Effective Descriptive Set Theory

what it is about

Lecture 2, Effective Borel, analytic and co-analytic pointsets

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Outline

Lecture 1. Recursion in Polish spaces

Lecture 2. Effective Borel, analytic and co-analytic pointsets

Lecture 3. Structure theory for pointclasses

- Definitions and basic facts in the first lecture:

– Recursive Polish space — just space from now on
– Pointset: a subset $P \subseteq \mathcal{X}$ of a space
– Pointclass: a collection Γ of pointsets, $\Gamma(\mathcal{X}) = \{P \subseteq \mathcal{X} : P \in \Gamma\}$
– Σ^0_1: the pointclass of semirecursive pointsets
– Locally recursive partial functions $f : \mathcal{X} \rightharpoonup \mathcal{Y}$
– The points of Γ: $y \in \Gamma \iff \mathcal{U}(y) = \{s : y \in N_s(\mathcal{Y})\} \in \Gamma(\mathbb{N})$
★ The Refined Surjection Theorem
★ Parametrized pointclasses, the 2nd Recursion Theorem
– The Kleene calculus for local recursion, the 2nd Recursion Theorem

Two basic facts from Lecture 1

- If a pointclass Γ is parametrized, then

(1) Γ is closed under total recursive substitutions, and

(2) every $\Gamma(\mathcal{X})$ has a parametrization, a pointset $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ such that for every $P \in \Gamma(\mathcal{N} \times \mathcal{X})$, there is a total recursive $\mathsf{S}^P : \mathcal{N} \to \mathcal{N}$ satisfying $\boxed{P(\alpha, x) \iff G(\mathsf{S}^P(\alpha), x)}$

$\Rightarrow$ 2nd RT: $P \in \Gamma(\mathcal{N} \times \mathcal{X}) \implies (\exists \text{ recursive } \widetilde{\varepsilon}) \boxed{P(\widetilde{\varepsilon}, x) \iff G(\widetilde{\varepsilon}, x)}$

- Refined Surjection Theorem *For every space $\mathcal{X}$, there is a total recursive function $\pi : \mathcal{N} \to \mathcal{X}$ and a Π^0_1 set $F \subseteq \mathcal{N}$ such that*

π is one-to-one on $F, \pi[F] = \mathcal{X}$,

and $\{(x, s) : \pi^{-1}(x) \in N_s(\mathcal{N}) \cap F\}$ *is* Σ^0_2

– Used to prove results for $\mathcal{N}$ and then transfer them to all $\mathcal{X}$

Relativized and boldface versions of pointclasses

If Γ *is parametrized, then*:

• The relativization $\Gamma[x]$ of Γ to a point $x \in \mathcal{X}$ is the pointclass of all x-sections of pointsets in Γ,

$$\Gamma[x](\mathcal{Y}) = \{P_x \subseteq \mathcal{Y} : P \in \Gamma(\mathcal{X} \times \mathcal{Y})\},$$

where $P_x(y) \iff P(x, y) \quad (\alpha \in \Sigma^0_1[\beta] \iff \alpha \textit{ is recursive in } \beta)$

$\Rightarrow$ *Each* $\Gamma[x]$ *is parametrized*

• The boldface version $\boldsymbol{\Gamma}$ of Γ is the union of all its relativizations,

$$\boldsymbol{\Gamma} = \bigcup_{\mathcal{X}, x \in \mathcal{X}} \Gamma[x] = \bigcup_{\varepsilon \in \mathcal{N}} \Gamma[\varepsilon]$$

• The ambiguous (self-dual) pointclass of Γ is $\Delta = \Gamma \cap \neg\Gamma$; this is not in general parametrized, and (by definition)

$$\Delta[x] = \Gamma[x] \cap \neg\Gamma[x], \quad \boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \neg\boldsymbol{\Gamma}$$

The analytical and projective pointclasses

- The arithmetical pointclasses are defined by induction on $k \geq 1$:

$$\Sigma^0_1, \quad \Pi^0_k = \neg\Sigma^0_k, \quad \Sigma^0_{k+1} = \exists^{\mathbb{N}}\Pi^0_k, \quad \Delta^0_k = \Sigma^0_k \cap \Pi^0_k$$

- The Borel pointclasses of finite order are their boldface versions

$$\boldsymbol{\Sigma}^0_k, \quad \boldsymbol{\Pi}^0_k, \quad \boldsymbol{\Delta}^0_k = \boldsymbol{\Sigma}^0_k \cap \boldsymbol{\Pi}^0_k$$

- The analytical pointclasses are defined by induction on $k \geq 1$:

$$\Sigma^1_1 = \exists^{\mathcal{N}}\Pi^0_2, \quad \Pi^1_k = \neg\Sigma^1_k, \quad \Sigma^1_{k+1} = \exists^{\mathcal{N}}\Pi^1_k, \quad \Delta^1_k = \Sigma^1_k \cap \Pi^1_k$$

$$\Sigma^1_1(\mathcal{X}) \ : \ P(x) \iff (\exists\alpha)(\forall t)Q(x,\alpha,t) \quad \text{with } Q \in \Sigma^0_1(\mathcal{X} \times \mathcal{N} \times \mathbb{N})$$

$$\Pi^1_1(\mathcal{X}) \ : \ P(x) \iff (\forall\alpha)(\exists t)Q(x,\alpha,t) \quad \text{with } Q \in \Pi^0_1(\mathcal{X} \times \mathcal{N} \times \mathbb{N})$$

- The (classical) projective pointclasses are their boldface versions,

$$\boldsymbol{\Sigma}^1_k, \quad \boldsymbol{\Pi}^1_k, \quad \boldsymbol{\Delta}^1_k = \boldsymbol{\Sigma}^1_k \cap \boldsymbol{\Pi}^1_k$$

$$\boldsymbol{\Pi}^1_1 \quad : \quad P(x) \iff (\forall\alpha)(\exists t)Q(\varepsilon,x,\alpha,t) \ \ (Q \in \Pi^0_1, \text{ some } \varepsilon \in \mathcal{N})$$

Elementary properties of the analytical pointclasses

$\Rightarrow$ $\Sigma^1_k, \Pi^1_k, \Delta^1_k$ *are closed under recursive substitutions,* $\&, \vee, \exists^{\mathbb{N}}, \forall^{\mathbb{N}}$

- $\alpha \mapsto \alpha^* = (\lambda t)\alpha(t+1), \quad (i,\alpha) \mapsto (\alpha)_i = (\lambda t)\alpha(\langle i, t\rangle)$ are recursive

$\Rightarrow$ Σ^1_k *is closed under* $\exists^{\mathcal{Y}}$, Π^1_k *is closed under* $\forall^{\mathcal{Y}}$, Δ^1_k *is closed under* $\neg$
(the proof uses the recursive surjection $\pi : \mathcal{N} \twoheadrightarrow \mathcal{Y}$)

$\Rightarrow$ $y \in \Delta^1_k[x] \iff y \in \Sigma^1_k[x] \iff$ *the singleton* $\{y\}$ *is in* $\Sigma^1_k[x]$

Theorem *For all* $k \geq 1$ *and* x, $\Pi^1_k, \Pi^1_1[x], \Sigma^1_k$ *and* $\Sigma^1_1[x]$ *are parametrized*

$\Rightarrow$ $P \in \mathbf{\Pi}^1_k(\mathcal{X}) \iff$ *P is a section* G_α *of* G; α is a $\mathbf{\Pi}^1_k$-code of P

$\Rightarrow$ $\mathbf{\Pi}^1_k(\mathcal{X})$ *is uniformly closed under countable unions*;

i.e., for some recursive $u : \mathcal{N} \to \mathcal{N}$, $\boxed{\bigcup_i G_{(\alpha)_i} = G_{u(\alpha)}}$

Proof. Set $P(\alpha, x) \iff (\exists i) G((\alpha)_i, x)$ and take $u(\alpha) = \mathsf{S}^P(\alpha)$

The arithmetical and analytical hierarchies

$$
\begin{array}{ccccccccccccc}
 & & \Sigma^0_1(\mathcal{X}) & & & & \cdots & & & \Sigma^1_1(\mathcal{X}) & & & \Sigma^1_2(\mathcal{X})\cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & & \subsetneq & & \subsetneq & \subsetneq & \\
\Delta^0_1(\mathcal{X}) & & & & \Delta^0_2(\mathcal{X}) & & & \subsetneq \Delta^1_1(\mathcal{X}) & & & \Delta^1_2(\mathcal{X}) & & \\
 & \subsetneq & & \subsetneq & & \subsetneq & & & \subsetneq & & \subsetneq & \subsetneq & \\
 & & \Pi^0_1(\mathcal{X}) & & & & \cdots & & & \Pi^1_1(\mathcal{X}) & & & \Pi^1_2(\mathcal{X})\ \cdots
\end{array}
$$

The Hierarchy Theorem for infinite $\mathcal{X}$

$\Rightarrow$ *In fact, for perfect $\mathcal{X}$ and every $k \geq 1$,*

$$\Sigma^1_k(\mathcal{X}) \setminus \boldsymbol{\Delta}^1_k(\mathcal{X}) \neq \emptyset$$

- Classical regularity results: *Every $\boldsymbol{\Sigma}^1_1$ set $P \subseteq \mathcal{R}$ is Lebesgue measurable; it has the property of Baire; and if it is uncountable, then it has a non-empty perfect subset*

- *This is most of what can be proved about projective pointsets and the analytical and projective pointclasses in* ZFC

The limits of ZFC in Descriptive Set Theory

• An almost complete theory was developed in 1905 - 1938 for the classical pointclasses

$$\mathbf{\Sigma}^1_1 \text{ (analytic)}, \quad \mathbf{\Pi}^1_1 \text{ (co-analytic)} \quad \text{and } \mathbf{\Sigma}^1_2 \text{ (PCA)}$$

and the pointsets in them, and effective versions of these results were quickly proved in the late 50's

• But this is as far as you can go in ZFC, for example

- in Gödel's L there is an uncountable Σ^1_2 set of real numbers which is not Lebesgue measurable, does not have the property of Baire and has no non-empty perfect subset (Gödel 1938, Addison 1959), and

- there are forcing models of ZFC in which all projective sets of real numbers have these regularity properties (Solovay 1970, assuming an inaccessible)

Determinacy and large cardinal hypotheses

- In the period 1966 - (roughly) 1990, all the basic facts about $\boldsymbol{\Sigma}^1_1, \boldsymbol{\Pi}^1_1$ and $\boldsymbol{\Sigma}^1_2$ were extended to all the projective pointclasses on the basis of large cardinal hypotheses

- A key step was the introduction in 1967 of determinacy (game theoretic) hypotheses which were used to establish these results; in 1988 it was shown by Martin, Steel and Woodin that these hypotheses follow from the existence of Woodin cardinals

- *The use of effective methods is essential in the derivation of consequences of* projective determinacy—a fact which encouraged the development of EDST

- In the sequel we will formulate and derive some of the basic results about $\Sigma^1_1, \Pi^1_1, \Delta^1_1$ and their boldface versions $\boldsymbol{\Sigma}^1_1, \boldsymbol{\Pi}^1_1, \boldsymbol{\Delta}^1_1$ on the basis of ZF+DC (the Axiom of Depended Choices)

- Whenever it is possible, we will use methods which can be used to extend these results to many other pointclasses

⋆ Borel and hyperarithmetical pointsets

- $\mathbf{B}(\mathcal{X})$ is the smallest family of subsets of $\mathcal{X}$ which contains all the open sets and is closed under complements and countable unions
- To get the effective lightface version of $\mathbf{B}(\mathcal{X})$, we code $\mathbf{B}(\mathcal{X})$ in $\mathcal{N}$:

Def Set $\mathsf{K}_1 = \{\alpha : \alpha(0) = 0\}$ and for each $\xi > 1$, by recursion

$$\mathsf{K}_\xi = \mathsf{K}_1 \cup \Big\{\alpha : \alpha(0) \neq 0 \;\&\; (\forall n)\Big[(\alpha^*)_n \in \bigcup\nolimits_{\eta<\xi} \mathsf{K}_\eta\Big]\Big\} \quad (\xi > 1)$$

Def For each $\mathcal{X}$, fix a parametrization $G^1 \subseteq \mathcal{N} \times \mathcal{X}$ of $\Sigma^0_1(\mathcal{X})$ and set

$$B^{\mathcal{X}}_{\alpha,\xi} = \begin{cases} G^1_{\alpha^*} = \{x : G^1(\alpha^*, x)\}, & \text{if } \alpha(0) = 0, \\ \bigcup_i \Big(\mathcal{X} \setminus B^{\mathcal{X}}_{(\alpha^*)_i,\eta(i)}\Big), & \text{otherwise,} \end{cases}$$

where $\eta(i) =$ least η so that $(\alpha^*)_i \in \mathsf{K}_\eta$

$\Rightarrow$ $\alpha \in (\mathsf{K}_\xi \cap \mathsf{K}_\zeta) \implies B^{\mathcal{X}}_{\alpha,\xi} = B^{\mathcal{X}}_{\alpha,\zeta} = B^{\mathcal{X}}_\alpha$; set $\mathsf{K} = \bigcup_\xi \mathsf{K}_\xi$

$\Rightarrow$ $\boxed{A \in \mathbf{B}(\mathcal{X}) \iff A = B^{\mathcal{X}}_\alpha \text{ for some } \alpha \in \mathsf{K}}$

Def $\boxed{A \in \mathrm{HYP}(\mathcal{X}) \iff A = B^{\mathcal{X}}_\alpha \text{ for some recursive } \alpha \in \mathsf{K}}$

Coded sets and uniformities

Def A coding of a set A on $I \subseteq \mathcal{N}$ is any surjection $\pi : I \twoheadrightarrow A$, and a coded set is any pair (A, π) of a set and a coding of it

- $\mathbf{\Pi}^1_1(\mathcal{X})$ on $\mathcal{N}$ by $\alpha \mapsto G_\alpha$, with G a parametrization of $\mathbf{\Pi}^1_1(\mathcal{X})$
- $\mathbf{\Delta}^1_1(\mathcal{X})$ on $\{\alpha \in \mathcal{N} : G_{(\alpha)_0} = \mathcal{X} \setminus G_{(\alpha)_1}\}$ by $\alpha \mapsto G_{(\alpha)_0}$ (same G)
- $\mathbf{B}(\mathcal{X})$ on K by $\alpha \mapsto B^{\mathcal{X}}_\alpha$

$\Rightarrow$ $\mathbf{B}(\mathcal{X})$ *is uniformly closed under complementation*, i.e., there is a locally recursive $u : \mathcal{N} \rightharpoonup \mathcal{N}$ such that

$$\boxed{\alpha \in \mathsf{K} \implies \Big(u(\alpha)\!\downarrow\ \&\ u(\alpha) \in \mathsf{K}\ \&\ B^{\mathcal{X}}_{u(\alpha)} = \mathcal{X} \setminus B^{\mathcal{X}}_\alpha\Big)}$$

Proof. Let $v(\alpha) = (\lambda n)\alpha((n)_1)$; then $v(\alpha)(\langle i, t\rangle) = \alpha(t)$ for all t, so

$$\alpha \in \mathsf{K} \implies (\forall i)[(v(\alpha))_i = \alpha \in \mathsf{K}] \implies B^{\mathcal{X}}_{(v(\alpha))_i} = B^{\mathcal{X}}_\alpha$$

and we can set $u(\alpha) = \langle 1\rangle \,\hat{}\, v(\alpha)$ • *In this case, the uniformity u is total*

Hyperarithmetical (effectively Borel) pointsets

- Each $\mathbf{B}(\mathcal{X})$ is coded on K by $\alpha \mapsto B_\alpha^{\mathcal{X}}$

Def A pointset $P \subseteq \mathcal{X}$ is hyperarithmetical (effectively Borel) if it has a recursive Borel code, i.e., $P = B_\alpha^{\mathcal{X}}$ with a recursive α

- HYP($\mathcal{X}$) is coded on $\{\alpha \in \mathsf{K} : \alpha \text{ is recursive}\}$ by $\alpha \mapsto B_\alpha^{\mathcal{X}}$

$\Rightarrow$ *The coded pointclass* **B** *is uniformly closed under* $\&, \vee, \neg, \exists^{\mathbb{N}}, \forall^{\mathbb{N}}$, *continuous substitutions and countable unions*

$\Rightarrow$ *The coded pointclass* HYP *is uniformly closed under* $\&, \vee, \neg, \exists^{\mathbb{N}}, \forall^{\mathbb{N}}$, *recursive substitutions and recursive countable unions*

$\Rightarrow$ *These facts hold independently of the choice of a parametrization of* $\Sigma^0_1(\mathcal{X})$ *used to define the map* $\alpha \mapsto B_\alpha^{\mathcal{X}}$, *because different choices produce* (suitably defined) *equivalent codings*

⋆ The easy half of the Suslin-Kleene Theorem

Theorem *For each $\mathcal{X}$, $\mathbf{B}(\mathcal{X}) \subseteq \mathbf{\Delta}^1_1(\mathcal{X})$ uniformly*,
i.e., there is a locally recursive $u : \mathcal{N} \rightharpoonup \mathcal{N}$ such that

$$(*) \qquad \alpha \in \mathsf{K} \implies \Big(u(\alpha)\!\downarrow \ \& \ u(\alpha) \text{ is a } \mathbf{\Delta}^1_1(\mathcal{X})\text{-code of } B^{\mathcal{X}}_{\alpha}\Big)$$

Proof. Define first a locally recursive $v : \mathcal{N} \times \mathcal{N} \rightharpoonup \mathcal{N}$ such that

$$(\forall i)[\{\varepsilon\}^{\mathcal{N} \rightharpoonup \mathcal{N}}(\alpha)(i)\!\downarrow \text{ and is a } \mathbf{\Delta}^1_1\text{-code of } A_i \subseteq \mathcal{X}]$$
$$\implies \Big(v(\varepsilon, \alpha)\!\downarrow \text{ and is a } \mathbf{\Delta}^1_1(\mathcal{X})\text{-code of } \textstyle\bigcup_i (\mathcal{X} \setminus A_i)\Big)$$

Set $u(\alpha) = \{\widetilde{\varepsilon}\}(\alpha)$, where by the 2nd RT for partial functions

$$\{\widetilde{\varepsilon}\}(\alpha) = \begin{cases} \text{a } \mathbf{\Delta}^1_1(\mathcal{X})\text{-code of } G^1_{\alpha^*}, & \text{if } \alpha(0) = 0, \\ v(\widetilde{\varepsilon}, \alpha^*) & \text{otherwise} \end{cases}$$

Proof of $(*)$ is by induction on *the least* ξ *such that* $\alpha \in \mathsf{K}_\xi$

- **Effective transfinite recursion**, the most basic tool of EDST

⋆ The Suslin-Kleene Theorem

Theorem *For each $\mathcal{X}$, $\mathbf{\Delta}^1_1(\mathcal{X}) \subseteq \mathbf{B}(\mathcal{X})$ uniformly*
i.e., there is a locally recursive $u : \mathcal{N} \rightharpoonup \mathcal{N}$ such that

if α is a $\mathbf{\Delta}^1_1$-code of $A \subseteq \mathcal{X}$, then $\Big(u(\alpha)\downarrow, u(\alpha) \in \mathsf{K}$ & $A = B^{\mathcal{X}}_{u(\alpha)}\Big)$

$\Rightarrow$ (Suslin 1916) *For every $\mathcal{X}$, $\mathbf{\Delta}^1_1(\mathcal{X}) = \mathbf{B}(\mathcal{X})$* Constructive proof!

$\Rightarrow$ (Kleene 1955) $\Delta^1_1(\mathbb{N}) = \mathrm{HYP}(\mathbb{N})$ *uniformly* (with his codings)

• There are several proofs. They all first prove the result for $\mathcal{N}$ using *Effective Transfinite Recursion* and the *Normal Form Theorem for $\Pi^1_1(\mathcal{N})$ pointsets* (coming up next) and then they appeal to the *Refined Surjection Theorem*

$\Rightarrow$ (Classical Corollary, may or may not be interesting) *There is a G_δ set $C \subseteq \mathcal{N}$ and a continuous $u : C \to \mathcal{N}$ such that*

if α is a $\mathbf{\Delta}^1_1$-code of $A \subseteq \mathcal{X}$, then $\Big(\alpha \in C$ & $A = B^{\mathcal{X}}_{u(\alpha)}\Big)$

• No proof of this is known which does not use effective methods but ...

⋆ The Normal Form Theorems for $\Pi^1_1(\mathcal{N}), \Sigma^1_1(\mathcal{N})$

Theorem *If $P \in \Pi^1_1(\mathcal{N})$, then for some recursive $R \subseteq \mathbb{N} \times \mathbb{N}$*

$$\boxed{P(\alpha) \iff (\forall \beta)(\exists t) R(\overline{\alpha}(t), \overline{\beta}(t))}$$

where $\boxed{\overline{\alpha}(t) = \langle \alpha(0), \dots, \alpha(t-1) \rangle = 2^{\alpha(0)+1} \cdots p_{t-1}^{\alpha(t-1)+1}}$

is the sequence code of $(\alpha(0), \dots, \alpha(t-1))$

– because if $Q \in \Sigma^0_1(\mathcal{N}^2)$, then $Q(\alpha, \beta) \iff (\exists t) R(\overline{\alpha}(t), \overline{\beta}(t))$

Theorem *If $P \in \Sigma^1_1(\mathcal{N})$, then for some recursive $R \subseteq \mathbb{N} \times \mathbb{N}$*

$$\alpha \in P \iff (\exists \beta)(\forall t) R(\overline{\alpha}(t), \overline{\beta}(t))$$

and so

$$\boxed{P = \text{proj}[C] \text{ with } C = \{(\alpha, \beta) : (\forall t) R(\overline{\alpha}(t), \overline{\beta}(t))\} \text{ in } \Pi^0_1}$$

so that, in particular, C is closed

- Similar equivalences (trivially) hold for $\Pi^1_1[\varepsilon](\mathcal{N}^n)$ and $\Sigma^1_1[\varepsilon](\mathcal{N}^n)$

The Effective Perfect Set Theorem

Theorem (Suslin 1916) *Every uncountable $\boldsymbol{\Sigma}^1_1$ pointset has a non-empty perfect subset* (and so has cardinality $2^{\aleph_0}$)

- This was previously proved for Borel sets by Hausdorff and Alexandroff (independently) and was a big deal at the time

  It is the strongest result about the Continuum Hypothesis which can be proved in ZFC

Theorem (Harrison 1967) *If $A \in \Sigma^1_1[x](\mathcal{Y})$ and A has a member $y \notin \Delta^1_1[x]$, then A has a non-empty perfect subset*

- Recall that

$$y \in \Delta^1_1[x] \iff \mathcal{U}(y) = \{s : x \in N_s(\mathcal{Y})\} \in \Delta^1_1[x](\mathbb{N}),$$

and $\Delta^1_1[x](\mathbb{N})$ is countable, so $\{y : y \in \Delta^1_1[x]\}$ is countable, and Harrison's Theorem implies—and "explains"—Suslin's result

Plan for proving the

Effective Perfect Set Theorem *If* $A \in \Sigma^1_1[x](\mathcal{Y})$ *and* A *has a member* $y \notin \Delta^1_1[x]$, *then* A *has a non-empty perfect subset*

Lemma 1 *If* $A \in \Sigma^1_1[x](\mathcal{Y})$, $A \neq \emptyset$ *and* A *has no* $\Delta^1_1[x]$ *member, then* A *has a non-empty perfect subset*

- Proof on the next slide, basically a proof of the classical theorem

Lemma 2 (Upper classification of $\Delta^1_1[x]$) *For each point* x, *the pointset set* $\{y \in \mathcal{Y} : y \in \Delta^1_1[x]\}$ *is* $\Pi^1_1[x]$

- We will derive Lemma 2 from some basic results of the effective theory in the next lecture

- *Proof of the Theorem from the two lemmas*. If $A \subseteq \mathcal{Y}$ is $\Sigma^1_1[x]$ and has at least one member not in $\Delta^1_1[x]$, then, by Lemma 2, $A \setminus \{y \in \mathcal{Y} : y \in \Delta^1_1[x]\}$ is $\Sigma^1_1[x]$, not empty and has no $\Delta^1_1[x]$ member; and so it has a non-empty perfect subset by Lemma 1

Proof of Lemma 1 for $\mathcal{N}$

Lemma *If* $A \in \Sigma^1_1[\varepsilon](\mathcal{N})$, $A \neq \emptyset$ *and* A *has no* $\Delta^1_1[\varepsilon]$ *member, then* A *has a non-empty, compact perfect subset*

- By the Normal Form Theorem for $\Sigma^1_1[\varepsilon](\mathcal{N})$,

$$A = \mathrm{proj}(C) \text{ with } C \subseteq \mathcal{N} \times \mathcal{N} \text{ in } \Pi^0_1[\varepsilon]$$

For any pair $w = (\pi_1(w), \pi_2(w))$ of sequence codes, let

$$C_w = \{(\alpha, \beta) \in C : (\exists t)[\pi_1(w) = \overline{\alpha}(t) \ \&\ \pi_2(w) = \overline{\beta}(t)]\} \in \Pi^0_1(\mathcal{N}^2)$$

$\Rightarrow$ $\boxed{\mathrm{proj}(C_w) \textit{ is never a singleton}}$; because if $\mathrm{proj}(C_w) = \{\alpha_0\}$, then

$$\alpha = \alpha_0 \iff (\exists \beta)[(\alpha, \beta) \in C_w] \text{ and so } \alpha_0 \text{ is } \Delta^1_1[\varepsilon]$$

- For any $w = (\pi_1(w), \pi_2(w))$, choose w^0, w^1 such that

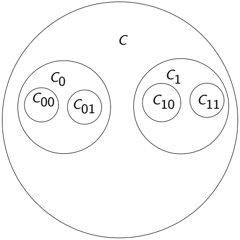
$$\mathrm{proj}(C_w) \neq \emptyset) \implies \Big(\mathrm{proj}(C_{w^0}) \neq \emptyset, \mathrm{proj}(C_{w^1}) \neq \emptyset,$$
$$\mathrm{proj}(C_{w^0}) \cup \mathrm{proj}(C_{w^1}) \subset \mathrm{proj}(C_w), \mathrm{proj}(C_{w^0}) \cap \mathrm{proj}(C_{w^1}) = \emptyset\Big)$$

$$A = \mathsf{proj}(C) \text{ with } C \subseteq \mathcal{N} \times \mathcal{N} \text{ in } \Pi^0_1[\varepsilon]$$

$$C_w = \{(\alpha, \beta) \in C : (\exists t)[\pi_1(w) = \overline{\alpha}(t), \pi_2(w) = \overline{\beta}(t)] \in \Pi^0_1(\mathcal{N}^2)$$

$$\mathsf{proj}(C_w) \neq \emptyset \implies \Big(\mathsf{proj}(C_{w^0}) \neq \emptyset, \mathsf{proj}(C_{w^1}) \neq \emptyset$$
$$\mathsf{proj}(C_{w^0}) \cup \mathsf{proj}(C_{w^1}) \subset \mathsf{proj}(C_w), \text{ and } \mathsf{proj}(C_{w^0}) \cap \mathsf{proj}(C_{w^1}) = \emptyset\Big)$$

- For each code $w = \langle w_0, w_1, \dots, w_k \rangle$ of a binary sequence, define C_w so that $C_{\langle\ \rangle} = C$, $C_{w*\langle 0\rangle} = C_{w^0}, C_{w*\langle 1\rangle} = C_{w^1}$



- $\bigcup_{\gamma:\mathbb{N}\to\{0,1\}} \bigcap_t C_{\overline{\gamma}(t)}$ is the required compact, perfect subset of $\mathsf{proj}(C)$