1. (10 points) Obtain the first four terms of the Laurent expansion around $z = 0$:

$$f(z) = \frac{e^z}{z(z^2 + 1)}.$$
2. (10 points) Classify the type of singularities for the following functions. In part (a), write the Laurent expansion centered at the singularity.

(a) (5 points)

\[ ze^{\frac{1}{z-1}} - e^{\frac{1}{z-1}}. \]

(b) (5 points)

\[ \frac{z \sin z^2}{z^3 + z^2}. \]
3. (10 points) Let $C$ be the unit circle counter-clockwise oriented, compute the line integrals:

(a) \[ \int_C \frac{e^{az}}{z^m} \, dz \]
for any complex number $a$ and any integer $m$.

(b) \[ \int_C e^z \, dz \]
4. (10 points) Evaluate the following integrals. Name the theorem, each time you apply one.

(a) 
\[ \int_{|z|=2} \frac{e^z}{(z^2+5)^2} \, dz \]

(b) 
\[ \int_{|z-i|=2} \frac{e^z}{(z^2+5)^2} \, dz \]
(c) \[ \int_\gamma \frac{\sin z}{z} dz, \]
where \( \gamma \) is the unit circle centered at the origin.

(d) \[ \int_\gamma |z|^2 dz, \]
where \( \gamma \) is the simicircle \( \{ z : |z| = 3, \text{Im} z < 0 \} \).
5. (10 points) Let \( f(z) \) be an entire function. Assume that for any \( z \) in \( \mathbb{C} \),

\[
|f(z) + 1| > \frac{1}{2}.
\]

Show that \( f \) is a constant.
6. (10 points) Prove that any degree \( n \geq 1 \) polynomial

\[ p(z) = a_n z^n + \cdots + a_0 \]

with \( a_n \neq 0 \) has a zero in \( \mathbb{C} \).
7. (10 points) Find the residue of

\[ f(z) = \frac{z^2}{(z + 1)(z - 1)^2} \]

at \( z = 1 \).
8. (10 points) Show that

\[ \int_{-\infty}^{\infty} \frac{\cos x}{(1 + x^2)^2} \, dx = \frac{\pi}{e} \]
9. (10 points) Show that

\[ \int_{-\infty}^{\infty} \frac{x^2}{(1 + x^2)^2} \, dx = \frac{\pi}{2\sqrt{2}} \]
10. (10 points) Let $f$ be analytic on a domain $D$, let $z_0 \in D$ and $f'(z_0) \neq 0$. Show that

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{1}{f(z) - f(z_0)} dz$$

where $C$ is a small circle inside $D$ centered at $z_0$. 