Countability
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Definitions: Let $X$ and $Y$ be sets.

A function $f : X \to Y$ is an injection if for any $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

A function $f : X \to Y$ is a surjection if for any $y \in Y$ there exists an $x \in X$ with $f(x) = y$.

A function $f : X \to Y$ is a bijection if it is both an injection and a surjection.

Intuitively, the existence of an injection $f : X \to Y$ shows that the size of $X$ is smaller than or equal to the size of $Y$; the existence of a surjection $f : X \to Y$ shows that the size of $X$ is greater than or equal to the size of $Y$; and the existence of a bijection $f : X \to Y$ shows that the size of $X$ equals the size of $Y$.

These notions can be rephrased using the concept of inverses:

Given functions $f : X \to Y$ and $g : Y \to X$, $g$ is a right-inverse of $Y$ if $f \circ g(y) = y$ for all $y \in Y$. And $g$ is a left-inverse of $Y$ if $g \circ f(x) = x$ for all $x \in X$. If $g$ is both a right-inverse and a left-inverse, then it is simply called the inverse of $f$.

We leave it as an exercise for you to check that a function $f : X \to Y$ is injective if and only if it has a left-inverse; the function $f$ is surjective if and only if it has a right-inverse; and the function $f$ is bijective if and only if it has an inverse.

We also leave it as an exercise to show that if $f : X \to Y$ is a bijection, then there is a unique inverse $g : Y \to X$, and that $g$ is itself a bijection. Thus there is a bijection from $X$ to $Y$ if and only if there is a bijection from $Y$ to $X$.

A set $X$ is said to be countable if there is an injection $f : X \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers.

Proposition 1: Let $f : X \to Y$ be an injection. Assume $Y$ is countable. Then $X$ is countable as well.
Proof: By definition, since $Y$ is countable, there is an injection $g : Y \to \mathbb{N}$. Since the composition of injections is an injection, the composition $g \circ f : X \to \mathbb{N}$ is an injection, showing that $X$ is countable.

Corollary 1: Let $f : X \to Y$ be a surjection. Assume $X$ is countable. Then $Y$ is countable as well.

Proof: Since $f$ is surjective, it has a right-inverse $g : Y \to X$. That is $f \circ g(y) = y$ for all $y \in Y$. This map $g$ is injective since it has a left-inverse $f$, so we can now apply Proposition 1 and the fact that $X$ is countable to conclude that $Y$ is countable.

Note that any finite set is countable. In fact, one way to define what it means for a set $X$ to be finite is that there is a bijection $f : X \to \{ n \in \mathbb{N} : n < m \}$ for some $m \in \mathbb{N}$. But every subset $S \subseteq \mathbb{N}$ is countable since the inclusion $i : S \to \mathbb{N}$ defined by $i(s) = s$ is an injection. By proposition 1 applied to the bijection $f : X \to \{ n \in \mathbb{N} : n < m \}$, we see that $X$ is countable.

To distinguish between finite countable sets and infinite ones, define a set $X$ to be countably infinite if it is infinite and countable.

Proposition 2: Let $S \subseteq \mathbb{N}$ be an infinite subset. Then there exists a bijection between $S$ and $\mathbb{N}$.

Proof: You might think this is intuitively clear. But let’s prove it carefully.

We define a map $f : \mathbb{N} \to S$ as follows. Note that any nonempty subset of the natural numbers has a least element (this is known as the well-ordering principle). Since the set $S$ is infinite, it is nonempty. So we let $f(0)$ be the least element in $S$. We now define $f$ recursively as follows. Let $n \in \mathbb{N}$ and assume we have defined $f$ on all numbers less than or equal to $n$. Then we note that the set $\{ f(0), \ldots, f(n) \} \subseteq S$ is not equal to $S$, or else $S$ would be finite. So the complement $S - \{ f(0), \ldots, f(n) \}$ is nonempty, and therefore has a least element. Define $f(n + 1)$ to be this least element. By recursion, this defines $f$ on all of $\mathbb{N}$.

We now argue that $f$ is a bijection. The injectivity of $f$ follows from the fact that, by construction, $f(n)$ differs from $f(m)$ for all $m < n$. To show $f$ is a surjection, suppose not. Then there exists $x \in S$ which is not in the image of $f$. But this implies $x \geq f(n)$ for all $n$ (because if $x$ were smaller than $f(n)$ then by construction it must be of the form $f(m)$ for some $m < n$). However, we know that each $f(n) \geq n$ for each $n$ so we conclude $x \geq f(n) \geq n$ for each $n$. In particular $x \geq x + 1$, a contradiction. So $f$ is in fact surjective, so it is a bijection.
Corollary 2: Let $X$ be a countably infinite set. Then there exists a bijection $h : X \to \mathbb{N}$.

Proof: Since $X$ is countable, fix an injection $g : X \to \mathbb{N}$. Let $S = g(X)$ be the image of $f$. Since $X$ is infinite and $g$ is injective, $S$ must be infinite as well. So by Proposition 2, there is a bijection $f : S \to \mathbb{N}$. Let $\tilde{g} : X \to S$ be given by $\tilde{g}(x) = g(x)$. Since $g$ is an injection, so is $\tilde{g}$. Moreover, $\tilde{g}$ is surjective by definition of $S$. So $\tilde{g}$ is a bijection. Since a composition of bijections is a bijection, the composition $f \circ \tilde{g} : X \to \mathbb{N}$ is a bijection. □

Corollary 2 shows a dichotomy: a countable set is either finite or equal in size to the natural numbers $\mathbb{N}$. There is no intermediate infinity. On the one hand, this makes sense. On the other hand, it shows that some of our intuitions for finite sets break down in the realm of infinite sets. In finite sets, we are used to the fact that if a set $A$ is contained but not equal to the set $B$, then $A$ is smaller in size than $B$. However, for infinite sets this is not true. Any infinite subset of $\mathbb{N}$ has the same size as $\mathbb{N}$ itself. For example, the set of even natural numbers has the same size as the set of natural numbers, even though naive intuition might tell us that there are much fewer even numbers.

We now show that, similarly, the set $\mathbb{N} \times \mathbb{N}$, consisting of pairs $(n, m)$ of natural numbers, is in fact equal in size to $\mathbb{N}$, even though it seems naively to be infinitely larger.

Proposition 5: The set $\mathbb{N} \times \mathbb{N}$ is countable.

Proof: The standard proof-by-picture looks like this:

![Diagram of \( \mathbb{N} \times \mathbb{N} \) enumeration]

The idea is that we can enumerate all the elements of $\mathbb{N} \times \mathbb{N}$ (which can be represented by a grid as above) by following the path illustrated by the arrows.
That is, we have a function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, mapping $n$ to the $n$th element of $\mathbb{N} \times \mathbb{N}$ visited in this path. The map $f$ is a bijection because every element of $\mathbb{N} \times \mathbb{N}$ is visited exactly once by this path.

Another proof: Another perspective on Proposition 5 is that we can partition the set $\mathbb{N}$ into infinitely many sets, each of which is itself infinite. Here is one way to do that. Let $A_n$ be the set of natural numbers whose decimal representation ends with exactly $n$ zeroes. (For example, $2100 \in A_2$, $31 \in A_0$ and $50 \in A_1$). The sets $A_n$ partition $\mathbb{N}$, in the sense that every natural number is in exactly one $A_n$. Each set $A_n$ is clearly infinite. Since each $A_n$ is a subset of $\mathbb{N}$ it is countable. Since it is countably infinite, by corollary 2 there is a bijection $f_n : \mathbb{N} \to A_n$. Now define a map $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $h(n, m) = f_n(m) \in A_n \subset \mathbb{N}$. The map $h$ is a bijection because each $f_n$ is a bijection and each natural number is in exactly one $A_n$.

Corollary 5: Let $A$ and $B$ be countable sets. Then

a) The union $A \cup B$ is countable.

b) The product $A \times B$ is countable.

Proof: Since $A$ is countable, fix an injection $f : A \to \mathbb{N}$. Since $B$ is countable, fix an injection $g : B \to \mathbb{N}$.

a) We define a map $h : A \cup B \to \mathbb{N} \times \mathbb{N}$ by $h(x) = (f(x), 0)$ if $x \in A$ and $h(x) = (g(x), 1)$ if $x \in B \setminus A$. We leave it as an exercise to check that the map $h$ is an injection. By Proposition 5, $\mathbb{N} \times \mathbb{N}$ is countable. By proposition 1, it follows from the fact that $h$ is an injection that $A \cup B$ is countable as well.

b) We define a map $i : A \times B \to \mathbb{N} \times \mathbb{N}$ by $i(a, b) = (f(a), g(b))$. We leave it as an exercise to check that the map $i$ is an injection. By Proposition 5, $\mathbb{N} \times \mathbb{N}$ is countable. By proposition 1, it follows from the fact that $i$ is an injection that $A \times B$ is countable as well.

Examples 5:

i) The set of integers $\mathbb{Z}$ is countable. Indeed, we can write $\mathbb{Z} = \mathbb{N} \cup \mathbb{Z}^-$, where $\mathbb{Z}^-$ is the set of negative numbers. But $\mathbb{N}$ is countable, and the set $\mathbb{Z}^-$ is countable as well, as can be exhibited by the injective map $f : \mathbb{Z}^- \to \mathbb{N}, x \mapsto -x$. By Corollary 5a), it follows that $\mathbb{Z}$, being the union of two countable sets, is countable as well.

ii) The set of rational numbers $\mathbb{Q}$ is countable. Indeed, we define a map $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$ by letting $f(a, b) = \frac{a}{b}$. The map $f$ is a surjection since every rational number is represented by a quotient of integers. By i), $\mathbb{Z}$ is countable. By Corollary 5b), it follows that so is $\mathbb{Z} \times \mathbb{Z}$ is countable as well. Since $f$ is surjective, it follows by Corollary 1 that $\mathbb{Q}$ is countable as well.
iii) Let $X$ be countable and let $n \in \mathbb{N}$. Then the set $X^n$ (that is, the set of $n$-tuples $(x_1, \ldots, x_n)$ with $x_i \in X$ is countable. Indeed, we can prove this using induction and the fact that $X^{n+1} = (X^n) \times X$.

So far, all the example we have seen were of countable sets. This raises the question: is every set countable? In fact, the answer is no, as shown by the following example.

Example 6: The set of real numbers $\mathbb{R}$ is not countable.

Proof: The classic proof uses Cantor's diagonalization argument.

Let $f : \mathbb{Z}^+ \to \mathbb{R}$ be a function, where $\mathbb{Z}^+$ is the set of positive integers. We show that $f$ cannot possibly be surjective.

Each real number has a decimal representation. This decimal representation is not quite unique, since for example, $1.000\cdots = 0.999\cdots$. To avoid this issue, we will require that the decimal representation of a number does not have trailing 9's at the end. This will ensure that decimal representation is unique.

Now, we will construct a number $x$ which fails to be in the image of $f$. We describe $x$ using its decimal representation. Let all the digits of $x$ to the left of the decimal point be 0. For any $n \in \mathbb{Z}^+$, define the $n$th digit of $x$ to the right of the decimal point to be any digit which is not equal to the $n$th digit of $f(n)$ to the right of the decimal point, and is also not equal to 9 (so as to avoid trailing 9's).

By construction, for any number $n \in \mathbb{Z}^+$, we see that $x$ differs from $f(n)$, because their decimal representations differ in the $n$th digit to the right of the decimal point, and since decimal representation is unique. Thus $x$ is not in the image of $f$.

Now we can conclude $\mathbb{R}$ is not countable. Indeed, suppose for contradiction $\mathbb{R}$ is countable. Since $\mathbb{R}$ is infinite, it follows that there is a bijection $\mathbb{N} \to \mathbb{R}$. Similarly, we know there is a bijection $\mathbb{Z}^+ \to \mathbb{N}$. Composing, we obtain a bijection $f : \mathbb{Z}^+ \to \mathbb{R}$. In particular, $f$ is a surjection, which we just showed cannot be the case, a contradiction.

The above example shows that there really are different sizes of infinity, though it is difficult to gauge them purely by intuition. For example, as we saw, the rational numbers $\mathbb{Q}$ may naively seem larger than the natural numbers $\mathbb{N}$, but are in fact the same size. On the other hand, the set of real numbers $\mathbb{R}$ is genuinely larger than the sets $\mathbb{Q}$ and $\mathbb{N}$, since they are countable but $\mathbb{R}$ is not. This raises the question: how many different sizes of infinity are there? It turns out that there is a vast infinity of infinities, of which $\mathbb{N}$ and $\mathbb{R}$ are only the beginning. The
study of these infinities is part of the field of set theory. But that’s a different story.

Application: As an application of the above theory, we give a proof that \( \mathbb{R} \) is infinite-dimensional as a vector space over \( \mathbb{Q} \).

Indeed, suppose for contradiction that there were a finite basis \( v_1,\ldots,v_n \) of \( \mathbb{R} \) as a vector space over \( \mathbb{Q} \). Consider the map \( f : \mathbb{R} \to \mathbb{Q}^n \) assigning to each real number \( x \in \mathbb{R} \) its coordinates with respect to the ordered basis \( v_1,\ldots,v_n \). That is, \( f(x) \) is the unique \( n \)-tuple \((c_1,\ldots,c_n)\), with \( c_i \in \mathbb{Q} \), such that \( x = c_1 v_1 + \cdots + c_n v_n \). The map \( f \) is a bijection.

But by example 5.ii), \( \mathbb{Q} \) is countable. By example 5.iii), it follows that so is the set \( \mathbb{Q}^n \). But since \( f : \mathbb{R} \to \mathbb{Q}^n \) is a bijection, it follows that \( \mathbb{R} \) is countable as well, a contradiction.

So \( \mathbb{R} \) cannot have a finite basis over \( \mathbb{Q} \), so it is in fact infinite-dimensional. \( \square \)

Note: In fact, \( \mathbb{R} \) does not even have a countable basis over \( \mathbb{Q} \). You can try to prove this as a challenging exercise.