NOTES FOR 197, SPRING 2018

We work in **ZFDC**, Zermelo-Frankel Theory with Dependent Choices, whose axioms are Zermelo's I - VII, the Replacement Axiom VIII and the axiom **DC** of dependent choices; when we need **AC**, we will list it among the hypotheses.

§1. Ordinal numbers.

Def. 1. A set α is an *ordinal number* if it is transitive, pure, grounded and connected, i.e.,

$$x = y \vee x \in y \vee y \in x \quad (x, y \in \alpha).$$

#1. The class ON of all ordinals is transitive,

$$\alpha \in \beta \in \text{ON} \Longrightarrow \alpha \in \text{ON}.$$

Def. 2. On each ordinal α we define the binary relation

$$x \leq_\alpha y \iff_{\text{df}} [x = y \vee x \in y]$$

#2. The relation $\leq_\alpha$ is a wellordering of α.

• When we say "ordinal" we will mean either the set α or the well ordered set $(\alpha, \leq_\alpha)$.

#3. For each ordinal α and $x \in \alpha$, $\mathbf{seg}_\alpha(x) = x$.

#4. Every well ordered set $U = (\text{Field}(U), \leq_U)$ is similar to a unique ordinal,

$$\text{ord}(U) = \text{the unique } \alpha \in \text{ON})[U =_o \alpha].$$

Def. 3. For any two ordinals α, β, we put

$$\begin{aligned} \alpha \leq \beta \iff_{\text{df}} & (\alpha, \leq_\alpha) \leq_o (\beta, \leq_\beta) \\ \iff & \text{there is an order-preserving bijection} \\ & \qquad \text{of } \alpha \text{ with an initial segment of } \beta. \end{aligned}$$

#5. For any two $\alpha, \beta \in \text{ON}$,

$$\alpha \leq \beta \iff (\exists \pi : \alpha \rightarrowtail \beta)[x \in y \in \alpha \Longrightarrow \pi(x) \in \pi(y)].$$

Notes for Math 197, 2018 Spring, Y. N. Moschovakis, May 2, 2018, 9:53.

#6 (Lemma 12.14 in NST). For any two ordinals α, β,

$$\alpha \leq \beta \iff \alpha = \beta \vee \alpha \in \beta \iff \alpha \sqsubseteq \beta \iff \alpha \subseteq \beta;$$

moreover, $\alpha < \beta \iff \alpha \in \beta$.

#7. The class ON is well ordered by the condition $\leq$, i.e.,

$$\begin{gathered}\alpha \leq \alpha,\ [\alpha \leq \beta \leq \gamma] \Longrightarrow \alpha \leq \gamma,\ [\alpha \leq \beta \leq \gamma] \Longrightarrow \alpha = \gamma,\\ \alpha < \beta \vee \alpha = \beta \vee \beta < \alpha,\\ (\exists \alpha)P(\alpha) \Longrightarrow (\exists \alpha)[P(\alpha) \,\&\, (\forall \beta < \alpha)[\neg P(\beta)]],\end{gathered}$$

where $P(\alpha)$ is any definite condition on ordinals.

#8. If $\mathcal{E}$ is a (non-empty) set of ordinals, then

$$\begin{aligned}\sup \mathcal{E} &= \text{the least } \beta\ (\forall \alpha \in \mathcal{E})[\alpha \leq \beta] = \bigcup \mathcal{E},\\ \min \mathcal{E} &= \bigcap \mathcal{E}.\end{aligned}$$

#9. The class ON is not a set.

#10. $0 = \emptyset$ is the least ordinal; $S(\alpha) = \alpha \cup \{\alpha\}$ is the *successor* of α, the least ordinal $> \alpha$; and if λ is not 0 and not the successor of any α, than λ is a *limit ordinal* and

$$\lambda = \sup\{\alpha \mid \alpha < \lambda\}.$$

- The least limit ordinal is called ω.

#11 (Proof by ordinal induction). If $P(\alpha)$ is a definite condition on ordinals and for all $\alpha \in \mathrm{ON}$

$$(\forall \beta \in \alpha)P(\beta) \Longrightarrow P(\alpha),$$

then $P(\alpha)$ is true for all $\alpha \in \mathrm{ON}$.

#12 (Definition by ordinal recursion). For every definite operation $H(w, \alpha)$, there is exactly one definite operation $F : \mathrm{ON} \to \mathrm{ON}$ such that

$$F(\alpha) = H(F\restriction\alpha, \alpha),$$

where $F\restriction\alpha$ is the restriction of F to α,

$$F\restriction(\alpha) = \{(\xi, F(\xi)) \mid \xi \in \alpha\}.$$

Similarly with a parameter: For every definite operation $H(w, \alpha, x)$, there is exactly one definite operation $F(\alpha, x)$ such that

$$F(\alpha, x) = H(\{(\xi, F(\xi, x) \mid \xi < \alpha\}, \alpha, x)).$$

Def. 4 (Addition of ordinals). By ordinal recursion with parameters,

$$\begin{aligned} \alpha + 0 &= \alpha, \\ \alpha + S(\beta) &= S(\alpha + \beta), \\ \alpha + \lambda &= \sup\{\alpha + \beta \mid \beta < \lambda\} \quad (\text{Limit}(\lambda)). \end{aligned}$$

#13 (Ord addition is associative). For all $\alpha, \beta, \gamma \in \text{ON}$,

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma.$$

#14. There are ordinals α, β, such that $\alpha + \beta \neq \beta + \alpha$.

#15. Problem x12.7 in NST.

#16. Problem x12.8 in NST.

#17. Problem x12.9 in NST.

Def. 5 (Multiplication of ordinals). By ordinal recursion with parameters,

$$\begin{aligned} \alpha \cdot 0 &= 0, \\ \alpha \cdot S(\beta) &= (\alpha \cdot \beta) + \alpha, \\ \alpha \cdot \lambda &= \sup\{\alpha \cdot \beta \mid \beta < \lambda\} \quad (\text{Limit}(\lambda)). \end{aligned}$$

#18 (Ord multiplication is associative). For all $\alpha, \beta, \gamma \in \text{ON}$,

$$\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma.$$

#19. There are ordinals α, β such that $\alpha \cdot \beta \neq \beta \cdot \alpha$.

#20. Problem x12.10 in NST.

#21. Problem x12.11 in NST.

#22 (Right distributive law). For all $\alpha, \beta, \gamma \in \text{ON}$,

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma.$$

#23 (Failure of left distributivity). Give an example where

$$(\alpha + \beta) \cdot \gamma \neq \alpha \cdot \gamma + \alpha \cdot \gamma.$$

- The first few ordinals are

$$0, 1, \dots, \omega, \omega + 1, \omega + 2, \dots \omega 2, \omega 2 + 1, \dots, \omega 3 \dots, \omega 4 \dots, \dots$$
$$\omega^2, \omega^2 + 1, \dots \omega^3, \dots \Omega_1 = \text{ the least uncountable ordinal}, \Omega_1 + 1, \dots\dots$$

§2. Cardinal numbers.

Def. 6. The cardinal number of a well-orderable set:

$$|A| = (\mu\xi \in \mathrm{ON})[A =_c \xi].$$

#24. For any well-orderable set A,

$$|A| = \mathrm{ord}(A, \leq),$$

where $\leq$ is any *best wellordering* of A.

#25. For any well-orderable sets A, B, $A =_c |A|$; $A =_c B \iff |A| = |B|$.

Def. 7. The class of cardinal numbers: $\mathrm{Card}(\kappa) \iff (\exists A)[\kappa = |A|]$.

#26. A set is a cardinal number if and only if it is an *initial ordinal*,

$$\mathrm{Card}(\kappa) \iff \alpha \in \mathrm{ON}\,\&\,(\forall\alpha \in \kappa)[\alpha <_c \kappa].$$

#27. The class Card is not a set.

#28 (**AC**). Every set A is well-orderable, and so $|A|$ is defined.

Def. 8 (**AC**). Cardinal arithmetic:

$$\begin{aligned}
\kappa + \lambda &= |\kappa \uplus \lambda|,\\
\kappa \cdot \lambda &= |\kappa \times \lambda|,\\
\kappa^\lambda &= |(\lambda \to \kappa)|,\\
\textstyle\sum_{i\in I}\kappa_i &= |\{(i,x) \in I \times \textstyle\bigcup_{i\in I}\kappa_i \mid x \in \kappa_i\}|,\\
\textstyle\prod_{i\in I}\kappa_i &= |\textstyle\prod_{i\in I}\kappa_i|,
\end{aligned}$$

(disregarding the double use of the same notation in the last def.)

#29 (**AC**). Cardinal addition and multiplication are associative and commutative; formulate and prove these laws for both the finite and infinite sums and products.

#30 (The absorption laws, **AC**).

$$(\kappa, \lambda \neq 0\,\&\,\max(\kappa,\lambda) \text{ infinite}) \Longrightarrow \kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda).$$

Def. 9 (The alephs). By ordinal recursion, we set

$$\begin{aligned}
\aleph_0 &= |\mathbb{N}| = \omega && = \text{the least infinite cardinal},\\
\alpha_{\beta+1} &= \aleph_\beta^+ && = \text{the least cardinal } > \aleph_\beta,\\
\aleph_\lambda &= \sup\{\aleph_\beta \mid \beta < \lambda\} && \quad (\mathrm{Limit}(\lambda)).
\end{aligned}$$

#31. Every $\aleph_\alpha$ is a cardinal number.

#32 (**AC**). Card $= \omega \bigcup \{\aleph_\alpha \mid \alpha \in \text{ON}\}$.

• (**AC**). The Continuum Hypothesis and the Generalized Continuum Hypothesis take the form

$$2^{\aleph_0} = \aleph_1; \quad 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

§3. Universes.

Def. 10 (The cumulative hierarchy of pure, grounded sets). The *partial von Neumann universes* are defined by the ordinal recursion

$$\begin{aligned} V_0 &= \emptyset, \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha), \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha \quad (\text{Limit}(\lambda). \end{aligned}$$

We also define the class

$$V = \bigcup_{\alpha \in \text{ON}} V_\alpha.$$

#33. $\alpha < \beta \Longrightarrow V_\alpha \subsetneq V_\beta$.

#34. V_ω comprises all the pure, grounded, hereditarily finite sets.

#35. The class $V = \bigcup_{\alpha \in \text{ON}} V_\alpha$ comprises exactly all sets which are pure and grounded.

#36. V satisfies all the axioms of **ZFDC**, and also the axioms of *Purity* and *Foundation*; and if we assume the Axiom of Choice, then it also satisfies **AC**.

• This means that if by "set" we understand "set in V", then all the axioms of **ZFDC** are true; and if we also assume **AC**, then we can prove that V *satisfies* **AC**, the proposition

> For every set $A \in V$, there is a function $\varepsilon : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ such that $\varepsilon \in V$ and
>
> $$(\forall x \subseteq A)[x \neq \emptyset \Longrightarrow \varepsilon(x) \in x].$$

Def. 11 (Zermelo universes, 11.19 in NST). A transitive class M is s *Zermelo universe* if it satisfies Zermelo's Axioms I – VI and **DC** and it contains Zermelo's set of natural numbers,

$$\mathbb{N}_0 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}.$$

Def. 12 (the least Zermelo universe); this is

$$Z = \bigcup_{n \in \mathbb{N}} Z_n,$$

where

$$Z_0 = \mathbb{N}_0, \quad Z_{n+1} = \mathcal{P}(Z_n).$$

#37. If λ is a limit ordinal, $\lambda > \omega$, then V_λ is a Zermelo universe.

#38. $V_\omega \notin Z$.

• This proposition has many consequences about the strength of the Zermelo axioms, for example:

#39. We cannot prove using only the Zermelo axioms I – VII that there is a set whose members are exactly all the pure, hereditarily finite sets. (And making this precise is part of the problem.)

§4. The Cantor-Bendixson Theorem. At this point we assume you have read the basic definitions about $\mathcal{N}$ and its topology, through 10.5 of Chapter 10.

#40. Proposition 10.6 in the book. This is a list of the basic properties (1) – (5) of the topology of Baire space $\mathcal{N}$, and their presentation will be probably split among two or three students. (Try to think of how to prove each part before reading the proofs; you may end up with a better argument or presentation than what the book has.)

#41. TFAE for a set $F \subseteq \mathcal{N}$:
(a) F is closed.
(b) There is a tree T on $\mathbb{N}$ such that $F = [T] =$ the body of T.
(c) There is a tree T on $\mathbb{N}$ such that $F = [T]$ and T has no finite branches.

#42. Every perfect, non-empty pointset $P \subseteq \mathcal{N}$ has cardinality $\mathfrak{c} = 2^{\aleph_0}$. This is 10.10 in NST; you may well think up a better solution than the one given there.

Def. 13. Let $A \subseteq \mathcal{N}$ be a pointset.

A point $x \in \mathcal{N}$ is a *limit point of* A if every nbhd of x contains some point in A other than x, i.e., for every nbhd $\mathcal{N}_u$,

$$x \in \mathcal{N}_u \Longrightarrow \text{ there is some } y \neq x \text{ in } (A \cap \mathcal{N}_u)$$

A point $x \in \mathcal{N}$ is a *condensation point of* A if for every nbhd $\mathcal{N}_u$,

$$x \in \mathcal{N}_u \Longrightarrow (A \cap \mathcal{N}_u) \text{ is uncountable.}$$

• Notice that every condensation point of A is a limit point of A.

#43. If x is a limit point of some $A \subseteq \mathcal{N}$, then for every nbhd $\mathcal{N}_u$,

$$x \in \mathcal{N}_u \Longrightarrow (A \cap \mathcal{N}_u) \text{ is infinite.}$$

#44. A pointset $F \subseteq \mathcal{N}$ is closed if and only if it contains all its limit points.

Def. 14. For any closed set $F \subseteq \mathcal{N}$, put

$$\text{kernel}(F) = \{x \in \mathcal{N} \mid x \text{ is a condensation point of } F\}.$$

Notice that $\text{kernel}(F) \subseteq F$.

#45. Give an example where $\text{kernel}(F) = F \neq \emptyset$ and another where $F \neq \emptyset$ but $\text{kernel}(F) = \emptyset$.

#46. Suppose T is a "pruned" tree on $\mathbb{N}$ (no finite branches) with body $[T] = F$ and let

$$kT = \{u \in T \mid [T_u] \text{ is uncountable}\};$$

then

$$kT \text{ is a tree and } [kT] = \text{kernel}(F).$$

#47 (existence of a Cantor-Bendixson decomposition). If $F \subseteq \mathcal{N}$ is closed, then there exists a perfect set P and a countable set S such that

$$F = P \cup S, \quad P \cap S = \emptyset.$$

#48 (uniqueness of the Cantor-Bendixson decomposition). If $F \subseteq \mathcal{N}$ is closed, P is perfect, S is countable and

$$F = P \cup S, \quad P \cap S = \emptyset,$$

then $P = \text{kernel}(F)$.

§5. Property P**.** A family Γ of pointsets has *property* P if every uncountable $A \in \Gamma$ has a perfect subset. For example, the family

$$\mathcal{F} = \{F \subseteq \mathcal{N} \mid F \text{ is closed}\}$$

has property P by the Cantor-Bendixson Theorem.

#49. $\mathcal{F} =_c \mathcal{N}$.

#50 (**AC**). $\mathcal{F}$ can be indexed on $\mathfrak{c} = 2^{\aleph_0}$, so

$$\mathcal{F} = \{F_\alpha \mid \alpha < \mathfrak{c}\}.$$

#51 (**AC**). There is a set $A \subset \mathcal{N}$ such that $|A| = \mathfrak{c}$ but A has no uncountable closed subset.

#52 (**AC**). The family $\mathcal{P}(\mathcal{N})$ of all subsets of $\mathcal{N}$ does not have property P.

§6. Another proof of the Cantor-Bendixson Theorem. The proof of #51 involves definition by transfinite (or ordinal) recursion, which can also be used to prove the Cantor-Bendixson Theorem as follows.

Def. 15. A point x is an *isolated point* of a pointset A if $x \in A$ but x is not a limit point of A.

Def. 16. We define the *derivative* F' of any closed pointset F by

$$F' = \{x \in F \mid x \text{ is a limit point of } F\} = F \setminus \{x \in F \mid x \text{ is isolated}\}.$$

#53. For every closed F, $F' \subseteq F$ and F' is closed.

#54. A closed set F is perfect if and only if $F' = F$.

Def. 17. For a given closed pointset F and every ordinal number ξ, we define F_ξ by recursion as follows:

$$\begin{aligned} F_0 &= F, \\ F_{\xi+1} &= (F_\xi)', \\ (\text{Limit}(\lambda)) \qquad F_\lambda &= \textstyle\bigcap_{\xi<\lambda} F_\xi. \end{aligned}$$

#55. Each F_ξ is closed and $\eta \leq \xi \Longrightarrow F_\eta \supseteq F_\xi$.

#56 (Cantor-Bendixson existence). For each closed pointset F, there is an ordinal μ such that

$$F_{\mu+1} = F_\mu \text{ and } \mu \text{ is countable.}$$

It follows that

(1) The set $P = F_\mu$ is perfect (perhaps empty).
(2) The set $S = (F \setminus P)$ is countable.
(3) $F = P \cup S$.

#57. Theorem 10.15 in the book, the characterization of continuous functions $f : \mathcal{N} \to \mathcal{N}$ in terms of functions on strings.

#58. For each of the following functions on $\mathcal{N}$, find a monotone string function $\tau : \mathbb{N}^* \to \mathbb{N}^*$ such that

$$f(x) = \sup\{\tau(u) \mid u \sqsubseteq x\}.$$

- $f_1(x) = \langle 0, 2, 7\rangle * x$.
- $f_2(x) = \text{tail}(\text{tail}(\text{tail}(x)))$.

#59. Suppose $f(x), g(x)$ are continuous functions on Baire space and let $h(x)$ be their "interweaving",

$$h(x) = (f(x)(0), g(x)(0), f(x)(1), g(x)(1), \cdots).$$

Define a string representation of $h(x)$ using given string representations of $g(x)$ and $h(x)$. (The idea is to explain neatly how to compute $\tau_h(u)$

using τ_f and τ_g; don't look for a formula, it's really a program that is needed.)

#60 (Problem 10.21 in the book). Prove that the following are equivalent for a set $K \subset \mathcal{N}$:

(1) $K = [T]$ for a finitely branching tree $T \subset \mathbb{N}^*$.

(2) For every family O of nbhds, if $K \subseteq \bigcup O$, then there is a finite subset $\mathcal{N}_{u_1}, \dots, \mathcal{N}_{u_k}$ of O such that

$$K \subseteq \mathcal{N}_{u_1} \cup \cdots \cup \mathcal{N}_{u_k}.$$

• The next two problems are the two parts of Theorem 10.19 in the book; so "solving" them means to read and understand he proofs well enough so you can present them in class.

#61. (1) of Theorem 10.19, that the continuous image of a compact set is compact. (You can do this using either of the two characterizations of compact sets in the preceding problem.)

#62. (2) of Theorem 10.19, that the continuous, injective image of a compact and perfect pointset if compact and perfect.

Def. 18. A pointset $A \subseteq \mathcal{N}$ is *analytic* if it is empty or the continuous image of $\mathcal{N}$.

• Again the next two problems together give us the Perfect Set Theorem 10.20, the second main goal of this class.

#63. The Lemma in the proof of Theorem 10.20, that the tree defined by (10-18) is splitting.

#64. Every uncountable analytic set has a perfect subset, assuming the Lemma.

• The remaining problems establish the basic closure properties of the family of analytic pointsets.

#65 (Lemma 10.21 of NST). Every closed pointset is analytic.

#66 (Lemma 10.22 of NST). The continuous image of an analytic pointset is analytic.

#67 (Lemma 10.23 of NST). If $f, g : \mathcal{N} \to \mathcal{N}$ are continuous, ten the pointset

$$E = \{x \in \mathcal{N} \mid f(x) = g(x)\}$$

is analytic.

#68. Countable unions of analytic sets are analytic.

#69. Every open pointset is analytic.

#70. Countable intersections of analytic sets are analytic.