transform of $h_k^\omega h_{k'}^{\omega'}$ whenever the product does not vanish.) To clarify this relationship, let us define:

$$U_1 : L^2(\mathbb{R}^n) \to l^2(\Gamma), \quad U_1(f) = \{c(\gamma)\}$$
$$U_2 : l^2(\Gamma) \to L^2(\mathbb{R}^n), \quad U_2(d(\gamma)) = \sum_\gamma d(\gamma)\phi_\gamma(x)$$

Then U_2 is the left inverse of U_1 and the mapping

$$\Pi = U_1 \circ U_2$$

is a non-orthogonal projection of $l^2(\Gamma)$ onto the range of U_1. Now given a mapping from Schwartz functions to tempered distributions, we associate the matrix $a(\gamma, \gamma')$ given by:

$$a(\gamma, \gamma') = \int_{\mathbb{R}^n} \overline{\phi_\gamma(y)}\, (T\phi_{\gamma'}(y))\, dy$$

which can also be given as:

$$a = U_1 \circ T \circ U_2 : l^2(\Gamma) \to l^2(\Gamma)$$

which gives since U_2 is the left inverse of U_1 that:

$$T = U_2 \circ a \circ U_1$$

or

$$Tf(y) = \sum_{\gamma,\gamma'} \phi_\gamma(y) a(\gamma, \gamma') c(\gamma')$$

whenever $f = \sum c(\gamma')\phi_{\gamma'}$. This maps bounded operators on l^2 to bounded operators on L^2 but is not a homomorphism (I think here the group operation is given by composition rather than addition...).

Of course, to prove L^2 bounded operator Schur's test is good tool. We state it with some weights:

Lemma 2.1. *Suppose that $a(\gamma, \gamma')$ is a matrix and there exists a strictly positive function $\rho(\gamma)$ such that:*

$$\sum_{\gamma'} a(\gamma, \gamma')\rho(\gamma') \leqslant C_a \rho(\gamma)$$
$$\sum_{\gamma'} a(\gamma', \gamma)\rho(\gamma') \leqslant C_a \rho(\gamma)$$

then the operator determined by a is bounded on $L^2(\mathbb{R}^n)$ and the operator norm is bounded by C_a.

Proof. Adapt the Cauchy-Schwartz proof of usual Schur's lemma rather than the interpolation proof (start by writing $|a(\gamma, \gamma')| = \sqrt{a(\gamma, \gamma')}\sqrt{a(\gamma, \gamma')}$) and apply Cauchy-Schwartz.... $\square$

2.2. **The Pseudo-distance function.** We define a pseudo-distance function that Smith says appeared in his previous paper on a Hardy space for FIO (but I remember seeing it once in Hörmander's paper on Fourier Integral Operators). This is defined on the cosphere bundle $S^*\mathbb{R}^n = \{(x,\xi) \in T^*\mathbb{R}^n : |\xi| = 1\}$ is given by:

$$d(x, w, \tilde{x}, \tilde{w}) = |\langle w, x - \tilde{x}\rangle| + |\langle \tilde{w}, x - \tilde{x}\rangle| + \min(|x - \tilde{x}|, |x - \tilde{x}|^2) + |w - \tilde{w}|^2 \tag{2.5}$$

A useful estimate to be observed is that:

$$d(x, w, \tilde{x}, \tilde{w}) \sim |\langle w, x - \tilde{x}\rangle| + \min(|x - \tilde{x}|, |x - \tilde{x}|^2) + |w - \tilde{w}|^2$$

(I didn't spend much time verifying this). The point is that this pseudo-distance is essentially preserved by canonical transformation. Here we consider canonical transformations determined by $C^{1,1}$ Hamiltonian:

Lemma 2.3. *(Invariance of pseudo-distance) Suppose that $H(t, x, \xi)$ is a real and homogeneous of degree 1 in ξ. Suppose also that $\nabla_x H(t, x, \xi)$ and $\nabla_\xi H(t, x, \xi)$ satisfy a Lipschitz condition in (x, ξ) with uniform Lipschitz constant for $|\xi| = 1$. Let χ_t be the transformation on the cosphere bundle $S^*(\mathbb{R}^n)$ induced by the projected Hamiltonian flow at time t (since H is homogeneous of degree 1 in ξ the Hamiltonian flow commutes with dilations) given by:*

$$\frac{dx}{dt} = H_\xi(t, x, \omega) \qquad \frac{d\omega}{dt} = -H_x(s, x, \omega) + \langle \omega, H_x(s, x, \omega)\rangle$$

(Notice that this flow on the cosphere bundle is the same as the flow of $(x, \frac{\xi}{|\xi|})$ whenever (x, ξ) follows that usual Hamiltonian flow given by H)

Than

$$d(\chi_t(x, \omega), \chi_t(\tilde{x}, \tilde{\omega}) \sim d(x, \omega, \tilde{x}, \tilde{\omega})$$

where the constant of proportionality is bounded by e^{Kt} for some K

Proof. Suppose that $(x, \omega) = \chi_t(y, \eta)$ and $(\tilde{x}, \tilde{\omega}) = \chi_t(\tilde{y}, \tilde{\eta})$. Since H_ξ and H_x are both Lipschitz on $S^*(\mathbb{R}^n)$ we get that:

$$|x - \tilde{x}| + |\omega - \tilde{\omega}| \sim |y - \tilde{y}| + |\eta - \tilde{\eta}|$$

with the constant of proportionality bounded by $e^{K|t|}$. The rest will follow from the following inequality:

$$|\langle \omega, x - \tilde{s}\rangle - \langle \eta, y - \tilde{y}\rangle| \leqslant e^{Kt} \left(|y - \tilde{y}|^2 + |\eta - \tilde{\eta}|^2\right)$$

This is done as follows:

$$\frac{d}{dt}\langle \omega, x - \tilde{x}\rangle = \langle \omega, H_\xi(s, x, \omega)\rangle - \langle \omega, H_\xi(s, \tilde{x}, \tilde{\omega})\rangle - \langle H_x(s, x, \omega), x - \tilde{x}\rangle + \langle \omega, H_x(s, x, \omega)\rangle\langle \omega, x - \tilde{x}\rangle$$

By homogeneity one recognizes that the first three terms as:

$$H(s,x,\omega)-H(s,\tilde{x},\tilde{\omega})+\langle\tilde{\omega}-\omega,H_\xi(s,x,\omega)\rangle-\langle H_x(s,\tilde{x},\tilde{\omega}),x-\tilde{x}\rangle+\langle H_x(s,x,\omega)-H_x(s,\tilde{x},\tilde{\omega}),x-\tilde{x}\rangle$$

The first four terms are a first order Taylor expansion around $(\tilde{x},\tilde{\omega})$ so they are $O(|x-\tilde{x}|^2+|\omega-\tilde{\omega}|^2)=O(|y-\tilde{y}|^2+|\eta-\tilde{\eta}|^2)$. The last term is bounded similarly since H_x is Lipschitz. Smith claims that the rest follows by Gronwall's inequality *but it seems to me that this requires some bound on H_x.....* □

2.4. **The weight function:** Now define a weight function $\mu_\delta(\gamma,\gamma')$ that will be used as a decaying bump function as γ and γ' become far apart.

Definition 2.5. If $\gamma=(x,\omega,k)$ and $\gamma'=(x',\omega',k')$, we set:

$$\mu_\delta(\gamma,\gamma')=(1+|k-k'|)^{-2}2^{-(\delta+\frac{n}{2})(k-k')}\left(1+\frac{d(x,\omega,x',\omega')}{2^{-k}+2^{-k'}}\right)^{-n-\delta} \tag{2.6}$$

It seems that this is closely related to the weight function used in defining the matrix for a Calderon-Zygmund operator. This weight function satisfies the following properties which are proved in the paper:

Lemma 2.6. *(Properties of μ_δ)*

a)

$$\sum_{\gamma'}\mu_\delta(\gamma,\gamma')2^{-nk'/2}\leqslant C(\delta)2^{-nk/2}$$

This says that μ_δ maps exponentially decaying vectors into exponentially decaying ones.

b)

$$\sum_{\gamma'}\mu(\gamma,\gamma')\mu_\delta(\gamma',\gamma_0)\leqslant C(\delta_0)\mu_\delta(\gamma,\gamma_0)$$

2.7. **The Classes of Operators:** We are now ready to define the classes of operators considered. In particular, we define operators in terms of what they do to the coherent wave packets.

Definition 2.8. Let χ be a mapping of $S^*(\mathbb{R}^n)$, we say that the matrix $a(\gamma,\gamma')$ belongs to a class $M^r_\delta(\chi)$ if:

$$a(\gamma,\gamma')\leqslant C_a 2^{rk'}\mu_\delta(\gamma,\chi(\gamma'))$$

where $\chi(\gamma')=(\chi(x_\gamma,\omega_\gamma),k')$. We also set

$$M^r(\chi)=\cap_{\delta>0}M^r_\delta(\chi)$$

Let us analyze what this means for the associated operator T. T takes $\phi_{\gamma'}$ to $\sum_\gamma a(\gamma,\gamma')\phi_\gamma$. But $a(\gamma,\gamma')\leqslant C_a2^{rk'}\mu_\delta(\gamma,\chi(\gamma'))$ so it decays sharply away from $chi(\gamma')$ so essentially T moves $\phi_{\gamma'}$ to $\phi_{\chi(\gamma)}$ which moves it phase-space concentration from (x',ω',k') to $(\chi(x',\omega'),k')$ while multiplying it by 2^k which has the almost the same effect as applying the operator $|\nabla|$ to it.

Definition 2.9. We say that an operator T from Schwartz function to tempered distributions belongs to class $I^r(\chi)$ if the associated matrix $a(\gamma,\gamma')$ belongs to $M^r(\chi)$.

Theorem 2.10. *Suppose that χ is an invertible transformation on $S^*(\mathbb{R}^n)$ such that for some constant C*

$$C^{-1}d(x,\omega,x',\omega') \leqslant d(\chi(x,\omega),\chi(x',\omega')) \leqslant Cd(x,\omega,x',\omega') \tag{2.7}$$

If the matrix $a(\gamma,\gamma') \in M^r(\chi)$, then the operator determined by a maps $H^\alpha(\mathbb{R}^n) \to H^{\alpha-r}(\mathbb{R}^n)$.

If $a_j \in M^{r_j}(\chi_j)$ for $j = 1,2$ where χ_j are invertible mappings on $S^*(\mathbb{R}^n)$ satisfying (2.7) then $a_1 \circ a_2 \in M^{r_1+r_2}(\chi_1 \circ \chi_2)$

Proof. Since $||f||_{H^\alpha(\mathbb{R}^n)} \sim \left(2^{2k\alpha}|c(\gamma)|^2\right)^{1/2}$ it is enough to show that the matrix a maps l^2_α to $l^2_{\alpha-r}$ The first statement follows from Schur's lemma, first property of the weight function, and the relation:

$$\mu_\delta(\gamma,\chi(\gamma')) \sim \mu_\delta(\chi^{-1}(\gamma),\gamma')$$

The second statement follows from the above statement and the second property of $a(\gamma,\gamma')$ mentioned. Here one has to use that M^r is defined as the intersection of M^r_δ. This is actually to establishing an algebra property involving Sobolev spaces. □

Smith mentions that standard Fourier integral operators of order r associated with a smooth canonical transformation χ, then $T \in I^r(\chi)$. Also one can define a class of operator I^r_δ with finite decay properties between its coefficients, but then the Banach space algebra only holds for operators on $L^2(\mathbb{R}^n)$.

The definition of $I^r(\chi)$ essentially says that given T, we say that it belongs to I^r if its matrix belongs to M^r. The following lemma essentially includes the converse statement:

Lemma 2.11. *Suppose that the matrix a belongs to M^r, then the operator defined from a belongs to I^r, in other words the map:*

$$a \mapsto \Pi \circ a \circ \Pi$$

is bounded on M^r. Where $\Pi = U_1 \circ U_2$.

Proof. By the last statement, it suffices to show that $\Pi \in M^0(I)$ where I is the identity transformation on $S^*(\mathbb{R}^n)$. The matrix coefficients of Π are given by:

$$\Pi(\gamma,\gamma') = \sum_{\mathbb{R}^n} \phi_\gamma(y)\phi_{\gamma'}(y)dy$$

which can be easily seen to be bounded by $\mu_\delta(\gamma,\gamma')$ since it vanishes unless $|k-k'| \lesssim 1$.... □

3. The Ansatz

In this section, we study how an element of the coherent wave packet frame should evolve in order to solve (rather approximate the solution to) the wave equation. For each $\gamma = (x,\omega,k)$, we will introduce the evolution $\phi_\gamma(x,t)$ that is essentially obtained from ϕ_γ by a rigid motion whose effect will be to change its phase space localization from (x,ω,k) to the coherent wave packet given by $(\chi_t(x_\gamma,\omega_\gamma),k)$ where χ_t is the Hamiltonian flow of $\sqrt{A_k(t,x,\xi)}$. The rigid motion is constructed so that the second order terms in the expansion:

$$(\partial_t^2 - A(t, x, \partial_y))\phi_\gamma(t, y)$$

cancel each other. As a result, it is proved that if $\mathbf{e_k}$ is the operator which takes $f = \sum_\gamma c_\gamma \phi_\gamma(y)$ to $\sum_{k_\gamma = k} c_\gamma \phi_\gamma(t, y)$ then *a)* $\mathbf{e_k}$ is an operator that belongs to $I^0(\chi_t)$ and *b)*

$$(\partial_t^2 - A(t, x, \partial_y))\mathbf{e_k}(\mathbf{t}, \mathbf{y})$$

is an operator of order 1. This will be later used to construct an approximate inverse for the wave operator.

Let $H_k(t, x, \xi) := \sqrt{A_k(t, x, \xi)}$ and notice that since H_k is homogeneous of degree 1, the Hamiltonian flow on $T^*(\mathbb{R}^n)$ commutes with dilations and hence induces a flow on the cosphere bundle $S^*(\mathbb{R}^n)$ (i.e. the induced flow in independent of the length $|\xi|$ of ξ but is completely determined by $\frac{\xi}{|\xi|}(t = 0)$) which is given by:

$$\frac{dx}{dt} = (H_k)_\xi(t, x, \xi) \qquad \frac{d\omega}{dt} = -(H_k)_x(t, x, \omega) + \langle (H_k)_x(t, x, \omega), \omega\rangle\omega$$

We will need an orthogonal matrix $\Theta(t)$ such that for every time t, $\Theta(t)\omega(t) = \omega(0)$, this gives that $\frac{d\Theta}{dt}\omega(t) = -\Theta(t)\frac{d\omega}{dt}$. I think that there are more than one matrix that does that, but the easiest one that satisfying the equation:

$$\frac{d\Theta}{dt} = -\Theta[-(H_k)_x(t, x, \omega) \otimes \omega + \omega \otimes (H_k)_x(t, x, \omega)]$$

where we used $v \otimes w$ to denote the map $x \mapsto v\langle w, x\rangle$. Notice that its transpose is $w \otimes v$. As a result, it is easy to see that the flow preserves $S^*(\mathbb{R}^n) \times O(n)$ (the term in braces is anti-symmetric). The initial data are of course given by

$$x(0) = x_\gamma \quad \omega(0) = \omega_\gamma \quad \Theta(0) = Id$$

With this we have (by construction or verification) that $\Theta(t)\omega(t) = \omega(0) = \omega_\gamma$

We now define the evolution of ϕ_γ as:

$$\phi_\gamma(t, y) = \phi_\gamma(\Theta(t)(y - x_\gamma(t)) + x_\gamma) \tag{3.1}$$

So we essentially rotate ϕ_γ and then translate to be centered around $x_\gamma(t)$. The rotation has the effect of rotating the Fourier support of ϕ_γ by $\Theta(t)$ thus making it essentialy supported around $\Theta(t)\omega_\gamma = \omega(t)$.

Definition 3.1. If $c(\gamma') = \int_{\mathbb{R}}^n \overline{\phi_\gamma(y)} f(y) dy$, we set

$$\mathbf{e}_k(t) = \sum_{\gamma' : k_{\gamma'} = k} c(\gamma')\phi_{\gamma'}(t, y)$$

Now if $\chi_t(x, \omega)$ is the time t Hamiltonian flow of $H(t, x, \xi) = \sqrt{A(t, x, \xi)}$ with initial data $(x_\gamma, \omega_\gamma)$ we have the following theorem:

Theorem 3.2. *The following holds:*

$$\sum_k \mathbf{e}_k(t) \in I^0(\chi_t)$$

$$\sum_k \left(\partial_t^2 - A(t,x,\partial)\right)\mathbf{e}_k(t) \in I^1(\chi_t)$$

Furthermore, the constant appearing in the definition of M_δ^r is uniformly bounded (for each δ) provided time varies over a compact interval.

Remark. Note that rigid motion by which we are moving ϕ_γ was chosen so that if one calculates $[\partial_t^2 - A_k(t,x,\partial_y)]\phi_\gamma(t,y)$ then the second order terms, i.e. terms that are not bounded by 2^k in some sense cancel each other. The motivation behind using this particular ansatz can be seen by first considering the homogeneous wave equation and finding the right rigid motion that needs to be taken there in order to cancel the highest order terms. Then one should consider the constant coefficient case and find the right rigid motion and then generalize. It is worth mentioning that it is here that one realizes the importance of the second dyadic decomposition as it is a very important part of the proof.

Remark. One particularly important way to think about the coherent wave packets ϕ_γ is that they are meant to generalized the exponentials on $\mathbb{R}^n$. So it is very important to keep in mind their frequency *and* space localizations throughout this proof and at some points pretend that applying differentiation operators to them just multiplies them with a factor of 2^k if one is differentiating radially and by a factor of $2^{k/2}$ when taking "angular" derivatives.

Remark. The most important derivative of ϕ_γ that should be cancelled by the choice of rigid motion is the second derivative in the direction of ω_t as it leads to a loss of 2^{2k} which is very large. In fact, the way to deal with this is exactly by choosing the rigid motion generated by the Hamiltonian flow of $A_k(t,x,\xi)$ as this will lead to the multiplication of the factor 2^{2k} with the second order remainder of the Taylor expansion of $A_k(t,x)$ which is $O(|y-x_\gamma|^2)$[1] thus leading to an acceptable bound as the relevant spacial scale is 2^{-k}

Remark. All error estimates and calculations are done to the truncated coefficients A_k rather than A_k. The way to move from A_k to A is by showing the Hamiltonian flows of them are close in the pseudo-metric introduced before. In fact, one shows that if χ_t is the flow by $\sqrt{A(t,x,\xi)}$ and $(x(t),\omega(t))$ that by $\sqrt{A_k}$ then

$$d(x(t),\omega(t),\chi_t(x_\gamma,\omega_\gamma)) \leqslant 2^{-k}(Ce^{|}t| - 1)$$

This means that in the final verification of Theorem 3.2 on can swap pseudo-distances with respect to $(x(t),\omega(t))$ with distances to points $\chi_t(x_\gamma,\omega_\gamma)$ without guilt. The proof of this stability result is very similar to the proof of invariance of the pseudo-distance sketched in the previous section.

Remark. As mentioned earlier, the regularity conditions on the coefficients $A(t,x)$ are dictated by the last steps of the verification of the proof of Theorem 3.2. The coefficients have to be $C_x^{1,1}$ in order for $\partial_t^2 - A_k(t,x,\partial)\phi_\gamma(t,y)$ to be an operator of order 1 rather than 2. Also one requires in the proof that the coefficients are $C_t^{0,1}$.

[1]Hence the great importance of having the coefficients $C^{1,1}$ and not less regular!

4. The Parametrix

In this section, we use the results prove so far to construct a parametrix to the wave equation (1.1) by which we mean operators $\mathbf{c}(t,s)$ and $\mathbf{s}(t,s)$ that satisfy the following properties:

(i) Behaviour when $t = s$:

$$\mathbf{c}(t,s)|_{t=s} = Id \quad \partial_t \, \mathbf{c}(t,s)|_{t=s} = 0$$

and

$$\mathbf{s}(t,s)|_{t=s} = 0 \quad \partial_t \, \mathbf{c}(t,s)|_{t=s} = Id$$

Here one notices the similarity with $\cos t|\nabla|$ and $\sin t|\nabla|$ given by the fundamental solutions to the wave equation.

(ii) Boundedness properties: $\partial_t^k \, \mathbf{c}(t,s)$ is a continuous family of operators from $H^\alpha \to H^{\alpha-k}$ for all α while $\partial_t^k \, \mathbf{s}(t,s)$ is a continuous family of operators from $H^\alpha \to H^{\alpha-1-k}$

(iii) Parametrix Property:

Defining

$$T_0(t,s) = (\partial_t^2 - A(t,x,\partial_x)) \, \mathbf{c}(t,s) \tag{4.1}$$

and

$$T_1(t,s) = (\partial_t^2 - A(t,x,\partial_x)) \, \mathbf{s}(t,s) \tag{4.2}$$

Then the following holds:

Theorem 4.1. *If $-1 \leqslant \alpha \leqslant 2$,then for each t, s:*

$$T_0(t,s) : H^{\alpha+1} \to H^\alpha$$

$$T_1(t,s) : H^\alpha \to H^\alpha$$

and the operator norms are uniformly bounded as $t-s$ varies over a compact interval.

This means that $\mathbf{c}(t,0)f + \mathbf{s}(t,0)g$ offers a first order approximation to the solution of the homogeneous ($F = 0$) wave equation (the error after applying $\Box$ being of order 1 rather than 2)

We will see that any family that satisfies the above three properties will allow us to construct solutions to (1.1) by simple iteration process for a Volterra type equation. In fact we will start by showing this:

Theorem 4.2. *If $-1 \leqslant \alpha \leqslant 2$ and $f \in H^{\alpha+1}, g \in H^\alpha, and F \in L_t^1 H^\alpha([-T,T] \times \mathbb{R}^n)$, then there exists $G \in L_t^1 H^\alpha([-T,T] \times \mathbb{R}^n)$ satisfying:*

$$||G||_{L_t^1 H^\alpha} \lesssim_T ||f||_{H^{\alpha+1}} + ||g||_{H^\alpha} + ||F||_{L_t^1 H^\alpha} \tag{4.3}$$

such that if $u(t,x)$ is given by:

$$u(t,x) = c(t,0)f(x) + s(t,0)g(x) + \int_0^t \mathbf{s}(t,s)G(s,.)(x)ds \tag{4.4}$$

is a weak solution to (1.1). *Moreover, if $f = g = 0$ and F vanishes for $t < 0$ (respectively $t > 0$) then G, and hence u, vanishes for $t < 0$(respectively $t > 0$).*

Proof.

First suppose the $G \in L_t^1 H^\alpha$ and consider the integral:

$$v(t,x) = \int_0^t \mathbf{s}(t,s)G(s,x)ds$$

This integral makes sense since s is bounded on H^α. Moreover, it is easy to see that $v \in C_t^0 H^{\alpha+1} \cap C_t^1 H^\alpha$ by the property (i) of the parametrix. (remember that a function is in $C_t^1 B$ for a Banach space B if its weak derivative is continuous in B. Exercise: verify this!) Also,

$$\partial_t v(t,x) = \int_0^t \partial_t \mathbf{s}(t,s)G(s,x)ds$$

and hence $v(0) = \partial_t v(0) = 0$. We now want to calculate $\partial_t^2 v$. It seems to me that this can be justified in more than one way. The best way seems to be a limiting argument working with nice function until the end and then taking limits using the estimate on G. Another way seems to use the definition of week derivative and use brute force on each term. Another is to argue that for $\phi \in C_c([-T,T] \times \mathbb{R}^n)$ then the function:

$$t \mapsto \langle \phi(t,.)\partial_t \mathbf{s}(t,s)G(s,.)\rangle$$

is smooth in t for each fixed s because $\partial_t^k \mathbf{s}(t,s)$ exists in the weak sense and we have:

$$\partial_t \langle \phi(t,.)\partial_t \mathbf{s}(t,s)G(s,.)\rangle = \langle \partial_t\phi(t,.)\partial_t \mathbf{s}(t,s)G(s,.)\rangle + \langle \phi(t,.)\partial_t^2 \mathbf{s}(t,s)G(s,.)\rangle$$

As a result we have:

$$\partial_t \left(\int_0^t \langle \phi(t,.)\partial_t \mathbf{s}(t,s)G(s,.)ds\rangle \right) = \langle \phi(t,.), G(t,.)\rangle + \int_0^t \langle \partial_t\phi(t,.)\partial_t \mathbf{s}(t,s)G(s,.)\rangle ds + \langle \phi(t,.)\partial_t^2 \mathbf{s}(t,s)G(s,.)\rangle$$

which follows from the fact that $G \in L_t^1 H^\alpha$ so one might need to use Lebesgue's differentiation theorem. This gives (since the left hand side vanishes once we integrate in t):

$$\partial_t^2 v(t,x) = G(t,x) + \int_0^t \partial_t^2 \mathbf{s}(t,s)G(s,.)ds$$

As a result of all this, we get that $u(t,x)$ defined as in the statement of the theorem will solve the wave equation weakly, iff:

$$G(t,x) + \int_0^t T_1(t,s)G(s,x)ds = F(t,x) - (\partial_t^2 - A(t,x,\partial_y)[\mathbf{c}(t,0)f + \mathbf{s}(t,0)g]$$

This is a Volterra type equation that is solved by iteration. Denote by $\tilde{F} \in L_t^1 H^\alpha$ the right hand side of this equation. We let $G_0 = \tilde{F}$, then if $G = G_0 + R_0$ then R_0 satisfies the equation:

$$R_0 + \int_0^t T_1(t,s)(R_0 + G_0)ds = 0$$

or

$$R_0 + \int_0^t T_1(t,s) R_0 ds = \int_0^t T_1(t,s) G_0 ds$$

Now letting $R_0 = \int_0^t T_1(t,s) G_0 ds + R_1$, we get:

$$R_1 + \int_0^t T_1(t,s) R_1(s,x) ds = \int_0^t T_1(t,s) \int_0^s T_1(s,r) G_0 ds$$

Notice that

$$|| \int_0^t T_1(t,s) \int_0^s T_1(s,r) G_0 ds ||_{L^1_t H^\alpha} \leqslant C^2(T) \int_0^t \int_0^s ds dt ||\tilde{F}||_{L^1_t H^\alpha} = C^2 t^2/2$$

so we are on the right track because this looks like the term in the expansion of $exp(C(T)T)||\tilde{F}||$

As result we get that

$$G(t,x) = \tilde{F}(t,x) + \sum_{n=1}^{\infty} G_n(t,x)$$

with

$$G_n(t,x) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} T_1(t,s_1) \dots T(s_{n-1},s_n) \tilde{F}(s_n,x) ds_n \dots ds_1$$

which is bounded in $L^1_t H^\alpha$ by $C(T)^n T^n/n!$ if I'm not mistaken with constants. □

It now remains to construct the parametrix. This is done using the ansatz of the previous section. For this we introduce a pair of families $\phi^\pm_\gamma(t,s,y)$ and $\vartheta^\pm(t,s,y)$ such that the solution to the Cauchy-problem:

$$\begin{aligned} \partial_t^2 u(t,x) - A(t,x,\partial_x) u(t,x) = 0 \\ u(t=s,x) = a\phi_\gamma(x) \\ \partial_t u(t=s,x) = b\phi_\gamma(x) \end{aligned} \tag{4.5}$$

is well approximated by

$$u(t,x) = \frac{a}{2}\left(\phi^+_\gamma(t,s,x) + \phi^-_\gamma(t,s,x)\right) + \frac{b}{2}\left(2^{-k}\vartheta^+(t,s,x) - 2^{-k}\vartheta^-(t,s,x)\right)$$

(Recall that $\phi_\gamma \to \phi_\gamma(t,x)$ should be thought of like $e^{it|\nabla|}\phi_\gamma$). The fact that we used the same ϕ_γ for the initial u and $\partial_t u$ is just for notational convenience and have no advantage whatsoever. In fact, we would like to define the operators $\mathbf{c}(t,s)$ and $\mathbf{s}(t,s)$ as sums in γ of the first and the second half of the left hand side of the above definition of u. The definition of $\mathbf{c}(t,s)$ will be more or less straightforward. Whereas that of $\mathbf{s}(t,s)$ will require a bit of work.

First, for $k > 0$, we define:

$$\widehat{\vartheta_\gamma}(\xi) = -2^k \langle \omega_\gamma, \xi \rangle^{-1} \widehat{\phi_\gamma}(\xi)$$

This should be thought of just as ϕ_γ itself for all practical purposes. Then we define the forward and backward flow given by the Hamiltonian system as before with initial time $t = s$. That is we define $(x^{\pm}(t,s), \omega_\gamma^{\pm}(t,s), \Theta^{\pm}(t,s))$ as the forward (+) and backward (-) flow of the Hamiltonian $H_k(t,x,\xi)$ with initial data at $t = s$ given by $(x_\gamma, \omega_\gamma, Id)$. Also we define the flow $\chi_{t,s}^{\pm}(x,\omega)$ corresponding to the forward and backward flow by $H(t,x,\xi)$. This will be used to define the canonical transformation used to define the class $M^r(\chi_{t,s}^{\pm})$ of matrices of operators.

As before we define

$$\phi_\gamma^{\pm}(t,s,x) = \phi_\gamma(\Theta^{\pm}(y - x_\gamma^{\pm}(t,s)) + x_\gamma)$$

and

$$\vartheta_\gamma^{\pm}(t,s,x) = \frac{1}{H_k(s,x_\gamma,\omega_\gamma)}\vartheta_\gamma(\Theta^{\pm}(y - x_\gamma^{\pm}(t,s)) + x_\gamma)$$

The extra factor in the front is needed to gaurantee the property that $\partial_t\theta$ (or the operators concocted from it) will approximate well the identity better (see coming calculations). As a result, one gets that:

$$\begin{aligned} \partial_t\phi_\gamma(t,s,y) &= \mp L(t, x_\gamma^{\pm}(t,s), \omega_\gamma^{\pm}(t,s), y, \partial_y)\phi_\gamma^{\pm}(t,s,y) \\ \partial_t\vartheta_\gamma^{\pm}(t,s,y) &= \mp L(t, x_\gamma^{\pm}(t,s), \omega_\gamma^{\pm}(t,s), y, \partial_y)\vartheta_\gamma^{\pm}(t,s,y) \end{aligned} \tag{4.6}$$

Definition 4.3. If $c(\gamma') = \int_{\mathbb{R}^n} \overline{\phi_{\gamma'}(y)} f(y)dy$, we set:

$$\mathbf{c}_k(t,s) = \frac{1}{2}\sum_{k_{\gamma'}=k} c(\gamma')\left(\phi_\gamma^{+}(t,s) + \phi_\gamma^{-}(t,s)\right)$$

$$\mathbf{s}_k(t,s) = \frac{1}{2}\sum_{k_{\gamma'}=k} c(\gamma')\left(2^{-k}\vartheta_\gamma^{+}(t,s) + 2^{-k}\vartheta_\gamma^{-}(t,s)\right)$$

As a consequence of what we have done in the previous section, we directly get:

Theorem 4.4. (i) $\sum_k \boldsymbol{c}_k(t,s) \in I^0(\chi_{t,s}^{+}) \oplus I^0(\chi_{t,s}^{-})$
(ii) $\sum_k \boldsymbol{s}_k(t,s) \in I^{-1}(\chi_{t,s}^{+}) \oplus I^{-1}(\chi_{t,s}^{-})$
(iii) $\sum_k \left(\partial_t^2 + A_k(t,x,\partial_y)\right)\boldsymbol{c}_k(t,s) : H^\alpha \to H^{\alpha-1}$
(iv) $\sum_k \left(\partial_t^2 + A_k(t,x,\partial_y)\right)\boldsymbol{s}_k(t,s) : H^\alpha \to H^\alpha$

Futhermore the constants appearing in the definition of M_δ^r are uniformly bounded (for each δ) provided s and t vary over compact intervals.

The properties are either done or follow by repeating the analysis we did in the previous sections.

We are now ready to define the operators

$$\mathbf{c}(t,s) = \sum_k \mathbf{c}(t,s)$$

Notice that this is a bounded operator on H^α for any α. It is actually continuous in t and s in the strong operator topology (this can be seen by the fact that $\mathbf{c}(t,s)f$ is the uniform limit in H^α of finite sums of the form $\sum_\gamma c(\gamma)\phi_\gamma(t,s,x)$ which are continuous in t and s). Similarly, one gets that the distributionally defined $\partial_t^k\,\mathbf{c}(t,s)$ defines a continuous in s and t family of operators from $H^\alpha \to H^{\alpha-k}$ and that:

$$\mathbf{c}(t,s)|_{t=s} = Id \qquad \partial_t\, \mathbf{c}(t,s)|_{t=s} = 0$$

(this last property follows from the opposite signs of the forward and backward flows.)

We now turn to the definition of $\mathbf{s}(t,s)$ which is a bit more complicated since we don't have exactly that $\partial_t \vartheta^{\pm}(t,s,x)|_{t=s} = \phi_\gamma$. However, we notice that if $\sum_k \partial_t\, \mathbf{s}(t,s)|_{t=s}$ is invertible we can multiply by this inverse and hope that it is bounded on Sobolev spaces. This will give the result. This can be done as long we truncate the sum at some k_0 and define the operator $\partial_t \tilde{\mathbf{s}}(t,s)$ to be the identity for frequencies less than k_0 and $\sum_{k \geqslant k_0} \mathbf{s}_k(t,s)$ otherwise.

In fact, if we denote:

$$\Delta_k f(y) = \sum_{k_\gamma = k} c(\gamma)\phi_\gamma(y)$$

where $c(\gamma) = \int_{\mathbb{R}^n} \overline{\phi_\gamma(y)} f(y) dy$

$$\tilde{\mathbf{s}}(t,s) = \sum_{k \geqslant k_0} \mathbf{s}_k(t,s) + (t-s) \sum_{k<k_0} \Delta_k$$

This operator will have an invertible partial derivative that satisfies all the needed properties.

Lemma 4.5. *There exists k_0 depending only on the bounds for $A(t,x)$ and in particular independent of s so that the operator $\partial_t \tilde{\mathbf{s}}(t,s)|_{t=s}$ admits a bounded inverse on $L^2(\mathbb{R}^n)$. Also this inverse extends to a bounded operator on H^α for all α and is continuous in s in the operator norm topology.*

Proof. This is done by showing that $R(s) = Id - \partial_t \tilde{\mathbf{s}}(t,s)|_{t=s}$ has an operator norm less than 1 on $L^2(R^n)$.The last property of the proof follows by noticing that the $R(s)$ is actually smoothing of order $\frac{1}{2}$ and that:

$$(Id - R(s))^{-1} = \sum_0^{2N-1} R(s)^j + R(s)^N (Id - R(s))^{-1} R(s)^N$$

Using this expression for N large enough and the fact that $(Id - R(s))^{-1}$ is bounded on $L^2(\mathbb{R}^n)$ we get the last statement (The fact that $R(s)$ is continuous in the operator norm topology as an operator from $H^\alpha \to H^{\alpha+1/2-\epsilon}$ gives the continuity in the operator norm of $(Id - R(s))^{-1}$ in the operator norm topology of operators from $H^\alpha \to H^\alpha$).

So the main thing is to show that $Id - \partial_t \tilde{\mathbf{s}}(t,s)|_{t=s}$ has an operator norm less than 1 on $L^2(R^n)$. For this we notice that by (4.6) and the equation (3.8 of the paper (the one that expresses L as an order 1 operator +operators of order $\leqslant \frac{1}{2}$ when applied to ϕ_γ) we get:

$$2^{-k} \partial_t \vartheta^{\pm}_\gamma(t,s,y)|_{t=s} = \phi_\gamma + \text{operators of order} \leqslant -\frac{1}{2} \text{ when applied to } \phi_\gamma$$

As a result, $R(s) \in I^{-1/2}(Id)$ with uniform bounds as s varies and for sufficiently large k_0 can be bounded in $L^2(\mathbb{R}^n)-$operator norm by < 1 which means it is invertible. □

Finally we set:

$$\mathbf{s}(t,s)=\tilde{\mathbf{s}}(t,s)(\tilde{\mathbf{s}}(t,s)|_{t=s})^{-1}$$

This family of operators is a strongly continuous family from $H^\alpha \to H^{\alpha+1}$ in both t and s. Moreover, $\partial_t^k \mathbf{s}(t,s)$ (defined in the sense of distributions) is a strongly continuous family of operators from $H^\alpha \to H^{\alpha+1-k}$ and we have:

$$\mathbf{s}(t,s)|t=s=0 \qquad \partial_t \mathbf{s}(t,s)|_{t=s}=Id$$

Finally we prove Theorem 4.1

Proof of Theorem 4.1:

By Theorem 4.4 it is enough to show that:

$$\sum_k (a_{ij}(t,x)-a_{ij}^k)\partial_i\partial_j \mathbf{c}(t,s): H^{\alpha+1}\to H^\alpha$$

$$\sum_k (a_{ij}(t,x)-a_{ij}^k)\partial_i\partial_j \mathbf{s}(t,s): H^{\alpha}\to H^\alpha$$

By splitting the sum into odd and even k, it suffices to prove that the operator:

$$T: \sum_{keven}(a_{ij}(t,x)-a_{ij}^k))\beta_k(D)$$

where $\beta_k(D)$ is a relevent Littlewood-Paley cutoff is bounded from $H^{\alpha-1}\to H^\alpha$. Suppressing the t and ij indices, we consider the operator

$$T_{ij}=\beta_j(D)(a_(x)-a^k(x))\beta_k(D)$$

Using the fact that $||a-a^k||_\infty \leqslant 2^{-k}$ as was already proved, the support properties of $a-a^k$ we get that $T_{ij}=\beta_j(D)(a-a^{2k-3})\beta_k(D)$ if $j\leqslant k-3$ and $=\beta_j(D)(a-a^{2j+3})\beta_k(D)$ if $j\geqslant k+3$ and hence the bounds $||T_{ij}||_{L^2\to L^2}\lesssim 2^{-2k}$ if $j\leqslant k-3$, 2^{-k} if $|j-k|\lesssim 1$ and 2^{-2j} if $j\geqslant k+3$ which gives the result for $\alpha=1$.

To the get the other $-1\leqslant\alpha\leqslant 2$ we use the identity:

$$[\partial_x,T]=\partial_x a(x)\sum_{keven}\beta_k(D)-\sum_{keven}(\partial_k a^k(x))\beta_k(D)$$

The fist term on the RHS is bounded on H^α since $\partial_x a$ is an H^α multiplier, whereas the second term is an $S^0_{1,1/2}$ pseudo-differential operator which is bounded on H^α. The result then follows using the fact that the result for $\alpha=1$ is true.

□

Remark. : Additional regularity on the coefficients would lead to increasing the range of α in the theorems above. In fact, if $a(t,x)\in L_t^\infty H^{s_0+1}$ for $s_0>n/2$ then the statement of the Theorem 4.2 holds for $s_0+1\geqslant\alpha\geqslant -s_0$. The proof is by a modification of the last analysis.

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