Kleene's Amazing Second Recursion Theorem
Extended Abstract*

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This little gem is stated unbilled and proved (completely) in the last two lines of §2 of the short note Kleene (1938). In modern notation, with all the hypotheses stated explicitly and in a strong form, it reads as follows:

Theorem 1 (SRT). *Fix a set* $\mathbb{V} \subseteq \mathbb{N}$*, and suppose that for each natural number* $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$*,* $\varphi^n : \mathbb{N}^{n+1} \rightharpoonup \mathbb{V}$ *is a recursive partial function of* $(n+1)$ *arguments with values in* $\mathbb{V}$ *so that* ***the standard assumptions*** (1) *and* (2) *hold with*

$$\{e\}(\vec{x}) = \varphi^n_e(\vec{x}) = \varphi^n(e, \vec{x}) \quad (\vec{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n).$$

(1) *Every* n*-ary recursive partial function with values in* $\mathbb{V}$ *is* φ^n_e *for some* e*.*

(2) *For all* m, n*, there is a recursive* (total) *function* $S = S^m_n : \mathbb{N}^{m+1} \to \mathbb{N}$ *such that*

$$\{S(e, \vec{y})\}(\vec{x}) = \{e\}(\vec{y}, \vec{x}) \quad (e \in \mathbb{N}, \vec{y} \in \mathbb{N}^m, \vec{x} \in \mathbb{N}^n).$$

Then, for every recursive, partial function $f(e, \vec{y}, \vec{x})$ *of* $(1+m+n)$ *arguments with values in* $\mathbb{V}$*, there is a total recursive function* $\tilde{z}(\vec{y})$ *of* m *arguments such that*

$$\{\tilde{z}(\vec{y})\}(\vec{x}) = f(\tilde{z}(\vec{y}), \vec{y}, \vec{x}) \quad (\vec{y} \in \mathbb{N}^m, \vec{x} \in \mathbb{N}^n). \tag{1}$$

Proof. Fix e so that $\{e\}(m, \vec{y}, \vec{x}) = f(S(m, m, \vec{y}), \vec{y}, \vec{x})$ and let $\tilde{z}(\vec{y}) = S(e, e, \vec{y})$. $\dashv$

Kleene states the theorem with $\mathbb{V} = \mathbb{N}$, relative to specific φ^n, S^m_n, supplied by his *Enumeration Theorem*, $m = 0$ (no *parameters* $\vec{y}$) and $n \geq 1$, i.e., not allowing *nullary partial functions*. And most of the time, this is all we need; but there are a few important applications where choosing "the right" φ^n, S^m_n, restricting the values to a proper $\mathbb{V} \subsetneq \mathbb{N}$ or allowing $m > 0$ or $n = 0$ simplifies the proof considerably. With $\mathbb{V} = \{\!\{0\}\!\}$ (singleton 0) and $m = n = 0$, for example, the characteristic equation

$$\{\tilde{z}\}(\,) = f(\tilde{z}) \tag{2}$$

is a rather "pure" form of self-reference, where the number $\tilde{z}$ produced by the proof (as a code of a nullary semirecursive relation) has the *property* $f(\tilde{z})$, at least when $f(\tilde{z})\downarrow$.

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* Part of an expository article in preparation, written to commemorate the passage of 100 years since the birth of Stephen Cole Kleene.

Kleene uses the theorem in the very next page to prove that there is a largest initial segment of the countable ordinals which can be given "constructive notations", in the first application of what we now call *effective grounded* (transfinite) *recursion*, one of the most useful methods of proof justified by SRT; but there are many others, touching most parts of logic and even classical analysis.

My aim in this lecture is to list, discuss, explain and in a couple of simple cases prove some of the most significant applications of the Second Recursion Theorem, in a kind of "retrospective exhibition" of the work that it has done since 1938. It is quite impressive, actually, the power of such a simple fact with a one-line proof; but part of its wide applicability stems precisely from this simplicity, which make it easy to formulate and establish it in many contexts outside ordinary recursion theory on $\mathbb{N}$. Some of the more important applications come up in *Effective Descriptive Set Theory*, where the relevant version of SRT is obtained by replacing $\mathbb{N}$ by the Baire space $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$ and applying recursion theory on $\mathcal{N}$—also developed by Kleene.

Speaking rather loosely, the identity (1) expresses a *self-referential* property of $\widetilde{z}(\vec{y})$ and SRT is often applied to justify powerful, self-referential definitions. I will discuss some of these first, and then I will turn to applications of effective grounded recursion.

The lecture will focus on some of the following consequences of SRT:[1]

A. Self reproducing Turing machines. It is quite simple to show using SRT that there is a Turing machine which prints its code when it is started on the blank tape. It takes just a little more work—and a careful choice of φ^n, S^m_n—to specify a Turing machine (naturally and literally) by a string of symbols in its own alphabet and then show

Theorem 2. *On every alphabet Σ with $N \geq 3$ symbols, there is a Turing machine M which started on the blank tape outputs itself.*

B. Myhill's characterization of r.e. complete sets. Recall that a recursively enumerable (r.e.) set $A \subseteq \mathbb{N}$ is *complete* if for each r.e. set B there is a recursive (total) function f such that $x \in B \iff f(x) \in A$.[2] An r.e. set A is *creative* if there is a unary recursive partial function $u(e)$ such that

$$W_e \cap A = \emptyset \implies u(e)\downarrow \ \& \ u(e) \notin (A \cup W_e). \tag{3}$$

The notion goes back to Post (1944) who showed (in effect) that every r.e.-complete set is creative and implicitly asked for the converse.

[1] The choice of these examples was dictated by what I know and what I like, but also by the natural limitations of space in an extended abstract and time in a lecture. A more complete list would surely include examples from *Recursion in higher types* and *Realizability theory.* (For the latter, see Moschovakis (2010) which is in some ways a companion article to this.)

[2] And then one can find a one-to-one f with the same property, cf. Theorem VII in Rogers (1967).

Theorem 3 (Myhill (1955)). *Every creative set is r.e.-complete.*

This clever argument of Myhill's has many applications, but it is also foundationally significant: it identifies *creativeness*, which is an intrinsic property of a set A but depends on the coding of recursive partial functions with *completeness*, which depends on the entire class of r.e. sets but is independent of coding. I believe it is the first important application of SRT in print by someone other than Kleene.

C. The Myhill-Shepherdson Theorem. One (modern) interpretation of this classical result is that algorithms which call their (computable, partial) function arguments *by name* can be simulated by non-deterministic algorithms which call their function arguments *by value*. It is a rather simple but interesting consequence of SRT.

Let $\mathcal{P}^1_r$ be the set of all unary recursive partial functions. A partial operation

$$\Phi : \mathbb{N}^n \times \mathcal{P}^1_r \rightharpoonup \mathbb{N} \tag{4}$$

is *effective* if its partial function *associate*

$$f(\vec{x}, e) = \Phi(\vec{x}, \varphi^1_e) \tag{5}$$

is recursive. In programming terms, an effective operation calls its function argument *by name*, i.e., it needs a code of any $p \in \mathcal{P}^1_r$ to compute the value $\Phi(\vec{x}, p)$.

There are cases, however, when we need to compute $\Phi(\vec{x}, p)$ without access to a code of p, only to its values. In programming terms again, this comes up when p is computed by some other program which is not known, but which can be asked to produce any values $p(\vec{y})$ that are required during the (otherwise effective) computation of $\Phi(\vec{x}, p)$. We can make this precise using a (deterministic or non-deterministic) *Turing machine M with an oracle* which can request values of the function argument p on a special *function input tape*: when M needs $p(y)$, it prints y on the function input tape and waits until it is replaced by $p(y)$ before it can go on—which, in fact, may cause the computation to stall if $p(y)\uparrow$. In these circumstances we say that *M computes Φ by value.*

Notice that the recursive associate f of an effective operation Φ as in (5) satisfies the following *invariance condition*:

$$\varphi_{e_1} = \varphi_{e_2} \implies f(\vec{x}, e_1) = f(\vec{x}, e_2). \tag{6}$$

This is used crucially in the proof of the next theorem, which involves two clever applications of SRT:

Theorem 4 (Myhill and Shepherdson (1955)). *A partial operation Φ as in* (4) *is effective if and only if it is computable by a non-deterministic Turing machine.*[3]

[3] The use of non-deterministic machines here is essential, because the operation

$$\Phi(p) = \begin{cases} 1, & \text{if } p(0)\downarrow \text{ or } p(1)\downarrow, \\ \bot, & \text{otherwise} \end{cases}$$

is effective but not computable by a deterministic Turing machine.

D. The Kreisel-Lacombe-Shoenfield-Ceitin Theorem. Let $\mathcal{F}_r^1$ be the set of all unary total recursive functions. By analogy with operations on $\mathcal{P}_r^1$, a partial operation

$$\Phi : \mathbb{N}^n \times \mathcal{F}_r^1 \rightharpoonup \mathbb{N}$$

is *effective* if there is a recursive partial function $f : \mathbb{N}^{n+1} \rightharpoonup \mathbb{N}$ such that

$$\varphi_e \in \mathcal{F}_r^1 \implies \Phi(\vec{x}, \varphi_e) = f(\vec{x}, e). \tag{7}$$

Notice that such a *recursive associate* f of Φ satisfies the *invariance condition*

$$\varphi_e = \varphi_m \in \mathcal{F}_r^1 \implies f(\vec{x}, e) = f(\vec{x}, m), \tag{8}$$

but is not uniquely determined.

The next theorem is a version of the Myhill-Shepherdson Theorem appropriate for these operations. Its proof is not quite so easy, and it involves applying SRT in the middle of a relatively complex construction:

Theorem 5 (Kreisel, Lacombe, and Shoenfield (1957), Ceitin (1962)). *Suppose $\Phi : \mathbb{N}^n \times \mathcal{F}_r^1 \to \mathbb{N}$ is a total effective operation.*

(1) *Φ is effectively continuous: i.e., there is a recursive partial function $g(e, \vec{x})$, such that when φ_e is total, then $g(e, \vec{x})\downarrow$ for all $\vec{x}$, and for all $p \in \mathcal{F}_r^1$,*

$$(\forall t < g(e, \vec{x}))[p(t) = \varphi_e(t)] \implies \Phi(\vec{x}, p) = \Phi(\vec{x}, \varphi_e).$$

(2) *Φ is computed by a deterministic Turing machine.*

The restriction in the theorem to total operations on $\mathcal{F}_r^1$ is necessary, because of a lovely counterexample in Friedberg (1958).

Ceitin (1962) proved independently a general version of (1) in this theorem: *every recursive operator on one constructive metric space to another is effectively continuous*. His result is the central fact in the school of *constructive analysis* which was flowering in Russia at that time, and it has played an important role in the development of constructive mathematics ever since.

E. Incompleteness and undecidability using SRT. We formulate in this section two basic theorems which relate SRT to incompleteness and undecidability results: a version of the so-called Fixed Point Lemma, and a beautiful result of Myhill's, which implies most simple incompleteness and undecidability facts about sufficiently strong theories and insures a very wide applicability for the Fixed Point Lemma.

Working in the language of Peano Arithmetic (PA) with symbols $0, 1, +, \cdot$, define first (recursively) for each $x \in \mathbb{N}$ a closed term Δx which denotes x, and for every formula χ, let

$$\#\chi = \text{the code (Gödel number) of } \chi, \quad \ulcorner \chi \urcorner \equiv \Delta\#\chi = \text{the name of } \chi.$$

We assume that the Gödel numbering of formulas is sufficiently effective so that (for example) $\#\varphi(\Delta x_1, \ldots, \Delta x_n)$ can be computed from $\#\varphi(v_1, \ldots, v_n)$ and $x_1, \ldots, x_n$. A *theory* (in the language of PA) is any set of sentences T and

$$\mathrm{Th}(T) = \{\#\theta \mid \theta \text{ is a sentence and } T \vdash \theta\}$$

is the set of (Gödel numbers of) the theorems of T. A theory T is *sound* if every $\theta \in T$ is true in the standard model $(\mathbb{N}, 0, 1, +, \cdot)$; it is *axiomatizable* if its *proof relation*

$$\begin{aligned}\mathrm{Proof}_T(e, y) \iff\ & e \text{ is the code of a sentence } \sigma \\ & \text{and } y \text{ is the code of a proof of } \sigma \text{ in } T\end{aligned}$$

is recursive, which implies that $\mathrm{Th}(T)$ is recursively enumerable; and it is *sufficiently expressive* if every recursive relation $R(\vec{x})$ is *numeralwise expressible* in T, i.e., for some formula $\varphi_R(v_1, \ldots, v_n)$,

$$\begin{aligned} R(x_1, \ldots, x_n) &\implies T \vdash \varphi_R(\Delta x_1, \ldots, \Delta x_n), \\ \neg R(x_1, \ldots, x_n) &\implies T \vdash \neg\varphi_R(\Delta x_1, \ldots, \Delta x_n). \end{aligned}$$

Theorem 6 (Fixed Point Lemma). *If T is axiomatizable in the language of* PA *and* $\mathrm{Th}(T)$ *is r.e.-complete, then for every formula $\theta(v)$ with at most v free, there is a sentence σ such that*

$$T \vdash \sigma \iff T \vdash \theta(\ulcorner \sigma \urcorner). \tag{9}$$

Proof. Let $\psi^0, \psi^1, \ldots$ be recursive partial functions satisfying the standard assumptions, let[4]

$$u \in A \iff (\exists n)[\mathrm{Seq}(u) \;\&\; \mathrm{lh}(u) = n + 1 \;\&\; \psi^n((u)_0, (u)_1, \ldots, (u)_n)\downarrow],$$

so that A is r.e., and so there is a total recursive function $\mathfrak{r}$ such that

$$u \in A \iff \mathfrak{r}(u) \in \mathrm{Th}(T).$$

It follows that for every n-ary semirecursive relation $R(\vec{x})$, there is a number e such that

$$R(\vec{x}) \iff \psi^n(e, \vec{x})\downarrow \iff \langle e, \vec{x}\rangle \in A \iff \mathfrak{r}(\langle e, \vec{x}\rangle) \in \mathrm{Th}(T). \tag{10}$$

We will use SRT with $\mathbb{V} = \{\!\{0\}\!\}$ and

$$\varphi^n(e, \vec{x}) = 0 \iff \mathfrak{r}(\langle e, \vec{x}\rangle) \in \mathrm{Th}(T),$$

[4] For any tuple $\vec{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{N}^n$, $\langle \vec{x}\rangle$ codes $\vec{x}$ so that for suitable recursive relations and functions,

$$\langle \vec{x}\rangle = f_n(x_0, \ldots, x_{n-1}),\ \mathrm{Seq}(w) \Longleftrightarrow w \text{ is a sequence code},\ \mathrm{lh}(\langle \vec{x}\rangle) = n,\ (\langle \vec{x}\rangle)_i = x_i,$$

and the code $\langle\ \rangle$ of the empty sequence is 1.

which (because of (10), easily) satisfy the standard assumptions.

Given $\theta(v)$, SRT (with $m = n = 0$) gives us a number $\widetilde{z}$ such that

$$\{\widetilde{z}\}(\ ) = 0 \iff \mathfrak{r}(\langle\widetilde{z}\rangle) \text{ is not the code of a sentence or } T \vdash \theta(\Delta\mathfrak{r}(\langle\widetilde{z}\rangle)). \quad (11)$$

Now $\mathfrak{r}(\langle\widetilde{z}\rangle)$ is the code of a sentence, because if it were not, then the right-hand-side of (11) would be true, which makes the left-hand-side true and insures that $\mathfrak{r}(\langle\widetilde{z}\rangle)$ codes a sentence, in fact a theorem of T; and if $\mathfrak{r}(\langle\widetilde{z}\rangle) = \#\sigma$, then

$$\begin{aligned} T \vdash \sigma \iff \mathfrak{r}(\langle\widetilde{z}\rangle) \in \mathrm{Th}(T) &\iff \{\widetilde{z}\}(\ ) = 0 \\ &\iff T \vdash \theta(\Delta\mathfrak{r}(\langle\widetilde{z}\rangle)) \iff T \vdash \theta(\ulcorner\sigma\urcorner). \end{aligned} \qquad \dashv$$

The conclusion of the Fixed Point Lemma is usually stated in the stronger form

$$T \vdash \sigma \leftrightarrow \theta(\ulcorner\sigma\urcorner),$$

but (9) is sufficient to yield the applications. For the First Incompleteness Theorem, for example, we assume in addition that T is sufficiently expressive, we choose σ such that

$$T \vdash \sigma \iff T \vdash \neg(\exists u)\mathsf{Proof}_T(\ulcorner\sigma\urcorner, u) \quad (12)$$

where $\mathsf{Proof}_T(v, u)$ numeralwise expresses in T its proof relation, and we check that if T is consistent, then $T \nvdash \sigma$, and if T is also sound, then $T \nvdash \neg\sigma$. The only difference from the usual argument is that (12) does not quite say that σ "expresses its own unprovability"—only that it is is provable exactly when its unprovability is also provable. For the Rosser form of Gödel's Theorem, we need to assume that T is a bit stronger (as we will explain below) and *consistent*, though not necessarily sound, and the classical argument again works with the more complex Rosser sentence and this same, small different understanding of what the Rosser sentence says.

There is a problem, however, with the key hypothesis in Theorem 6 *that* $\mathrm{Th}(T)$ *is r.e.-complete.* This is trivial for sufficiently expressive and sound theories, including, of course, PA, but not so simple to verify for theories which are consistent but not sound. In fact it holds for every axiomatizable theory T which extends the system Q from Robinson (1950)—which is the standard hypothesis for incompleteness and undecidability results about consistent theories which need not be sound.[5]

Theorem 7 (Myhill (1955)). *If T is any consistent, axiomatizable extension of* Q*, then* $\mathrm{Th}(T)$ *is creative, and hence r.e.-complete.*

[5] For a description of Q and its properties, see (for example) Boolos, Burgess, and Jeffrey (1974) or even Kleene (1952), §41. Notice also that Theorem 7 does not lose much of its foundational interest or its important applications if we replace Q by PA in its statement—and the properties of Q that are used in the proof are quite obvious for PA.

The proof uses SRT with $\mathbb{V} = \{\!\{0, 1\}\!\}$, and (like all arguments of Myhill), it is very clever.

So consistent, axiomatizable extensions of Q are undecidable and hence incomplete; moreover, the Fixed Point Lemma Theorem 6 applies to them, and so we can construct specific, interesting sentences that they cannot decide, a la Rosser.

A minor (notational) adjustment of the proofs establishes Theorems 6 and 7 for any consistent, axiomatizable theory T, in any language, provided only that Q can be *interpreted in* T,[6] including, for example, ZFC; and then a third fundamental result of Myhill (1955) implies that the sets of theorems of any two such theories are *recursively isomorphic*.[7]

F. Solovay's theorem in provability logic. The (propositional) *modal formulas* are built up as usual using variables $p_0, p_1, \ldots$; a constant $\perp$ denoting falsity; the binary implication operator $\rightarrow$ (which we use with $\perp$ to define all the classical propositional connectives); and a unary operator $\Box$, which is usually interpreted by "it is necessary that". Solovay (1976) studies interpretations of modal formulas by sentences of PA in which $\Box$ is interpreted by "it is provable in PA that" and establishes some of the basic results of the logic of provability. His central argument appeals to SRT at a crucial point.

An *interpretation* of modal formulas is any assignment π of sentences of PA to the propositional variables, which is then extended to all formulas by the structural recursion

$$\pi(\perp) \equiv 0 = 1, \quad \pi(\varphi \rightarrow \psi) \equiv (\pi(\varphi) \rightarrow \pi(\psi)), \quad \pi(\Box\varphi) \equiv (\exists u)\mathsf{Proof}_{\mathsf{PA}}(\ulcorner \pi(\varphi) \urcorner, u).$$

A *modal formula* φ is PA-*valid* if $\mathsf{PA} \vdash \pi(\varphi)$ for every interpretation π.

The axiom schemes of the system GL are:

(A0) All tautologies;
(A1) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ (transitivity of provability);
(A2) $\Box\varphi \rightarrow \Box\Box\varphi$ (provable sentences are provably provable); and
(A3) $\Big(\Box(\Box\varphi \rightarrow \varphi)\Big) \rightarrow \Box\varphi$ (Löb's Theorem).

The inference rules of GL are:

(R1) $\varphi \rightarrow \psi, \varphi \implies \psi$ (Modus Ponens); and
(R2) $\varphi \implies \Box\varphi$ (Necessitation).

Theorem 8 (Solovay (1976)). *A modal formula φ is PA-valid if and only if it is a theorem of GL.*

6 A (weak) interpretation of T_1 in T_2 is any recursive map $\chi \mapsto \chi^*$ of the sentences of T_1 to those of T_2 such that $T_1 \vdash \chi \implies T_2 \vdash \chi^*$ and $T_2 \vdash (\neg\chi)^* \leftrightarrow \neg(\chi^*)$.

7 Pour-El and Kripke (1967) have interesting, stronger results of this type, whose proofs also use the Second Recursion Theorem.

Solovay shows also that the class of PA-valid modal formulas is decidable, and he obtains a similar decidable characterization of the modal formulas φ such that every interpretation $\pi(\varphi)$ is true (in the standard model), in terms of a related axiom system GL'.

The proof of Theorem 8 is long, complex, ingenious and depends essentially (and subtly) on the full strength of PA. It is nothing like the one-line derivations of Theorems 2 and 6 from standard facts about the relevant objects by a natural application of SRT or even the longer, clever proofs of Theorems 3, 4, 5 and 7 in which SRT still yields the punch lines. Still, I cannot see how one could possibly construct (or even think up) the key, "self-referential" closed term l of Solovay's Lemma 4.1 directly, without appealing to the Second Recursion Theorem,[8] and so, in that sense, SRT is an essential ingredient of his argument.

Next we turn to effective, grounded recursion, and perhaps the best way to explain it it to describe how it was introduced in Kleene (1938).

G. Constructive and recursive ordinals. Ordinal numbers can be viewed as the order types of well ordered sets, but also as *extended number systems*, which go beyond $\mathbb{N}$ and can be used to count (and regulate) transfinite iteration. Church and Kleene developed in the 1930s an extensive theory of such systems, aiming primarily at a *constructive theory of ordinals*—which, however, was only partly realized since many of their basic results could only be proved using classical logic. Kleene (1938) formulated the Second Recursion Theorem to solve one of the basic problems in this area.

A *notation system for ordinals* or $\mathfrak{r}$-*system* (in Kleene (1938)) is a set $S \subseteq \mathbb{N}$, together with a function $x \mapsto |x|_S$ which assigns to each x in S a countable ordinal so that the following conditions hold:

(ON1) There is a recursive partial function $K(x)$ whose domain of convergence includes S and such that, for $x \in S$,

$$\begin{aligned} |x|_S = 0 &\iff K(x) = 0, \\ |x|_S \text{ is a successor ordinal} &\iff K(x) = 1, \\ |x|_S \text{ is a limit ordinal} &\iff K(x) = 2. \end{aligned}$$

(ON2) There is a recursive partial function $P(x)$, such that if $x \in S$ and $|x|_S$ is a successor ordinal, then $P(x)\downarrow, P(x) \in S$ and $|x|_S = |P(x)|_S + 1$.

(ON3) There is a recursive partial function $Q(x,t)$, such that if $x \in S$ and $|x|_S$ is a limit ordinal, then for all t, $Q(x,t)\downarrow, |Q(x,t)|_S < |Q(x,t+1)|_S$ and $|x|_S = \lim_t |Q(x,t)|_S$.

In short, an $\mathfrak{r}$-system assigns *S-names* (number codes) to some ordinals, so that we can effectively recognize whether a code x names 0, a successor ordinal or a limit ordinal, and we can compute an S-name for the predecessor of

[8] Which Solovay invokes to define a function $h : \mathbb{N} \to \{0, \dots, n\}$ by the magical words "Our definition of h will be in terms of a Gödel number e for h. The apparent circularity is handled, using the recursion theorem, in the usual way."

each S-named successor ordinal and (S-names for) a strictly increasing sequence converging to each S-named limit ordinal.

A countable ordinal is *constructive* if it gets a name in some $\mathfrak{r}$-system.

The empty set is an $\mathfrak{r}$-system, as is $\mathbb{N}$, which names the finite ordinals, and every $\mathfrak{r}$-system (obviously) assigns names to a countable, initial segment of the ordinal numbers. It is not immediately clear, however, whether the set of constructive ordinals is countable or what properties it may have: the main result in Kleene (1938) clarifies the picture considerably by constructing a single $\mathfrak{r}$-system which names all of them.

This "maximal" $\mathfrak{r}$-system is defined by a straightforward transfinite recursion which yields the following

Lemma. *There is an $\mathfrak{r}$-system $(S_1, |\ |_1)$ such that:*

(i) $1 \in S_1$ *and* $|1|_1 = 0$.
(ii) *If* $x \in S_1$, *then* $2^x \in S_1$ *and* $|2^x|_1 = |x|_1 + 1$.
(iii) *If* φ_e^1 *is total and for all* t, $\varphi_e(t) \in S_1$ *and* $|\varphi_e(t)|_1 < |\varphi_e(t+1)|_1$, *then* $3 \cdot 5^e \in S_1$ *and* $|3 \cdot 5^e|_1 = \lim_t |\varphi_e(t)|$.

Theorem 9 (Kleene (1938)). *For every $\mathfrak{r}$-system $(S, |\ |_S)$, there is a unary recursive function ψ such that*

$$x \in S \implies \Big(\psi(x) \in S_1 \ \&\ |x|_S = |x|_1\Big).$$

In particular, the system $(S_1, |\ |_1)$ names all constructive ordinals.

Proof. Let K, P, Q be the recursive partial functions that come with $(S, |\ |_S)$, choose a number e_0 such that

$$\{S(e_0, z, x)\}(t) = \{e_0\}(z, x, t) = \{z\}(Q(x, t)),$$

fix by SRT (with $\mathbb{V} = \mathbb{N}, m = 0, n = 1$) a number $\widetilde{z}$ such that

$$\varphi_{\widetilde{z}}(x) = \begin{cases} 1, & \text{if } K(x) = 0, \\ 2^{\varphi_{\widetilde{z}}(P(x))}, & \text{if } K(x) = 1, \\ 3 \cdot 5^{S(e_0, \widetilde{z}, x)}, & \text{otherwise,} \end{cases}$$

and set $\psi(x) = \varphi_{\widetilde{z}}(x)$. The required properties of $\psi(x)$ are proved by a simple (possibly transfinite) induction on $|x|_S$. ⊣

In effect, the map from S to S_1 is defined by the obvious transfinite recursion on $|x|_S$, which is made effective by appealing to SRT—hence the name for the method.

The choice of numbers of the form $3 \cdot 5^e$ to name limit ordinals was made for reasons that do not concern us here, but it poses an interesting question: which ordinals get names in $(S_1', |\ |_1')$, defined by replacing $3 \cdot 5^e$ by (say) 7^e in the definition of S_1? They are the same constructive ordinals, of course, and the proof is by defining by effective grounded recursion (exactly as in the proof of Theorem 9) a pair of recursive functions ψ, ψ' such that

$$x \in S_1 \implies \Big(\psi'(x) \in S'_1 \ \&\ |x|_1 = |\psi'(x)|'_1\Big),$$
$$x \in S'_1 \implies \Big(\psi(x) \in S_1 \ \&\ |x|'_1 = |\psi(x)|_1\Big).$$

(And I do not know how else one could prove this "obvious" fact.)

The constructive ordinals are "constructive analogs" of the classical countable ordinals, and so the constructive analog of the first uncountable ordinal Ω_1 is

$$\omega_1^{\mathrm{CK}} = \sup\{|x|_1 \mid x \in S_1\} = \text{the least non-constructive ordinal},$$

the superscript standing for *Church-Kleene*. This is one of the most basic "universal constants" which shows up in logic—in many parts of it and under many guises. We list here one of its earliest characterizations.

A countable ordinal ξ is *recursive* if it is the order type of a recursive well ordering on some subset of $\mathbb{N}$.

Theorem 10 (Markwald (1954), Spector (1955)). *A countable ordinal ξ is constructive if and only if it is recursive.*

Both directions of the theorem are proved by (fairly routine) effective, grounded recursions.

H. The hyperarithmetical hierarchy. Kleene (1955c) associates with each $a \in S_1$ a set $H_a \subseteq \mathbb{N}$ so that:[9]

(H1) $H_1 = \mathbb{N}$.

(H2) $H_{2^b} = H'_b$.

(H3) If $a = 3 \cdot 5^e$, then $t \in H_a \iff (t)_1 \in H_{\varphi_e((t)_0)}$.

A relation $P \subseteq \mathbb{N}^n$ is *hyperarithmetical* if it is recursive in some H_a.[10]

This natural extension of the arithmetical hierarchy was also defined independently (and in different ways) by Davis (1950) and Mostowski (1951) who knew most of its basic properties, but not the central Theorem 11 below. To formulate it, we need to refer to the *arithmetical* and *analytical* relations on $\mathbb{N}$ which were introduced in Kleene (1943, 1955a); without repeating the definitions, we just record here the fact that they fall into two *hierarchies*

$$\Delta^0_1 \subsetneq \cdots \subsetneq \Sigma^0_\ell \cup \Pi^0_\ell \subsetneq \Delta^0_{\ell+1} \subsetneq \cdots \subsetneq \Delta^1_1 \subsetneq \cdots \subsetneq \Sigma^1_k \cup \Pi^1_k \subsetneq \Delta^1_{k+1} \subsetneq \cdots$$

[9] For each $A \subseteq \mathbb{N}$, A' is the *jump* of A.

[10] Actually, Kleene defines H_a only when $a \in O$, a subsystem of S_1 which has more structure and is "more constructive". I will disregard this fine point here, as many basic facts about O can only be proved classically and the attempt to use intuitionistic logic whenever it is possible clouds and complicates the arguments.

where Δ^0_1 comprises the recursive relations. Above all the arithmetical relations and at the bottom of the analytical hierarchy lie the Δ^1_1 relations which satisfy a double equivalence

$$P(\vec{x}) \iff (\exists\beta)Q(\vec{x},\beta) \iff (\forall\beta)R(\vec{x},\beta)$$

where β ranges over the Baire space $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$ and Q, R are arithmetical (or just Δ^0_2) relations on $\mathbb{N}^n \times \mathcal{N}$, suitably defined.

Theorem 11 (Kleene (1955c)). *A relation $P \subseteq \mathbb{N}^n$ is Δ^1_1 if and only if it is hyperarithmetical.*

Notice that the theorem characterizes only the Δ^1_1 relations on $\mathbb{N}$, so we will revisit the question later.

This is the most significant, foundational result in the sequence of articles Kleene (1935, 1943, 1944, 1955a,b,c, 1959) in which Kleene developed the theory of arithmetical, hyperarithmetical and analytical relations on $\mathbb{N}$, surely one of the most impressive bodies of work in the theory of definability.[11] Starting with the 1944 article, Kleene uses effective, grounded recursion in practically every argument: it is the key, indispensable technical tool for this theory.

From the extensive work of others in this area, we cite only one, early but spectacular result:

Theorem 12 (The Uniqueness Theorem, Spector (1955)). *There is a recursive function $u(a,b)$, such that if $|a|_1 \leq_o |b|_1$, then $u(a,b)$ is the code of a Turing machine M which decides H_a using H_b as oracle.*

In particular, if $|a|_1 = |b|_1$, then H_a and H_b have the same degree of unsolvability.

The definition of $u(a,b)$ is naturally given by effective grounded recursion.

Classical and effective descriptive set theory. Kleene was primarily interested in relations on $\mathbb{N}$, and he was more-or-less dragged into introducing quantification over $\mathcal{N}$ and the analytical hierarchy in order to find *explicit forms* for the hyperarithmetical relations. Once they were introduced, however, the analytical relations on Baire space naturally posed new problems: is there, for example, a *construction principle* for the Δ^1_1 relations which satisfy

$$P(\boldsymbol{x}) \iff (\exists\beta)Q(\boldsymbol{x},\beta) \iff (\forall\beta)R(\boldsymbol{x},\beta) \tag{13}$$

where $\boldsymbol{x} = \vec{x}, \vec{\alpha}$ varies over $\mathbb{N}^n \times \mathcal{N}^m$ and Q, R are arithmetical—a useful and interesting analog of Theorem 11?

In fact, these were very old problems, initially posed (and sometimes answered, in different form, to be sure) by Borel, Lebesgue, Lusin, Suslin and

[11] Kleene was the first logician to receive in 1983 the *Steele Prize for a seminal contribution to research* of the American Mathematical Society, specifically for the articles Kleene (1955a,b,c).

many others, primarily analysts and topologists who were working in *Descriptive Set Theory* in the first half of the 20th century. The similarity between what they had been doing and Kleene's work was first noticed by Mostowski (1946) and (especially) Addison (1954, 1959), and later work by many people created a common generalization of the classical and the new results now known as *Effective Descriptive Set Theory*; Moschovakis (2009) is the standard text on the subject and it gives a more careful history and a detailed introduction to this field.

The main inheritance of effective descriptive set theory from recursion theory is the propensity to *code*—assign to objects names that determine their relevant properties and then *compute*, *decide* or *define* functions and relations on these objects by operating on the codes rather than the objects coded—except that now we use points in $\mathcal{N}$ rather than numbers for codes. We operate on $\mathcal{N}$-codes using *recursion on* $\mathcal{N}$, by which a partial function $f : \mathcal{N}^n \rightharpoonup \mathcal{N}$ is recursive if $f(\vec{\alpha}) = (\lambda s) f^*(\vec{\alpha}, s)$ where $f^* : \mathcal{N}^n \times \mathbb{N} \rightharpoonup \mathbb{N}$ is recursive, i.e., there is a recursive relation $R \subseteq \mathbb{N}^{n+2}$ such that

$$f^*(\vec{\alpha}, s) = w \iff (\exists t) R(\overline{\alpha}_1(t), \ldots, \overline{\alpha}_n(t), s, w)$$

where $\overline{\alpha}(t) = \langle \alpha(0), \alpha(1), \ldots, \alpha(t-1) \rangle$. It is easy to show (and important) that for these partial functions,

$$f(\vec{\alpha}) = \beta \implies \beta \text{ is recursive in } \alpha_1, \ldots, \alpha_n.$$

A partial function $g : \mathcal{N}^n \rightharpoonup \mathcal{N}$ is *continuous* if

$$g(\vec{\alpha}) = f(\delta_0, \vec{\alpha})$$

with some recursive $f : \mathcal{N}^{n+1} \rightharpoonup \mathcal{N}$ and some $\delta_0 \in \mathcal{N}$.[12]

And here is the relevant version of SRT:

Theorem 13. *There is a recursive partial functions $\varphi^n : \mathcal{N}^{n+1} \rightharpoonup \mathcal{N}$ for each $n \in \mathbb{N}$, so that* (1) *and* (2) *hold with*

$$\{\varepsilon\}(\vec{\alpha}) = \varphi^n_\varepsilon(\vec{\alpha}) = \varphi^n(\varepsilon, \vec{\alpha}) \quad (\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathcal{N}^n).$$

(1) *Every continuous $g : \mathcal{N}^n \rightharpoonup \mathcal{N}$ is φ^n_ε for some ε, and every recursive $f : \mathcal{N}^n \rightharpoonup \mathcal{N}$ is φ^n_ε for some recursive ε.*

(2) *For all m, n, there is a recursive* (total) *function $S = S^m_n : \mathcal{N}^{m+1} \to \mathcal{N}$ such that*

$$\{S(\varepsilon, \vec{\beta})\}(\vec{\alpha}) = \{\varepsilon\}(\vec{\beta}, \vec{\alpha}) \quad (\varepsilon \in \mathcal{N}, \vec{\beta} \in \mathcal{N}^m, \vec{\alpha} \in \mathcal{N}^n).$$

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[12] Notice that the domain of convergence of a recursive, partial $f : \mathcal{N}^n \rightharpoonup \mathcal{N}$ is not (in general) Σ^0_1 but Π^0_2,

$$f(\vec{\alpha})\downarrow \iff (\forall s)(\exists w)(\exists t) R(\overline{\alpha}_1(t), \ldots, \overline{\alpha}_n(t), s, w).$$

It is not difficult to check that a partial $g : \mathcal{N}^n \rightharpoonup \mathcal{N}$ is continuous if its domain of convergence D_g is a G_δ ($\boldsymbol{\Pi}^0_2$) set and g is topologically continuous on D_g.

It follows that for every recursive, partial function $f(\varepsilon, \vec{\beta}, \vec{\alpha})$ of $(1+m+n)$ arguments, there is a total recursive function $\widetilde{\gamma}(\vec{\beta})$ of m arguments such that

$$\{\widetilde{\gamma}(\vec{\beta})\}(\vec{\alpha}) = f(\widetilde{\gamma}(\vec{\beta}), \vec{\beta}, \vec{\alpha}) \quad (\vec{\beta} \in \mathcal{N}^m, \vec{\alpha} \in \mathcal{N}^n). \tag{14}$$

I will confine myself here to formulating just two results from effective descriptive set theory. Both are proved by effective, grounded recursion justified by this version of SRT, and they suggest the power of the method and the breadth of its applicability.

I. The Suslin-Kleene Theorem. Classical descriptive set theory is the study of definable subsets of an arbitrary *Polish* (metrizable, separable, complete topological) *space* $\mathcal{X}$, including the real numbers $\mathbb{R}$ and products of the form $\mathcal{X} = \mathbb{N}^n \times \mathcal{N}^m$. The class $\mathbf{B} = \mathbf{B}_{\mathcal{X}}$ of the *Borel subsets* of $\mathcal{X}$ is the smallest class which contains all the open balls and is closed under countable unions and complements, and starting with this and iterating projection on $\mathcal{N}$ and complementation we get the *projective hierarchy*,

$$\mathbf{B} \subseteq \utilde{\boldsymbol{\Delta}}^1_1 \subsetneq \utilde{\boldsymbol{\Sigma}}^1_1 \cup \utilde{\boldsymbol{\Pi}}^1_1 \subsetneq \utilde{\boldsymbol{\Delta}}^1_2 \subsetneq \utilde{\boldsymbol{\Sigma}}^1_2 \cup \utilde{\boldsymbol{\Pi}}^1_2 \subsetneq \cdots$$

The class $\utilde{\boldsymbol{\Delta}}^1_1$ comprises the sets which satisfy (13) with Q and R Borel (or even $\utilde{\boldsymbol{\Delta}}^0_3$) subsets of the product space $\mathcal{X} \times \mathcal{N}$, and their identification with the Borel sets is a cornerstone of the theory:

Theorem 14 (Suslin (1917)). *For every Polish space $\mathcal{X}$, $\utilde{\boldsymbol{\Delta}}^1_1 = \boldsymbol{B}$.*

There is an obvious resemblance in form between this result of Suslin and Kleene's Theorem 11, which led Mostowski and Addison to talk first of "analogies" between descriptive set theory and the "hierarchy theory" of Kleene, as it was then called; but neither of these results implies the other, as Suslin's Theorem is trivial on $\mathbb{N}^n$ and Kleene's Theorem says nothing about subsets of $\mathcal{N}$—not to mention the real numbers or other Polish spaces which are very important for the classical theory. One of the first, substantial achievements of the "marriage" of the classical and the new theory was the derivation of a basic fact which extends (and refines) both Theorems 11 and 14.

We code the $\utilde{\boldsymbol{\Sigma}}^1_k$ and $\utilde{\boldsymbol{\Pi}}^1_k$ subsets of each Polish space in a natural way, so that the usual operations on them (countable unions and intersections, for example) are *recursive in the codes*, also in a natural way. A $\utilde{\boldsymbol{\Delta}}^1_1$-code of a set A is the (suitably defined) pair $\langle \alpha, \beta \rangle$ of a $\utilde{\boldsymbol{\Sigma}}^1_1$ and a $\utilde{\boldsymbol{\Pi}}^1_1$ code for A. Finally, we code the Borel subsets of each $\mathcal{X}$, so that a Borel code of a set A supplies all the information necessary to construct A from open balls by iterating the operations of countable union and complementation. And with these definitions at hand:

Theorem 15 (The Suslin-Kleene Theorem, see Moschovakis (2009)). *For each Polish space $\mathcal{X}$ which admits a recursive presentation, there are recursive functions $u, v : \mathcal{N} \to \mathcal{N}$ such that if α is a Borel code of a set $A \subseteq \mathcal{X}$, then $u(\alpha)$ is a $\utilde{\boldsymbol{\Delta}}^1_1$-code of A, and if β is a $\utilde{\boldsymbol{\Delta}}^1_1$-code of A, then $v(\beta)$ is a Borel code of A.*

In particular, the Δ^1_1 subsets of $\mathcal{X}$ are exactly the Borel subsets of $\mathcal{X}$ which have recursive codes.[13]

The Suslin-Kleene Theorem implies immediately Suslin's Theorem and (with just a little extra work) Kleene's Theorem 11. It is shown by adapting one of the classical proofs of Suslin's Theorem rather than Kleene's much more difficult argument, and, of course, effective grounded recursion.

J. The Coding Lemma. The last example is from the exotic world of *determinacy*, about as far from recursion theory as one could go—or so it seems at first.

Theorem 16 (The Coding Lemma, Moschovakis (1970, 2009)). *In* $\mathsf{ZFDC}+\mathsf{AD}$: *if there exists a surjection $f : \mathbb{R} \twoheadrightarrow \kappa$ of the continuum onto a cardinal κ, then there exists a surjection $g : \mathbb{R} \twoheadrightarrow \mathcal{P}(\kappa)$ of the continuum onto the powerset of κ.*

Here ZFDC stands for ZFC with the Axiom of (full) Choice AC replaced by the weaker *Axiom of Dependent Choices* DC, and AD is the *Axiom of Determinacy*, which is inconsistent with AC. It has been shown by Martin and Steel (1988) and Woodin (1988) that (granting the appropriate large cardinal axioms), AD holds in $L(\mathbb{R})$, the smallest model of ZFDC which contains all ordinals and all real numbers. Long before that great (and reassuring!) result, however, AD was used systematically to uncover the structure of the analytical and projective hierarchies—so it has something to do with recursion theory after all!

It is not possible to give here a brief, meaningful explanation of all that goes into the statement of Theorem 16 which, in any case, is only a corollary of a substantially stronger and more general result. Notice, however, that in a world where it holds, $\mathbb{R}$ is immense in size, if we measure size by surjections: it can be mapped onto $\aleph_1$ (classically), and so onto $\aleph_2$, and inductively onto every $\aleph_n$ and so onto $\aleph_\omega$, etc., all the way onto every $\aleph_\xi$ for $\xi < \aleph_1$: and it can also be mapped onto the powerset of each of these cardinals. This *surjective size* of $\mathbb{R}$ is actually immense in the world of AD, the Coding Lemma is one of the important tools in proving this—and it does not appear possible to prove the Coding Lemma without using SRT, which creeps in this way into the study of cardinals, perhaps the most purely set-theoretic part of set theory.

The hypothesis AD of full determinacy is covered in Section 7D, *The completely playful universe* of Moschovakis (2009), part of Chapter 7 whose title is *The Recursion Theorem.*

[13] The restriction to *recursively presented spaces* is inessential, because every Polish space can be presented *recursively in some* $\varepsilon_0 \in \mathcal{N}$, and then the whole theory "relativises" to this ε_0. Notice also that Moschovakis (2009) develops the effective theory primarily for countable and perfect Polish spaces, but the definitions makes sense for arbitrary Polish spaces, and then each such $\mathcal{X}$ is isometric with the Π^0_1 subset $\mathcal{X} \times \{\!\!\{\lambda t0\}\!\!\}$ of the perfect $\mathcal{X} \times \mathcal{N}$; because of this representation, the Suslin-Kleene Theorem holds for all recursively presented Polish spaces—as do many (though not all) results in Moschovakis (2009).

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