MASS CRITICAL NLS BELOW ENERGY NORM

ZAHER HANI

Abstract. These notes are loosely written to accompany a couple of talks at the UCLA participating analysis seminar. We will address the problem of proving global well-posedness of $2D$ cubic NLS below the energy norm H^1 as a prototype of the more general problem in the title. Despite the fact that there is more than one strategy to get global well-posedness below the energy norm, we will-due to time restrictions- only restrict our attention to the first of which, namely the so-called "Fourier Truncation" Argument of Bourgain.

1. Introduction and Setting up the problem

The general idea behind the argument is rather simple. We are dealing with a problem that is supercritical w.r.t. to energy norm H^1[1], the critical regularity is $s_c = 0$. We are also trying to prove global well-posedness at a regularity s that is also supercritical w.r.t. energy, i.e. $s < 1$. As a result of this the energy of the solution is usually infinite and hence one cannot exploit the Hamiltonian conservation directly to get results (Energy conservation is enough to give GWP for H^1 data). However, if we truncate the initial data in frequency at N, then the low frequency component is in H^1 and hence we can flow it by our equation globally. Bourgain observed that if we flow the low frequency component non-linearly and flowed the high frequency component via the unitary linear flow, then the difference between their sum and the real solution is in H^1 and we can obtain polynomial bounds on the H^1 norm of this difference.

To be concrete let us consider the cubic NLS on $\mathbb{R}^2$:

$$i\partial_t u(t,x) + \Delta u(t,x) = |u(t,x)|^2 u(t,x),\ (t,x) \in \mathbb{R} \times \mathbb{R}^2 \tag{1.1}$$

$$u(0,x) = \phi \in H^s \tag{1.2}$$

The equation is locally well-posed for $s > 0$ in a subcritical manner, i.e. the time of existence guaranteed by the local theory depends only on the H^s norm of the initial data rather than its profile [2]

The equation enjoys the following conservation laws:

[1]The reference norm need not be H^1, in fact [CS] use the mass conservation to get GWP results for KdV equation for negative Sobolev exponents, whose local well-posedness was proved for $s > -3/4$ by Kenig, Ponce, and Vega [KPV] in the non-periodic case and $s > -1/2$ in the periodic case

[2]The equation is locally wellposed in L^2, however the local wellposedness result we have depends on the profile of the initial data in addition to its size in L^2.

- Conservation of mass

$$M[u](t) := \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M[u](0) \tag{1.3}$$

- Conservation of momentum

$$\overrightarrow{p}[u](t) = \Im \int_{\mathbb{R}^2} \nabla u(t,x) \bar{u(t,x)} dx = \overrightarrow{p}[u](0) \tag{1.4}$$

- Conservation of the Hamiltonian

$$E[u](t) := \int_{\mathbb{R}^d} |\nabla u(t,x)|^2 + \frac{1}{p}|u(t,x)|^p dx = E[u](0) \tag{1.5}$$

These are easily checked formally and can be justified for more general strong solution by usual limiting arguments (see [T]).

The conservation of the Hamiltonian above gives instantly global well-posedness in H^1. Standard persistence of regularity arguments give global wellposedness in H^s for all $s \geqslant 1$.

Bourgain's Fourier Truncation argument which we present below was the first GWP result in H^s for $0 < s < 1$. One should note that global well-posedness for $s = 0$ is still open (in fact it would imply GWP for all $s \geqslant 0$) and is widely regarded as a relatively difficult problem[3].

As mentioned above, the rough idea is to write $\phi = \phi_N + \phi^N$ where $\phi_N = P_{\leqslant N}\phi$ and $\phi^N = \phi - \phi_N = P_{>N}\phi_N$ where $P_{\leqslant N}$ denotes the "usual" Littlewood-Paley projection on frequencies $|\xi| \lesssim N$ and N is to be chosen later depending of the time of existence T which is fixed but arbitrary. We will flow ϕ_N using the nonlinear equation (1.1) to get $u_0(t)$ which is exists globally since $\phi_N \in H^1$ and flow ϕ^N using the linear equation:

$$i\partial_t u + \Delta u = 0 \tag{1.6}$$

to get $e^{it\Delta}\phi^N$ which is in H^s and $||e^{it\Delta}\phi||_{H^s} = ||\phi^N||_{H^s}$. While the sum of those two solutions is far from being a solution to (1.1), the difference between the real solution u and the sum of these two solutions $u_0(t) + e^{it\Delta}\phi^N$ can be shown to be smoother, in fact in H^1, which means that $||u(t)||_{H^s}$ cannot blowup on $[0,T]$. Since T was arbitrary, we get global wellposedness in H^s. This shows among other things that the power-type nonlinearity has some sort of a smoothing effect and that the only part of the solution that is in H^s rather than H^1 is the linear flow of high frequency components.

While this rough idea works directly for some equations like the string equation [B], the argument for the $2D$ cubic NLS is more involved.

The theorem we will prove is the following:

[3]Conservation of mass does not directly give GWP as with the Hamiltonian because the local well-posedness at $s = 0$ is critical, i.e. depends on the profile of the data rather than solely on its L^2 norm.

Theorem 1.1. [B1] *The equation* (1.1),(1.2) *for* $s > 2/3$ *is globally wellposed and satisfies* $u(t) - e^{it\Delta}\phi \in H^1$. *In fact,*

$$||u(t) - e^{it\Delta}\phi||_{H^1(\mathbb{R}^2)} \lesssim (1+|t|)^{\frac{1-s+}{3s-2}} \tag{1.7}$$

Technical refinements which we won't present here (see [B1]) allow one to weaken the assumption $s > 2/3$ to $s > 3/5$.

One should mention that after Bourgain results, Colliander, Keel, Staffilani, Takaoka, and Tao [CKSTT] were able to push the exponent down to $s > 4/7$ using the so-called "I-method". Several attempts succeeded at pushing the exponent closer to $s = 0$ using improvements on the I-method. I think that the record now is due to Dodson [D] at $s > 1/4$.

2. The Argument

Fix a time $T > 0$. $\phi = \phi_N + \phi^N$ as above where N is to be chosen later depending on T. We will first flow ϕ_N using the nonlinear equation (1.1). Indeed, if we denote this flow by u_0, then:

$$i\partial_t u_0(t,x) + \Delta u_0(t,x) = |u_0(t,x)|^2 u_0(t,x) \tag{2.1}$$

$$u_0(0,x) = \phi_N =: \phi_0 \tag{2.2}$$

Since $||\phi_N||_{H^1} \lesssim N^{1-s}||\phi_N||_{H^s} \lesssim N^{1-s}$, $u_0(t,x)$ exists globally and satisfies thanks to the conservation of mass and energy $||u_0(t)||_{L^2} \lesssim 1$ and $||u_0(t)||_{H^1} \lesssim N^{1-s}$.

Now we will restrict our attention to an interval $I = [0,\delta]$ such that $||u_0||_{L^4_{t,x}(I)} = o(1)$. In fact, since $||u_0(t)||_4 \lesssim ||u(t)||_2^{1/2}||u(t)||_{H^1}^{1/2} \lesssim N^{\frac{1-s}{2}}$ we just need to take $\delta \ll N^{-2(1-s)}$ say $\delta \sim N^{-2(1-s)-}$. This actually implies that $u(t) \in C_t^0 H^s([0,\delta]\times\mathbb{R}^2)$, since $\delta \ll ||\phi||_{H^s}^{-2}$.

For $t \in I$, write $u(t) = u_0(t) + v(t)$. Then $v(t)$ satisfies the difference equation given by:

$$i\partial_t v + \Delta v = 2|u_0|^2 v + u_0^2\overline{v} + \overline{u_0}v^2 + 2u_0|v|^2 + |v|^2 v \tag{2.3}$$

$$v(t) = \phi^N \tag{2.4}$$

By Duhamel's formula we have:

$$v(t) = e^{it\Delta}\phi^N + w \tag{2.5}$$

where:

$$w(t) = i\int_0^t e^{i(t-s)\Delta}\left(2|u_0|^2 v + u_0^2\overline{v} + \overline{u_0}v^2 + 2u_0|v|^2 + |v|^2 v\right) \tag{2.6}$$

and hence:

$$u(t) = u_0(t) + e^{it\Delta}\phi^N + w(t) \tag{2.7}$$

Thus we have written u as the sum of the nonlinear flow of ϕ_N, the linear flow of ϕ^N, plus an error $w(t)$. As mentioned in our description of the strategy, we will show that this error is actually smooth, in fact in H^1. This will allow us to define for $t_1 = \delta$:

$$\phi_1 = u_0(t_1) + w(t_1) \tag{2.8}$$

and repeat the above replacing $\phi_0 = \phi_N$ with ϕ_1 and $\psi_0 = \phi^N$ with $\psi_1 = e^{it_1\Delta}\psi_0$. However, in order to be able to do that we need to make sure that ϕ_1 satisfies the same type of estimates that ϕ_0 did. This will be the case if we are able to prove the following:

Proposition 2.1. *w satisfies the following for all $t \in I$:*

(i) $w(t) \in H^1$
(ii) $||w(t)||_{L^2} \lesssim N^{-s}$
(iii) $||w(t)||_{H^1} \lesssim N^{1-2s+}$

Notice that this proposition can be interpreted by saying that (1.1) acts on the high frequency components almost linearly which is a manifestation of the smoothing effect of the nonlinearity.

In fact, with this proposition we are guaranteed that we can run the second iteration for a time interval of length δ since $||\phi_1||_2 \lesssim 1$ and $||\phi_1||_{H^1} \lesssim N^{1-s}$. Actually, we will be iterate the process as long as we can guarantee this. After, $\sim T/\delta$ iterations, the L^2 norm of the ϕ_i increases by $\sim \frac{T}{\delta}N^{-s} \lesssim TN^{2(1-s)+}N^{-s} = TN^{2-3s+}$ whereas the H^1 norm increases by an amount $TN^{2(1-s)+}N^{1-2s+} = TN^{3-4s+}$ which is still $\ll N^{1-s}$ as long as $s > 2/3$ and N is taken to be equal to $N \sim T^{\frac{1}{3s-2-}}$. One can also calculate the increase in the Hamiltonian at each step of the iteration:

$$\begin{aligned} H(\phi_1) - H(\phi_0) =& H(u_0(t_1) + w(t_1)) - H(u_0(t_1)) \\ \lesssim& (||u_0(t_1)||_{H^1} + ||w(t_1)||_{H^1})||w(t_1)||_{H^1} + ||(|u_0(t_1)|^3 + |w(t_1)|^3)w(t_1)||_{L^1} \\ \lesssim& N^{1-s}N^{1-2s+} + (||u_0(t_1)||_{L^4}^3||w(t_1)||_{L^4} + ||w(t_1)||_{L^4}^3||w(t_1)||_{L^4}) \\ \lesssim& N^{2-3s+} + N^{\frac{3(1-s)}{2}}||w(t_1)||_{L^2}^{1/2}||w(t_1)||_{H^1}^{1/2} \\ \lesssim& N^{2-3s+} + N^{\frac{3(1-s)}{2}}N^{-s/2}N^{\frac{1-2s+}{2}} \lesssim N^{2-3s+} \end{aligned}$$

As a result of this calculation, we get that after $\sim T/\delta = TN^{2(1-s)+}$ iterations, the Hamiltonian has become possibly as large as $TN^{2(1-s)+}N^{2-3s+} = TN^{4-5s+}$ which is still

$\lesssim N^{2(1-s)}$ (which is the condition that guarantees the repetition of the process) as long as $TN^{2-3s+} \ll 1$ which means that we can take $N \sim T^{\frac{1}{3s-2-}}$. Finally, as a result of this $||u(t) - e^{it\Delta}\phi||_H^1(\mathbb{R}^2) \lesssim T^{\frac{1-s}{3s-2-}}$ which completes the proof of the theorem.

3. Proof of Proposition 2.1

In this section we will prove proposition 2.1. The key estimate will be Proposition 3.1 below which is a key to proving the smoothing effect of the nonlinearities appearing in (2.6).

We start by proving the second part of the proposition which is the easiest. By Strichartz estimates, we have:

$$||w||_{L_t^\infty L_x^2(I\times\mathbb{R}^2)} \lesssim || \left(2|u_0|^2 v + u_0^2\overline{v} + \overline{u_0}v^2 + 2u_0|v|^2 + |v|^2 v\right) ||_{L^{4/3}_{t,x}}$$
$$\lesssim ||u_0||^2_{L^4_{t,x}} ||v||_{L^4_{t,x}} + ||u_0||_{L^4_{t,x}} ||v||^2_{L^4_{t,x}} + ||v||^3_{L^4_{t,x}}$$

Also by Strichartz, we have:

$$||v||_{L^4_{t,x}} \lesssim ||\phi^N||_{L^2} + ||u_0||^2_{L^4_{t,x}} ||v||_{L^4_{t,x}} + ||u_0||_{L^4_{t,x}} ||v||^2_{L^4_{t,x}} + ||v||^3_{L^4_{t,x}}$$

since $||\phi^N||_{L^2} \lesssim N^{-s}$ and $||u_0||_{L^4_{t,x}} = o(1)$, we get that $||v||_{L^4_{t,x}} \lesssim N^{-s}$, which gives when substituting in the $||w||_{L_t^\infty L_x^2}$ inequality that $||w||_{L_t^\infty L_x^2} \leqslant N^{-s}$.

In order to prove the rest of the proposition, we will need to use $X^{s,b}$ norms. We define $X^{s,b}$ as the closure of Schwartz functions under the norm:

$$||u||_{X^{s,b}} = \left(\int_{\mathbb{R}} \int_{\mathbb{R}^2} \langle \xi \rangle^{2s} \langle \tau + |\xi|^2 \rangle^{2b} |u(\tau,\xi)|^2 d\tau d\xi\right)^{1/2} \tag{3.1}$$

Most often $X^{s,b}$ is used as a restriction norm to a time interval of finite length, i.e. we define $||u||_{X_I^{s,b}} = inf||\psi||_{X^{s,b}}$ where the infimum is taken over all ψ that agree with u on I.

The solution u_0 of (2.1) satisfies:

$$||u_0||_{X_I^{0,1/2+}} \lesssim 1 \text{ and } ||u_0||_{X_I^{1,1/2+}} \lesssim ||\phi_0||_{H^1} \lesssim N^{1-s} \tag{3.2}$$

which gives by interpolation:

$$||u_0||_{X^{1/2+,1/2+}} \lesssim N^{(1-s)/2+} \tag{3.3}$$

Similarly, one can show using standard arguments:

$$||v||_{X^{0,1/2+}} \lesssim N^{-s} \text{ and } ||v||_{X^{s,1/2+}} \lesssim 1 \tag{3.4}$$

The key point in being able to prove that $w \in H^1$ and the more general theme that power-type nonlinearity is somehow smoothing is the following refinement of Strichartz estimate:

Proposition 3.1. *Let $\psi_1, \psi_2 \in L^2(\mathbb{R}^2)$ and suppose that their Fourier transforms are supported on $|\xi| \sim N_1$ and $|\xi| \sim N_2$ for some dyadic integers $N_1 \leqslant N_2$. Then*

$$||e^{it\Delta}\psi_1 e^{it\Delta}\psi_2||_{L^2(\mathbb{R}\times\mathbb{R}^2)} \lesssim \frac{N_1}{N_2}||\psi_1||_{L^2(\mathbb{R}^2)}||\psi_2||_{L^2(\mathbb{R}^2)} \tag{3.5}$$

Note that old Strichartz does not give the factor of $\frac{N_1}{N_2}$. An immediate corollary of this proposition is the following:

Corollary 3.2. (i) *For $0 \leqslant s < 1/2$:*

$$||D_x^s\left(e^{it\Delta}\psi_1 e^{it\Delta}\psi_2\right)||_{L^2(\mathbb{R}\times\mathbb{R}^2)} \lesssim ||\psi_1||_H^s||\psi_2||_L^2 \tag{3.6}$$

(ii) *For $1/2 \leqslant s$:*

$$||D_x^s\left(e^{it\Delta}\psi_1 e^{it\Delta}\psi_2\right)||_{L^2(\mathbb{R}\times\mathbb{R}^2)} \lesssim ||\psi_1||_H^s||\psi_2||_L^2 + ||\psi_1||_H^{1/2+}||\psi_2||_H^{s-1/2} \tag{3.7}$$

(iii)

$$||D_x^{1/2-}u_1u_2||_L^2(I) \lesssim ||u_1||_X^{1/2,1/2+}||u_2||_{X^{0,1/2-}} \tag{3.8}$$

(iv)

$$||D_x^{1/2+}u_1u_2||_L^2(I) \lesssim ||u_1||_X^{1/2+,1/2+}||u_2||_{X^{0+,1/2+}} \tag{3.9}$$

The proof of this corollary is done by decomposing into Littlewood-Paley pieces and using Cauchy-Schwartz quite often. In order to prove the third part on need to use the estimate $||u||_{L_t^4L_x^2} \lesssim ||u||_{X^{0,1/4}}$ which results from Hausdorff-Young.

With these estimates at hand we are ready to complete the proof of proposition 2.1.

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant\delta} ||w||_{H^1} \lesssim ||w||_{X_\delta^{1,1/2+}} \\
&\lesssim ||\left(2|u_0|^2v + u_0^2\overline{v} + \overline{u_0}v^2 + 2u_0|v|^2 + |v|^2v\right)||_{X^{1,-1/2+}} \\
&= \sup_{||W||_{X^{0,1/2-}}} \langle D_x\left(2|u_0|^2v + u_0^2\overline{v} + \overline{u_0}v^2 + 2u_0|v|^2 + |v|^2v\right), W\rangle
\end{aligned}$$

It turns out that the hardest terms to bound are the ones that are linear in v. Others are bounded similarly making use of the fact that their smallness.

$$\langle D_x(|u_0|^2v), W\rangle = \langle D_xu_0v, u_0W\rangle + \langle D_x(u_0v), \overline{u_0}W\rangle$$

The first term above is bounded easily as follows:

$$|\langle D_x u_0 v, u_0 W\rangle| \leqslant ||D_x u_0||_{L^4}||v||_{L^4}||u_0||_{L^{4+}}||W||_{L^{4-}} \lesssim ||u_0||_{X^{1,1/2+}}||v||_{X^{0,1/2+}}||u_0||_{X^{0+,1/2+}}||W||_{X^{0,1/2-}}$$

where in the last inequality we have used Strichartz estimates and Sobolev embedding to get that $||u_0||_{L^{4+}} \lesssim ||u_0||_{L_t^{4+}W_x^{0+,q}} \leqslant ||u_0||_{X^{0+,1/2+}}$ where $q = 4-$ is chosen so that $(4+, q)$ is Schrodinger admissible. As for the bound on W, we just interpolated Strichartz estimates with the trivial $X^{0,0} = L^2$ equality. Hence, we get:

$$|\langle D_x u_0 v, u_0 W\rangle| \lesssim N^{1-s}N^{-s}N^0+ = N^{1-2s+}$$

For the second term, we have:

$$\langle D_x(u_0 v), \overline{u_0}W\rangle = \langle D_x^{1/2+}(u_0 v), D_x^{1/2-}\overline{u_0}W\rangle$$
$$\lesssim ||u_0||_{X^{1/2+,1/2+}}||v||_{X^{0+,1/2+}}||u_0||_{X^{1/2,1/2+}} \leqslant N^{1-2s+}$$

where we have used Corollary 3.2 and interpolated the two estimates of (3.4). The bounds on the other terms are done similarly.

References

[B] J. Bourgain, *Global Solutions of Nonlinear Schrödinger Equations*, AMS Colloquim Publications 46.

[B1] J. Bourgain, *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, IMRN 1998, N5, 253-283

[CS] J. Colliander, G. Staffilani, H. Takaoka *Global Wellposedness of KdV below L^2*, Math. Res. Lett. 6 (1999), no. 5-6, 755–778.

[C] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of mathematical Sciences, AMS, 2003

[CKSTT] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Almost conservation laws and global rough solutions to a nonlinear Schrodinger equation*, Math. Res. Letters 9 (2002), 659-682

[D] B. Dodson, *Improved almost Morawetz estimates for the cubic nonlinear Schrodinger equation*, preprint arXiv:0909.0757v2

[KPV] C. Kenig, G. Ponce, L. Vega, *A bilinear estimate with applications to the KdV equation*, J. AMS, 9:573-603, 1996

[T] T. Tao, *Nonlinear Dispersive Equation: Local and Global Analysis*, Regional Conference Series in Mathematics, 106. American Mathematical Society, Providence, RI, 2006

UCLA Department of Mathematics, Los Angeles, CA 90095-1555.

E-mail address: zhani@math.ucla.edu