**Mathematics M114S, Spring 2018**
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**Second Midterm, March 6, 2018**

**Name (last name first):** ________________________

**Signature:** ________________________

There are 2 multiple choice problems (with several parts) worth 54 points altogether—except that wrong solutions will subtract points, so the sum may be negative. There are also 5 other problems worth 60 points for a total of 114 points with 100 considered a "perfect score" for purposes of curving.

Some of the questions are completely trivial while others require some thinking—including a couple of the multiple choice ones; so do first those that you know immediately how to do.

You may use all the results we have covered in Chapters 1 – 9 of the book.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

## Solved!

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Problem 6. ________

Problem 7. ________

Total: ____________

**Problem 1**. For each of the statements $P$ below, mark the correct answer, where

- [T] means that P can be proved in Z+DC (i.e., without the full Axiom of Choice);
- [F] means that the negation of P can be proved in Z+DC;
- [T with AC] means that P can be proved in Z+AC;
- [F with AC] means that the negation can be proved in Z+AC;
- [None] means P cannot be proved or disproved in ZF+AC.

You do not need to prove your answer—which in the case of [None] you cannot do.

**(1a)** [T] [F] [T with AC] [F with AC] [None]

$(\forall\, A, B)\Big(\text{if } B \text{ is wellorderable and } A \leq_c B, \text{ then } A \text{ is wellorderable}\Big).$

SOLUTION. [T].

**(1b)** [T] [F] [T with AC] [F with AC] [None]

$(\forall\, A, B)\Big(\text{if } B \text{ is wellorderable and } B \leq_c A, \text{ then } A \text{ is wellorderable}\Big).$

SOLUTION. [T with AC].

**(1c)** [T] [F] [T with AC] [F with AC] [None]

$(\forall A)\Big(\text{if } h(A) \text{ is the Hartogs set of } A, \text{ then } A \leq_c h(A)\Big).$

SOLUTION. [T with AC].

**(1d)** [T] [F] [T with AC] [F with AC] [None]

$(\forall A)\Big(\text{if } h(A) \text{ is the Hartogs set of } A, \text{ then } h(A) \leq_c A\Big).$

SOLUTION. [F].

**(1e)** [T] [F] [T with AC] [F with AC] [None]

$(\forall \text{ infinite } A)\Big(\text{if } h(A) \text{ is the Hartogs set of } A, \text{ then } h(A) \leq_c \mathcal{P}(A)\Big).$

SOLUTION. [T with AC]. Each member of $h(A)$ is the equivalence class of a set of wellordered subsets of $A$, and we can use AC to get from this an injection of $h(A)$ into $\mathcal{P}(A \times A)$.

**(1f)** [T] [F] [T with AC] [F with AC] [None]

$(\forall \text{ infinite } A)\Big(\text{if } h(A) \text{ is the Hartogs set of } A, \text{ then } \mathcal{P}(A) \leq_c h(A)\Big).$

SOLUTION. [None]. This is equivalent to the Generalized Continuum Hypothesis.

**Problem 2**. In each of the three linear orderings drawn, mark all the answers that apply:

| | | | | |
|---|---|---|---|---|
| (1) $0, 1, \dots$ | wellordering | not a wellordering | inductive | not inductive |
| (2) $1, 2, \dots, 0$ | wellordering | not a wellordering | inductive | not inductive |
| (3) $0, \dots, -2, -1, 1, 2, \dots$ | wellordering | not a wellordering | inductive | not inductive |

SOLUTION. (1) This is the usual ordering of $\mathbb{N}$ and it is a wellordering and not inductive as a poset.

(2) This is the usual ordering of $\mathbb{N}$ with a largest element added, and it is a wellordering and inductive as a poset.

(3) This is the ordering on the rational integers $\mathbb{Z}$ with a least element added. It is not a wellordering, because the set of negative numbers has no minimum, and it is not inductive because the entire poset does not have a sup.

**Problem 3**. Give an example of two wellordered sets $U$ and $V$ such that

$$U +_o V \neq_o V +_o U.$$

SOLUTION. Take $U = \mathbb{N}$, the natural numbers with their usual ordering and $V = \{a\}$ the (only) ordering of a single point $a$ where $a \notin \mathbb{N}$. Now the two sums look like

$$\mathbb{N} +_o \{a\} = 0, 1, \dots, a, \quad \{a\} +_o \mathbb{N} = a, 0, 1, \dots,$$

so $\{a\} +_o \mathbb{N} =_o \mathbb{N}$, which is a proper initial segment of $\mathbb{N} +_o \{a\}$, so

$$\{a\} +_o \mathbb{N} <_o \mathbb{N} +_o \{a\}.$$

**Problem 4**. Let $U$ be a wellordered set and let $E$ some set, $a \in E$, and

$$g : E \to E, \quad h : U \to E,$$

two functions on the indicated domain and range.

**(4a)** Prove that there is exactly one function

$$f : U \to E,$$

such that

$$f(x) = \begin{cases} a, & \text{if } x = 0_U, \text{ the least point in } U, \\ g(f(y)), & \text{if } x = S(y), \text{ the successor of some } y \in U, \\ h(x), & \text{if } x \text{ is a limit point of } U. \end{cases}$$

SOLUTION. This is an application of Definition by Transfinite Recursion, especially simple because when $x$ is a limit point, $f(x)$ does not depend on any values $f(y)$ for $y <_U x$.

**(4b)** Consider the special case where $U$ is similar to $\mathbb{N} +_o \mathbb{N}$,

$$U : 0 < 2 < 4 < \cdots < 1 < 3 < 5 < \cdots,$$

$E = \mathbb{N}$, $a = 0$, $g(y) = y + 5$ and $h(1) = 17$, and compute the values

(1) $f(4) =$

(2) $f(3) =$

SOLUTION. (1) $f(4) = f(S(2)) = g(f(2)) = f(2)+5 = g(S(0))+5 = f(0)+5+5 = 0 + 5 + 5 = 10$.

(2) $f(3) = f(S(1)) = g(f(1)) = f(1) + 5 = h(1) + 5 = 17 + 5 = 22$.

**Problem 5**. With the customary meanings of *ancestor* and *descendant* and assuming that the current population of the world is finite (about 7.6 billion), prove that

**either** *humans have existed for infinitely many years*

**or** *there is a finite set* $\{b_1, \dots, b_n\}$ *of (ancient) humans with no human parents such that all of us living today are their descendants.*

Note. The prevailing hypothesis is the second one, defended either on religious grounds ($b_1$ and $b_2$ are Adam and Eve) or by the theory of evolution, where the main task is to make precise the meaning of "human"—which, of course, we assume. Set theory comes in only to show that there cannot be infinitely many "original humans" unless there were infinitely many generations of humans before our own, or (equivalently), some living human has an infinitely long "ancestry line".

Solution. Consider the tree $T$ on the set $H$ of all humans who are alive now or were alive at some previous time, where

$$(h_0, \dots, h_n) \in T \iff h_0 \text{ is a living person } \& (\forall i < n)[h_i \text{ is a child of } h_{i+1}].$$

This is a finitely branching tree: there are some 7.6 billion choices for $h_0$, but no more than two choices for $h_{i+1}$ after that, since every human has (at most) one father and one mother. By König's Lemma, either $T$ is finite, in which case we are all descendants of the finitely many (original) humans at the leaves; or $T$ has an infinite branch, which means that there were humans who lived arbitrary long ago, in fact one of us has an infinite "ancestry line".

**Problem 6** (AC). Prove that for every infinite cardinal $\kappa$,

$$2^\kappa =_c 3^\kappa.$$

You may use all the results we have proved about cardinal equations and inequalities without citing each of them specifically, but you should point out where the Axiom of Choice is used.

SOLUTION.

$$2^\kappa \leq_c 3^\kappa \leq_c 4^\kappa =_c \left(2^2\right)^\kappa =_c 2^{2\kappa} =_c 2^\kappa,$$

using the trivial, basic properties of cardinal arithmetic and the absorption law

$$2\kappa =_c \kappa + \kappa =_c \kappa$$

in the last step, which requires the Axiom of Choice. Now this shows that $2^\kappa \leq_c 3^\kappa \leq c2^\kappa$, and then the Schrder-Bernstein Theorem gives $2^\kappa =_c 3^\kappa$.

**Problem 7** (AC). Assume that $\kappa$ and $\lambda$ are infinite cardinals, and

$$2^\kappa = \bigcup_{i\in\lambda} A_i \text{ with each } A_i <_c 2^\kappa;$$

prove that

$$2^{\kappa\lambda} >_c 2^\kappa$$

and infer that $\lambda >_c \kappa$.

Hint: Use König's Theorem.

Solution. Using the assumption and König's Theorem we have

$$2^\kappa = \bigcup_{i\in\lambda} A_i <_c \prod_{i\in\lambda} 2^\kappa =_c \left(2^\kappa\right)^\lambda =_c 2^{\kappa\lambda};$$

now, if $2^{\kappa\lambda} \leq_c 2^\kappa$, this would imply that $2^\kappa <_c 2^\kappa$, which is absurd, so we infer that $2^{\kappa\lambda} >_c 2^\kappa$.

This implies that $\lambda >_c \kappa$; because if $\lambda \leq_c \kappa$, then $\kappa\lambda = \kappa$ by the Absorption Laws, and so $2^{\kappa\lambda} =_c 2^\kappa$, contradicting what we have proved.