Mathematics M114S, Spring 2018
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First Midterm, February 6, 2018

Name (last name first): ______________________

Signature: ______________________

There are 9 questions (organized in 4 problems) and each is worth 15 points for a total of 135; 100 points count for a "perfect" score, so you can skip two questions or afford to make some errors.

Some of the questions are completely trivial while others require some work; so do first those that you know immediately how to do.

You may use all the results we have covered in Chapters 1 - 5 of the book.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

Some notations:

$$A \setminus B = \{x \in A \mid x \notin B\}, \quad \Delta = (\mathbb{N} \to \{0,1\}), \quad \aleph_0 = |\mathbb{N}|, \quad \mathfrak{c} = |\mathcal{P}(\mathbb{N})|$$

Solved!

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Total: ______________

Problem 1. For each of the following claims, determine whether it is true or false and prove your answer.

(1a) If $f : X \to Y$ is a function and $A, B \subseteq X$, then $f[A \setminus B] = f[A] \setminus f[B]$.

Solution. FALSE. For a counterexample, fix a non-empty set C and put

$$A = \{0\} \times C, \quad B = \{1\} \times C, \quad X = A \cup B, \quad f(i, x) = x.$$

Now check:

$$f[A \setminus B] = f[A] = C, \text{ because } A \setminus B = A,$$

but $f[B] = C$, so $f[A] \setminus f[B] = C \setminus C = \emptyset$.

(1b) $f : X \rightarrowtail Y$ is an injection and $A, B \subseteq X$, then $f[A \setminus B] = f[A] \setminus f[B]$.

Solution. TRUE. We prove the two inclusions that are required.

If $y \in f[A \setminus B]$, then there is an $x \in A \setminus B$ such that $f(x) = y$, so, in particular, $y \in f[A]$. Now, it cannot also be that $y \in f[B]$; because this would mean that there is a $z \in B$ such that $f(z) = y$, and $z \neq x$ (since $x \notin B$) and this contradicts the hypothesis that f is an injection.

If $y \in f[A] \setminus f[B]$, then there is some $x \in A$ such that $f(x) = y$; now it cannot be that $x \in B$, since this implies that $y \in f[B]$, so $x \in A \setminus B$ and $y \in f[A \setminus B]$. (This second part does not need the hypothesis that f is an injection.)

(1c) If A is countable and $x_0 \in A$, then $\{x \in A \mid x \neq x_0\}$ is countable.

Solution. TRUE. $A \setminus \{x_0\} \subset A$, and so $A \setminus \{x_0\} \leq_c A$; the hypothesis gives $A \leq_c \mathbb{N}$; and so $A \setminus \{x_0\} \leq_c \mathbb{N}$ by the transitivity of $\leq_c$.

Problem 2. Compute the following cardinals; you may use the definitions and any results we have covered, and *every answer is one of* $\aleph_0$, $\mathfrak{c}$, *or* $2^{\mathfrak{c}}$. (You get half credit for guessing the correct answer and full credit for proving it.)

(2a) $\aleph_0^{\aleph_0} =_c \boxed{\mathfrak{c}}$

Solution. Using known or easy $\leq_c$-inequalities, we compute:

$$\begin{aligned}\aleph_0^{\aleph_0} =_c (\mathbb{N} \to \mathbb{N}) \leq_c \mathcal{P}(\mathbb{N} \times \mathbb{N}) &\text{ (because every function is a set of ordered pairs)}\\ =_c \mathcal{P}(\mathbb{N}) &\text{ (because } \mathbb{N} \times \mathbb{N} =_c \mathbb{N})\\ =_c (\mathbb{N} \to \{0,1\}) \leq_c (\mathbb{N} \to \mathbb{N}) =_c \aleph_0^{\aleph_0},\end{aligned}$$

so all these sets have the same cardinality by the Schröder-Bernstein Theorem, so

$$\aleph_0^{\aleph_0} =_c \mathcal{P}(\mathbb{N}) =_c \mathfrak{c}.$$

(2b) $\mathfrak{c} \cdot \mathfrak{c} =_c \boxed{\mathfrak{c}}$.

Solution. We use the definitions and known equinumerosities:

$$\mathfrak{c} \cdot \mathfrak{c} =_c 2^{\aleph_0} \cdot 2^{\aleph_0} =_c 2^{\aleph_0 + \aleph_0} =_c 2^{\aleph_0} =_c \mathfrak{c}.$$

(2c) $\mathfrak{c}^{\mathfrak{c}} =_c \boxed{2^{\mathfrak{c}}}$.

Solution. The most obvious way to start this is

$$\mathfrak{c}^{\mathfrak{c}} =_c (2^{\aleph_0})^{\mathfrak{c}} =_c 2^{\aleph_0 \cdot \mathfrak{c}}.$$

To finish the computation we need the value of $\aleph_0 \cdot \mathfrak{c}$ (which we suspect is $\mathfrak{c}$), and we need Schröer-Bernstein for this once more:

$$\mathfrak{c} \leq_c \aleph_0 \cdot \mathfrak{c} =_c \aleph_0 \cdot \aleph_0^{\aleph_0} =_c \aleph_0^{1+\aleph_0} =_c \aleph_0^{\aleph_0} =_c \mathfrak{c},$$

Problem 3. Suppose $g : A \to A$ is a function and we define the sets $A_n \subseteq A$ by the recursion

$$A_0 = A, \quad A_{n+1} = g[A_n] \qquad (n \in \mathbb{N}).$$

(3a) Prove that for every n, $A_{n+1} \subseteq A_n$.

SOLUTION. The simplest way (for both parts) is to prove first that

$$(*) \qquad Y \subseteq X \subseteq A \Longrightarrow g[Y] \subseteq g[X];$$

in full detail, this because

$$\begin{aligned} y \in g[Y] &\Longrightarrow y = g(x) \text{ for some } x \in Y \\ &\Longrightarrow y = g(x) \text{ for some } x \in X \text{ because } Y \subseteq X \Longrightarrow y \in g[X]. \end{aligned}$$

We use this to prove the required $A_{n+1} \subseteq A_n$ by induction on n.

Basis, $n = 0$: $A_1 \subseteq A = A_0$, because $g : A \to A$.

Induction step: The induction hypothesis is $A_{n+1} \subseteq A_n$, so by $(*)$,

$$A_{n+2} = g[A_{n+1}] \subseteq g[A_n] = A_{n+1}.$$

(3b) Prove that if $A^* = \bigcap_n A_n = \{x \in A \mid \text{ for all } n, x \in A_n\}$, then $g[A^*] = A^*$.

Note! This is false, unless we assume that $g : A \rightarrowtail A$ is an injection. (Try to get a counterexample!)

SOLUTION. One inclusion is correct: for any $g : A \to A$, $\boxed{g[A^*] \subseteq A^*}$, as follows: $A_* \subseteq A_n$ for every n; so $g[A^*] \subseteq g[A_n] = A_{n+1}$, for every n by $(*)$; and hence

$$g[A^*] \subseteq \bigcap_{n=0}^{\infty} A_{n+1} = \bigcap_{n=1}^{\infty} A_n \subseteq A_0 \cap \bigcap_{n=1}^{\infty} A_n = A^*.$$

For the converse inclusion $\boxed{A^* \subseteq g[A^*]}$, we assume that $g : A \rightarrowtail A$ is an injection and we argue as follows: if $x \in A^*$, then $x \in A_1 = g[A]$, and hence $x = g(u_0)$ for some $u \in A$; moreover, for every n, $x \in A_{n+1}$, and so there is some $u_n \in A_n$ such that $x = g(u_n)$; but since g is an injection, we must have that $u_n = u_0$; and so $u_0 \in A^*$ and $x \in g[A^*]$.

Problem 4. Prove that if there is an surjection $f : A \twoheadrightarrow B$, then $B \leq_c \mathcal{P}(A)$.

Solution. Given $f : A \twoheadrightarrow B$, define $g : B \to \mathcal{P}(A)$ by

$$g(y) = f^{-1}[\{y\}] = \{x \in A \mid f(x) = y\}.$$

Notice that for every $y \in B$, $g(y) \neq \emptyset$, because f is a surjection; and

$$y_1 \neq y_2 \Longrightarrow g(y_1) \cap g(y_2) = \emptyset,$$

since if some $x \in (g(y_1) \cap g(y_2))$, then $f(x) = y_1$ and $f(x) = y_2$ which cannot happen if $y_1 \neq y_2$. In particular, $y_1 \neq y_2 \Longrightarrow g(y_1) \neq g(y_2)$, so g is an injection.