1 Introduction

This article serves to illustrate the connection between the shape of a domain for integration, the bounds for integration and system of inequalities. The three are really the same thing. Well, I lied a little bit. I don’t care about shape in this article. I only care about the other two.

When we perform double, triple or multiple integrations in general, we are usually given a domain. The usual tactic is to observe the shape of the domain, and deduce the bounds for integration from the shape.

In practice and in most of the exercises, the domain is usually given by a bunch of inequalities. This system of inequalities will define your domain of integration. In theory, they illustrate some shape in the corresponding dimensions, of course. However, more often than not, you might not be able to visualize the domain, making your life very miserable and confusing.

In Section 2 I shall illustrate the general philosophy using a very simple example. In Section 3, 4, 5 I shall explain how to go from a system of inequalities to obtain bounds for integration, without visualization, purely by manipulating inequalities, using a fairly reasonable example. Finally in Section 6 I shall present another example where we need to cut the domain into smaller pieces.

The prerequisite of the skills presented here are minimal. However, I will assume that you have basic math accuracy about inequalities. For example, if I say $x^2 < 4$, you must know that this means $−2 < x < 2$. If I say $xy > y$, you shall know that this does NOT necessarily mean $x > 1$. If you don’t know these, you shall stop right here. You shall come to my office hour and ask me to review basic inequality manipulations with you. They are far more useful than calculus in real life.

I typed these up in a hurry, so there might be many mistakes. Contact me if you see any or if you have any confusions.

2 Why bounds are naturally inequalities

You can skip this section freely. It only serves to illustrate some simple philosophy.

Suppose we see an integration with bounds like this:

$$\int_0^1 \int_0^z \int_0^{z-y} \text{function } dx dy dz$$

What is the shape? Well to start, this is a triple integration. Think of it as three steps of integrations.

The first integration is against $dx$, and it has bounds from 0 to $z−y$. These are the $x$-bounds. By the very definition of bounds, they restrict the possible range of $x$. So we get the following inequalities for free:

$$0 \leq x \leq z−y$$

The second integration is against $dy$, with bounds from 0 to $z$. So we have $0 \leq y \leq z$. Finally the third integration tells me that $0 \leq z \leq 1$.

To sum up, the bounds for triple integration tells me the following inequalities:
1. $0 \leq x$
2. $x \leq z - y$
3. $0 \leq y$
4. $y \leq z$
5. $0 \leq z$
6. $z \leq 1$

This is in fact your domain of integration. You can draw it if you want.

Conversely, if you are given six inequalities of the following kind, then they readily translate into bounds:

1. An upper bound of $x$ independent of $x$ (i.e. inequality like $x \leq$ something)
2. A lower bound of $x$ independent of $x$ (i.e. inequality like $x \geq$ something)
3. An upper bound of $y$ independent of $x, y$
4. A lower bound of $y$ independent of $x, y$
5. An upper bound of $z$ independent of $x, y, z$
6. A lower bound of $z$ independent of $x, y, z$

I want you to remember that bounds and inequalities are the same thing. When you are given a system of inequalities for your domain, the most trustworthy way is to manipulate those inequalities, until you get some better looking inequalities directly usable as bounds. (This is the most trustworthy way, but might not be the fastest. The fastest is always the visualization.)

### 3 Starting our example

Suppose our domain is given by the following inequalities:

1. $xy + yz + zx \leq 2$
2. $y \geq 1$
3. $z \geq 1$
4. $y^2 + z^2 \leq 3$
5. $x \geq 0$
6. $x + e^y \geq 1$

How shall we find the bounds for triple integration?

We start by choosing any variable. Let’s say we choose $x$ to be the first variable to integrate. Then what we want to obtain is an upper bound of $x$ independent of $x$, and a lower bound of $x$ independent of $x$.

Look at all of our inequalities. If an inequality does not contain $x$, then leave it be. Otherwise, solve the inequality to be on $x$ (i.e. we want $x$ to be on one side, and everything else should be on the other side and independent of $x$). We shall get the following:

1. $x \leq \frac{2 - yz}{y + z}$
2. $y \geq 1$
3. $z \geq 1$
4. $y^2 + z^2 \leq 3$
5. $x \geq 0$
6. $x \geq 1 - e^y$

Lo and behold, we have one upper bound for $x$ and two lower bound for $x$. The upper bound is, of course, the upper bound of choice. Which lower bound shall we use?

Remember, the starting inequalities are simultaneously true. Therefore, both lower bounds are simultaneously true. If two inequalities are both true, then we shall use the more restrictive one, and the other one is in fact redundant. Therefore, we always want the LARGEST lower bound, and the SMALLEST upper bound. Stop for a minute and wrap your mind around these two slightly twisted concepts.

In our case, the two lower bounds are $0$ and $1 - e^y$, and we want the larger one. Since we know $y \geq 1$, we have $1 - e^y \leq 1 - e \leq 0$. Therefore $x \geq 0$ is more restrictive than $x \geq 1 - e^y$, and the latter is in fact redundant. So we discard it, and keep $0$ as our lower $x$-bound.

Now we have established our bounds for $x$. The innermost integration should look like this:

$$\int_0^{\frac{2-yz}{y+z}} \text{function } dx$$

## 4 Projections

Now we proceed to find the bounds for $y$. Remember, after throwing away the redundant inequality, we are left with the following inequalities:

1. $x \leq \frac{2-yz}{y+z}$
2. $y \geq 1$
3. $z \geq 1$
4. $y^2 + z^2 \leq 3$
5. $x \geq 0$

We would like to project everything to the $yz$-plane, and try to figure out the $y$-bounds from there. How would projections change inequalities?

Remember, for this moment, every inequality involving $x$ should be already solved to be on $x$. When the projection happens, we simply discard them, with one special requirement: The upper bound of $x$ shall be larger than or equal to the lower bound of $x$. If you think about this a little bit, this will make a lot of sense.

Now the inequalities will look like this:

1. $y \geq 1$
2. $z \geq 1$
3. $y^2 + z^2 \leq 3$
4. $0 \leq \frac{2-yz}{y+z}$

This system is the original shape projected to the $yz$-plane. We can simplify a little bit to get the following, remember that $y + z > 0$, so we can multiply this on both sides of the last inequality.
1. \( y \geq 1 \)
2. \( z \geq 1 \)
3. \( y^2 + z^2 \leq 3 \)
4. \( yz \leq 2 \)

Now, to find the \( y \)-bounds, we solve them to be on \( y \). Note that the degree 2 inequality on \( y, z \) would split into two if you solve it to be on \( y \), since every degree 2 polynomial has two roots by the quadratic formula.

1. \( y \geq 1 \)
2. \( z \geq 1 \)
3. \( y \leq \sqrt{3 - z^2} \)
4. \( y \geq -\sqrt{3 - z^2} \)
5. \( y \leq \frac{2}{z} \)

Now \( y \) has two upper bounds and two lower bounds. Remember that the more restrictive ones shall apply. It turns out, after calculation, the smallest upper bound is \( \sqrt{3 - z^2} \), and the largest lower bound is 1. Discarding the redundant inequalities, we have the following:

1. \( y \geq 1 \)
2. \( z \geq 1 \)
3. \( y \leq \sqrt{3 - z^2} \)

So the \( y \)-bounds are from 1 to \( \sqrt{3 - z^2} \). The inner two layers of integration shall look like this:

\[
\int_1^{\sqrt{3 - z^2}} \int_0^{\frac{2 - yz}{x+y}} \text{function } dx\,dy
\]

5 Final Step

At the beginning, we start off with a 3D shape. After finding the \( x \)-bounds, we project the shape to the \( yz \)-plane to have a 2D shape. Now that we know the \( y \)-bounds, we shall project this 2D shape to the \( z \)-axis, to obtain a 1D shape, i.e. an interval.

At the end of last section, we have the following inequalities:

1. \( y \geq 1 \)
2. \( z \geq 1 \)
3. \( y \leq \sqrt{3 - z^2} \)

Discarding the \( y \)-inequalities while maintaining the upper bound \( \geq \) lower bound relation, we have the following after projection:

1. \( z \geq 1 \)
2. \( 1 \leq \sqrt{3 - z^2} \)
Solving them to be on $z$, we obtain:

1. $z \geq 1$
2. $z \leq \sqrt{2}$
3. $z \geq -\sqrt{2}$

Taking the largest lower bound and smallest upper bound, we have:

1. $z \geq 1$
2. $z \leq \sqrt{2}$

So the bounds for our integration shall be like this:

$$\int_{1}^{\sqrt{2}} \int_{1}^{\sqrt{3-z^2}} \int_{0}^{\frac{2-yz}{y+z}} \text{function } dxdydz$$

And we have solved our problem.

6 Branching out

Let’s say our domain for integration is given by the following inequalities:

1. $x + y \geq z$
2. $x \geq 0$
3. $y^2 + z^2 \leq 1$
4. $x \leq 4$
5. $z \geq 0$

Now we first solve them to be on $x$. We have:

1. $x \geq z - y$
2. $x \geq 0$
3. $y^2 + z^2 \leq 1$
4. $x \leq 4$
5. $z \geq 0$

Now we have one upper bound, which will be our upper bound of choice for $x$. For lower bound, however, it is unclear which one of 0 or $z - y$ is our largest lower bound. Both $z - y \geq 0$ and $z - y \leq 0$ might happen. This type of phenomena is our cue to divide up our domain.

Let us call our original domain to be $A$. We define $A^+$ to be the domain $A$ with the extra condition that $z - y \geq 0$. We define $A^-$ to be the domain $A$ with the extra condition that $z - y \leq 0$.

Let us first focus on $A^-$. In this domain, clearly 0 is the largest lower bound. So our bounds for $x$ will be from 0 to 4. Now we project to the $yz$-plane and obtain the following inequalities:

1. $y^2 + z^2 \leq 1$
2. \( z \geq 0 \)
3. \( 0 \leq 4 \)
4. \( z - y \leq 0 \)

The second to last inequality comes from the lower bound \( \leq \) upper bound for \( x \). It is of course trivial, so we can discard it. The last inequality comes from our definition of \( A^- \). We now solve everything else to be on \( y \). We have

1. \( y \leq \sqrt{1 - z^2} \)
2. \( y \geq -\sqrt{1 - z^2} \)
3. \( z \geq 0 \)
4. \( y \geq z \)

The \( y \)-upper bound is clearly \( \sqrt{1 - z^2} \), and because of \( z \geq 0 \), \( z \) is the largest lower bound for \( y \). Finally we project to \( z \)-axis, and we have the following:

1. \( z \leq \sqrt{1 - z^2} \)
2. \( z \geq 0 \)

After calculation, we see that the bounds for \( z \) are from 0 to \( \frac{1}{\sqrt{2}} \).

To sum up, over the region \( A^- \), the integration have \( x \)-bounds 0 to 4, \( y \)-bounds \( z \) to \( \sqrt{1 - z^2} \), and \( z \)-bounds 0 to \( \frac{1}{\sqrt{2}} \).

Now we move on to \( A^+ \). Now the largest lower bound for \( x \) will be \( z - y \). So the \( x \)-bounds are \( z - y \) to 4. After projection to \( yz \)-plane, we have:

1. \( y^2 + z^2 \leq 1 \)
2. \( z \geq 0 \)
3. \( z - y \leq 4 \)
4. \( z - y \geq 0 \)

Solve them to be on \( y \) and discarding redundant inequalities, we have:

1. \( y \leq \sqrt{1 - z^2} \)
2. \( y \geq -\sqrt{1 - z^2} \)
3. \( z \geq 0 \)
4. \( y \leq z \)

Now \( y \)-lower bound is clear. We have two \( y \)-upper bounds and it is unclear which one is the smallest upper bound. So we branch out again. We cut \( A^+ \) into yet another two pieces.

We define \( A^+_1 \) to be the domain \( A^+ \) with the extra condition that \( z \leq \sqrt{1 - z^2} \) (i.e. \( z \leq \frac{1}{\sqrt{2}} \)). We define \( A^+_2 \) to be the domain \( A^+ \) with the extra condition that \( z \geq \sqrt{1 - z^2} \) (i.e. \( z \geq \frac{1}{\sqrt{2}} \)).

Now on the domain \( A^+_1 \), the \( y \)-bounds are clearly \(-\sqrt{1 - z^2}\) to \( z \). And after calculation we see that the \( z \)-bounds are 0 to \( \frac{1}{\sqrt{2}} \).
On the domain $A^+_2$, the $y$-bounds are clearly $-\sqrt{1-z^2}$ to $\sqrt{1-z^2}$. And after calculation we see that the $z$-bounds are $\frac{1}{\sqrt{2}}$ to 1. (Note that here $z \leq 1$ comes from the fact that $\sqrt{1-z^2}$ has to be defined. This is a hidden inequality all along.)

So we now have divided our original domain $A$ into three pieces, $A^+_1$, $A^+_2$, and $A^-$, and we have bounds on all of these domains. We have the final answer like the following:

$$
\int \int \int_A \text{function } dxdydz = \int \int \int_{A^+_1} \text{function } dxdydz + \int \int \int_{A^+_2} \text{function } dxdydz + \int \int \int_{A^-} \text{function } dxdydz
$$

$$
= \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{z-y}^{z} \text{function } dxdydz + \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{z-y}^{z} \text{function } dxdydz + \int_0^1 \int_{\sqrt{1-z^2}}^{z} \int_{z-y}^{\sqrt{1-z^2}} \text{function } dxdydz
$$