1 SOME NOTES

I first planned to write this list with one specific example to every method or definition. But then I just found out that time is far from enough for me to do this. Anyway the examples in our textbook are very suitable. So I provide a list showing you where exactly you can find the important definitions, theories or summaries of method as well as examples in your book. I strongly suggest you go back to your textbook for those problems and exercises. And also, being familiar with every method and formula to every kind of differential equations will make life much easier. It is better you may recite some formulas. And for time limited, it is very possible I made some mistakes somewhere and also I may forget some points. Please feel free to tell me.

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Wish everybody will get what you deceive to get after hard working!
At last good luck to your finals week!

2 FIRST-ORDER EQUATIONS

2.1 SEPARABLE EQUATIONS

• Summary of the Method on Page 30.

Example(Page 35.12): Find the general solution for \( y' = \frac{2xy + 2x}{x^2 - 1} \).
from the equation:

\[
\frac{y'}{y+1} = \frac{2x}{x^2-1}
\]

(do integration)

\[
\log(y + 1) = \int \frac{2x\,dx}{x^2-1}
\]

\[
\log(y + 1) = \int \frac{dx^2}{x^2-1}
\]

\[
\log(y + 1) = \log(x^2 - 1) + C'
\]

\[
y + 1 = C(x^2 - 1)
\]

2.2 LINEAR EQUATION \( y' = a(t)x + f(t) \)

- Summary of the First Method on Page 49.
- Example 4.23 on Page 50.
- Structure of Solutions on Page 54. THEOREM 4.1.

2.3 EXACT DIFFERENTIAL EQUATIONS \( P\,dx + Q\,dy = 0 \)

- How to tell a differential form is exact? on Page 66 THEOREM 6.20.
- Solving exact differential equations on Page 67.
- Definition of integrating factor and how we use it to solve a differential equation on Page 69-70.

Example 6.39 on Page 72.

Example (Page 75.23) Given integrating factor \( \mu(x, y) = \frac{1}{xy} \), find the solution of

\[
(x^2 y^2 - 1)\,y\,dx + (1 + x^2 y^2)\,x\,dy = 0.
\]

After multiplying \( \mu \), we get:

\[
\frac{x^2 y^2 - 1}{x} \,dx + \frac{(1 + x^2 y^2)}{y} \,dy = 0
\]

\[
\frac{d \left( \frac{x^2 y^2 - 1}{x} \right)}{dy} = xy = \frac{d \left( \frac{1 + x^2 y^2}{y} \right)}{dx}
\]

So \( \frac{x^2 y^2 - 1}{x} \,dx + \frac{(1 + x^2 y^2)}{y} \,dy \) is exact. Suppose it equals \( dF(x, y) \). Then

\[
\frac{dF}{dx} = \frac{x^2 y^2 - 1}{x}
\]

\[
F(x, y) = \int \frac{x^2 y^2 - 1}{x} \,dx = \int xy^2 - \frac{1}{x} \,dx = \frac{1}{2} x^2 y^2 - \log x + \phi(y).
\]
We have \( \frac{dF}{dy} = \frac{1+x^2y^2}{y} \), then

\[
\frac{dF}{dy} = x^2 y + \phi'(y) = x^2 y + \frac{1}{y}
\]

\[
\phi = \int \frac{1}{y} dy = \log y + C'
\]

\[
F(x, y) = \frac{1}{2} x^2 y^2 - \log x + \log y + C'
\]

The solution is: \( F(x, y) = C'' \), which is

\[
\frac{1}{2} x^2 y^2 - \log x + \log y = C
\]

- Homogeneous equations. (You can also solve for equations which can be transformed to homogeneous equations.)

Example 6.40 on Page.

### 2.4 EXISTENCE AND UNIQUENESS OF SOLUTIONS

- Be familiar with THEOREM 7.6 and THEOREM 7.16 on Page 78 and Page 82 respectively.
- Some equations have no solution or more than one solutions. Make sure you know that how they do not contradict with our Theorems.
- Dependence of solutions on initial data. THEOREM 7.15 on Page 81.

Example 7.1 on Page 77.
Example 7.18 on Page 83.

### 2.5 STABILITY OF AUTONOMOUS EQUATIONS

- Make sure you know how to find equilibrium points and draw a graph for phase line to show their stability.

Example 9.6 on Page 95.

### 3 SECOND-ORDER EQUATIONS \( y'' + py' + qy = f \)

This chapter is very important!!

#### 3.1 GENERAL THEORY

- THEOREM 1.23 on Page 142 very important!
3.2 Homogeneous Equations with Constant Coefficients

- Summary of the method can be found in PROPOSITION 3.3 PROPOSITION 3.11 PROPOSITION 3.18 on Page 156.

3.3 Inhomogeneities Equations

- Method of undetermined coefficients on Page 172 Table 1. You can try this method for non-constant coefficients equations.

Examples can be found on Page 166-171.

- Variation of Parameters. Better if you can remember the formula 6.16 on Page 176. (Or you may know need to how to derive this formula.)

Example (on Page 177. 13) Verify $y_1(t) = t$, $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2 y'' + 3t y' - 3y = 0.$$ 

Use variation of parameters to find the general solution to

$$t^2 y'' + 3t y' - 3y = \frac{1}{t}$$

First divide the equation by $t^2$ to get a formal form of second order differential equation.

$$y'' + \frac{3}{t} y' - \frac{3}{t^2} y = \frac{1}{t^3}$$

Checking for $y_1$ is left for you as exercise. Checking for $y_2$:

$$y'' + \frac{3}{t} y' - \frac{3}{t^2} y = (12t^{-5}) + \frac{3}{t}(-3t^{-4}) - \frac{3}{t^2}(t^{-3}) = 0$$

Calculate the Wronskian: $W(t, t^{-3}) = -4t^{-3}$.

Then use the formula 6.16:

$$v_1(t) = \int \frac{-y_2 g dt}{W} = -\frac{1}{8} t^{-2}.$$ 

(usually from integration there will be a constant, but here we only need one $v_1$, so choose the constant to be 0.)

$$v_2(t) = \int \frac{y_1 g dt}{W} = -\frac{1}{8} t^2.$$
Here $g(t) = \frac{1}{t^2}$. Then

$$yp = v_1y_1 + v_2y_2 = -\frac{1}{4t}.$$ 

Then general solution is

$$y(t) = C_1t + C_2\frac{1}{t^3} - \frac{1}{4t}.$$ 

## 4 Linear Systems with Constant Coefficients

- Definition for characteristic polynomial on Page 374.

### 4.1 Planar Systems

To solve the planar systems, you need to look at the characteristic equation and then according to their different eigenvalues we have 3 cases.

- Distinct real eigenvalues. THEOREM 2.9 on Page 380.
  
  Example 2.10 on Page 380.

- Complex eigenvalues. THEOREM 2.25 on Page 383.
  
  Example 2.26 on Page 384.

- One real eigenvalue of multiplicity 2. Look at THEOREM 2.40 on Page 388. Important case!
  
  Example of a simple case. Find the general solution of the system $y' = Ay$. Where

  $$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$ 

  Then from $\|AI - A\| = 0$, we have $\lambda_1 = \lambda_2 = 2$.

  $$2I - A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

  Choose two independent eigenvectors $e_1 = (1,0)^T, e_2 = (0,1)^T$, the general solution:

  $$y(t) = C_1e^{2t}(1,0)^T + C_2e^{2t}(0,1)^T.$$ 

### 4.2 Phase Plane Portraits

- Five types of Generic Equilibria points on Page 404.

- Make sure you know how to sketch a graph (or phase plane) for 4 cases above. Below is a picture I find on wiki, I think it is very good.
On Page 401, our textbook provide us a good way to decide the direction of rotation.

Some students asked me that if the two eigenvalues are purely imaginary numbers, then how can we draw a graph for this. So first we know that the picture should be ellipse. And my idea is that fix the two constants and draw maybe 4 points of one potential ellipse and sketch it out roughly.

4.3 Higher-Dimensional Systems and Exponential of a Matrix

Again homework problem is very important.
- You need to know the formula of the solutions for higher dimensional system containing the eigenvectors and eigenvalues of the matrix. And what happens when the algebraic dimension and the geometric dimension of eigenvalues are different!
  (So usually we don’t bother to use $e^{At}$ to find solutions. Just old way but a little bit more generalized one.)

- You need to know how to calculate $e^{At}$ when $A$ is a very simple matrix or when $(A - \lambda I)^n = 0$. In the latter case, by definition we only have finite sums.
  Make sure you know how to do examples in our textbook of this chapter.

5 Applications

Last but not least. Applications of first order linear differential equations are important.
To be specific, you need to know how to set up the system for "dropping solutions systems" and "string systems"!!

6  THANK YOU VERY MUCH!