There are 9 questions (organized in 4 problems) and each is worth 15 points for a total of 135; 100 points count for a “perfect” score, so you can skip two questions or afford to make some errors.

Some of the questions are completely trivial while others require some work: so do first those that you know immediately how to do.

You may use all the results we have covered in Chapters 1 - 5 of the book.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the “architecture” of your argument.

Some notations:

\[ A \setminus B = \{ x \in A \mid x \notin B \} \quad \Delta = (\mathbb{N} \to \{0, 1\}) \quad \aleph_0 = |\mathbb{N}| \quad c = |\mathcal{P}(\mathbb{N})| \]

Solved!
Problem 1. For each of the following claims, determine whether it is true or false and prove your answer.

(1a) If \( f : X \to Y \) is a function and \( A, B \subseteq X \), then \( f[A \setminus B] = f[A] \setminus f[B] \).

Solution. FALSE. For a counterexample, fix a non-empty set \( C \) and put
\[
A = \{0\} \times C, \quad B = \{1\} \times C, \quad X = A \cup B, \quad f(i, x) = x.
\]
Now check:
\[
f[A \setminus B] = f[A] = C, \quad \text{because } A \setminus B = A,
\]
but \( f[B] = C \), so \( f[A] \setminus f[B] = C \setminus C = \emptyset \).

(1b) \( f : X \to Y \) is an injection and \( A, B \subseteq X \), then \( f[A \setminus B] = f[A] \setminus f[B] \).

Solution. TRUE. We prove the two inclusions that are required.

If \( y \in f[A \setminus B] \), then there is an \( x \in A \setminus B \) such that \( f(x) = y \); so, in particular, \( y \in f[A] \). Now, it cannot also be that \( y \in f[B] \); because this would mean that there is a \( z \in B \) such that \( f(z) = y \), and \( z \neq x \) (since \( x \notin B \)) and this contradicts the hypothesis that \( f \) is an injection.

If \( y \in f[A] \setminus f[B] \), then there is some \( x \in A \) such that \( f(x) = y \); now it cannot be that \( x \in B \), since this implies that \( y \in f[B] \), so \( x \in A \setminus B \) and \( y \in f[A \setminus B] \). (This second part does not need the hypothesis that \( f \) is an injection.)

(1c) If \( A \) is countable and \( x_0 \in A \), then \( \{x \in A \mid x \neq x_0\} \) is countable.

Solution. TRUE. \( A \setminus \{x_0\} \subset A \), and so \( A \setminus \{x_0\} \leq_c A \); the hypothesis gives \( A \leq_c \mathbb{N} \); and so \( A \setminus \{x_0\} \leq_c \mathbb{N} \) by the transitivity of \( \leq_c \).
Problem 2. Compute the following cardinals: you may use the definitions and any results we have covered, and every answer is one of $\aleph_0$, $c$, or $2^c$. (You get half credit for guessing the correct answer and full credit for proving it.)

(2a) $\aleph_0^{\aleph_0} = c$ [Proof]

Solution. Using known or easy $\leq_c$-inequalities, we compute:

$\aleph_0^{\aleph_0} = (\aleph_0 \to \aleph_0) \leq_c \mathcal{P}(\aleph_0 \times \aleph_0)$ (because every function is a set of ordered pairs)

$= c \mathcal{P}(\aleph_0)$ (because $\aleph_0 \times \aleph_0 = c$)

$= c (\aleph_0 \to \{0, 1\}) \leq_c (\aleph_0 \to \aleph_0) = c^{\aleph_0}$.

so all these sets have the same cardinality by the Schröder-Bernstein Theorem, so

$\aleph_0^{\aleph_0} = c \mathcal{P}(\aleph_0) = c$.

(2b) $c \cdot c = c$ [Proof]

Solution. We use the definitions and known equinumerosities:

$c \cdot c = c^{\aleph_0} \cdot c^{\aleph_0} = c^{\aleph_0 + \aleph_0} = c^{\aleph_0} = c$.

(2c) $c^c = 2^c$ [Proof]

Solution. The most obvious way to start this is

$c^c = c (2^{\aleph_0})^c = c^{\aleph_0}.$

To finish the computation we need the value of $\aleph_0 \cdot c$ (which we suspect is $c$), and we need Schröder-Bernstein for this once more:

$c \leq c \cdot c = c \cdot \aleph_0^{\aleph_0} = c^{\aleph_0 + \aleph_0} = c^{\aleph_0} = c$. 
Problem 3. Suppose \( g : A \to A \) is a function and we define the sets \( A_n \subseteq A \) by the recursion
\[
A_0 = A, \quad A_{n+1} = g[A_n] \quad (n \in \mathbb{N}).
\]

(3a) Prove that for every \( n \), \( A_{n+1} \subseteq A_n \).

Solution. The simplest way (for both parts) is to prove first that
\[
Y \subseteq X \subseteq A \implies g[Y] \subseteq g[X];
\]
in full detail, this because
\[
y \in g[Y] \implies y = g(x) \text{ for some } x \in Y
\implies y = g(x) \text{ for some } x \in X \text{ because } Y \subseteq X \implies y \in g[X].
\]

We use this to prove the required \( A_{n+1} \subseteq A_n \) by induction on \( n \).

Basis, \( n = 0 \): \( A_1 \subseteq A = A_0 \), because \( g : A \to A \).

Induction step: The induction hypothesis is \( A_{n+1} \subseteq A_n \), so by \((*)\),
\[
A_{n+2} = g[A_{n+1}] \subseteq g[A_n] = A_{n+1}.
\]

(3b) Prove that if \( A^* = \bigcap_n A_n = \{ x \in A \mid \text{ for all } n, x \in A_n \} \), then \( g[A^*] = A^* \).

Note! This is false, unless we assume that \( g : A \to A \) is an injection. (Try to get a counterexample!)

Solution. One inclusion is correct: for any \( g : A \to A \), \( g[A^*] \subseteq A^* \) as follows:
\( A_* \subseteq A_n \) for every \( n \): so \( g[A^*] \subseteq g[A_n] = A_{n+1} \), for every \( n \) by \((*)\); and hence
\[
g[A^*] \subseteq \bigcap_{n=0}^\infty A_{n+1} = \bigcap_{n=1}^\infty A_n \subseteq A_0 \cap \bigcap_{n=1}^\infty A_n = A^*.
\]

For the converse inclusion \( A^* \subseteq g[A^*] \) we assume that \( g : A \to A \) is an injection and we argue as follows: if \( x \in A^* \), then \( x \in A_1 = g[A] \), and hence \( x = g(u_0) \) for some \( u \in A \); moreover, for every \( n, x \in A_{n+1} \), and so there is some \( u_n \in A_n \) such that \( x = g(u_n) \); but since \( g \) is an injection, we must have that \( u_n = u_0 \); and so \( u_0 \in A^* \) and \( x \in g[A^*] \).
Problem 4. Prove that if there is an surjection $f : A \twoheadrightarrow B$, then $B \leq c \mathcal{P}(A)$.

Solution. Given $f : A \twoheadrightarrow B$, define $g : B \rightarrow \mathcal{P}(A)$ by

$$g(y) = f^{-1}\{y\} = \{x \in A \mid f(x) = y\}.$$  

Notice that for every $y \in B$, $g(y) \neq \emptyset$, because $f$ is a surjection; and

$$y_1 \neq y_2 \implies g(y_1) \cap g(y_2) = \emptyset,$$

since if some $x \in (g(y_1) \cap g(y_2))$, then $f(x) = y_1$ and $f(x) = y_2$ which cannot happen if $y_1 \neq y_2$. In particular, $y_1 \neq y_2 \implies g(y_1) \neq g(y_2)$, so $g$ is an injection.