

Effective Descriptive Set Theory

what it is about

Lecture 1, Recursion in Polish spaces

Yiannis N. Moschovakis

UCLA and University of Athens

www.math.ucla.edu/~ynm

Winter school in Hejnice, Czech Republic,
30 January – 6 February, 2016

The child of two fields

- **Classical descriptive set theory**, 1895 –
Borel, Baire, Hadamard, **Lebesgue 1905**, Lusin, Suslin, Novikov, ...
Definability theory on the continuum at first represented by

\mathcal{R} = the real numbers, \mathcal{N} = Baire space = $(\mathbb{N} \rightarrow \mathbb{N})$

with $\mathbb{N} = \{0, 1, \dots\}$, later studied on **Polish spaces**

- **Hyperarithmetical computability on \mathbb{N}** , 1950 –
Martin Davis, Mostowski, **Kleene 1955**, Spector, ...

Common motivation (after Lebesgue):

★ ***Constructively defined** sets and functions should have special properties that distinguish them from arbitrary ones*

- **Effective descriptive set theory** (EDST): a common extension, on **recursive Polish spaces**, with applications to both (and other fields)

Outline

Lecture 1. **Recursion in Polish spaces**

Lecture 2. Effective Borel, analytic and co-analytic pointsets

Lecture 3. Structure theory for pointclasses

- Primary sources for the lectures (posted on my homepage):

Descriptive set theory, ynm, 1980, 2nd edition 2009

Classical descriptive set theory as a refinement of effective descriptive set theory, ynm, 2010

Kleene's amazing second recursion theorem, ynm, 2010

Notes on effective descriptive set theory, Gregoriades and ynm
(in preparation)

- I will try to give an **elementary introduction** to some of the fundamental notions, ideas and methods of proof **specific to EDST** **not** to cover a large part of the field, recent results or applications
- There are several proofs on the slides that I will skip in the lectures

EDST as a recursion theory: what comes first?

- In classical recursion (computability) theory on \mathbb{N} and \mathcal{N} , we typically define
 - first** the recursive partial functions $f : \mathbb{N}^n \times \mathcal{N}^k \rightarrow \mathbb{N}$
 - next** the semirecursive (r.e.) relations $P \subseteq \mathbb{N}^n \times \mathcal{N}^k$
(the domains of convergence of recursive partial functions)
 - and then** the arithmetical and analytical relations, etc
- In Polish recursion theory we must reverse the order: define
 - first** the semirecursive relations (pointsets) $P \subseteq \mathcal{X}$
 - next** the **locally recursive** partial functions $f : \mathcal{X} \rightarrow \mathcal{Y}$
(whose domains of convergence are arbitrary)
 - and then** the arithmetical and analytical relations, etc(and we must define **recursive Polish spaces**, which include $\mathbb{N}, \mathcal{N}, \mathcal{R}$)
- Emil Post followed this second order of definitions for recursion on \mathbb{N}

★ Recursively presented Polish metric spaces

- Fix a **recursive enumeration** q_0, q_1, \dots of the rational numbers \mathbb{Q} , i.e., such that $k \mapsto \text{sign}(q_k), \text{num}(q_k), \text{den}(q_k)$ are recursive

Def A **recursive presentation** of a Polish (= separable, complete) metric space (\mathcal{X}, d) is a sequence $\mathbf{r} = (r_0, r_1, \dots)$ of points which is dense in \mathcal{X} and such that the following two relations are recursive:

$$P^{\mathbf{r}}(i, j, k) \iff d(r_i, r_j) \leq q_k, \quad Q^{\mathbf{r}}(i, j, k) \iff d(r_i, r_j) < q_k$$

- Recursively presented Polish metric space:** $(\mathcal{X}, d, \mathbf{r})$

\Rightarrow The relations $P^{\mathbf{r}}, Q^{\mathbf{r}}$ determine $(\mathcal{X}, d, \mathbf{r})$ up to isometry

- Relativization:** For any $\varepsilon \in \mathcal{N}$, \mathbf{r} is an ε -**recursive** presentation of (\mathcal{X}, d) if the relations $P^{\mathbf{r}}, Q^{\mathbf{r}}$ are **recursive in** ε

\Rightarrow Every Polish metric space has an ε -recursive presentation, for some $\varepsilon \in \mathcal{N}$ (Used to apply results of EDST to all Polish metric spaces)

Examples (with natural metrics and presentations)

- $\Rightarrow \{0, \dots, m\}$ and \mathbb{N} with $d(n, k) = 1$ for $n \neq k$,
 \mathcal{R} , Baire space \mathcal{N} , Cantor space $\mathcal{C} = (\mathbb{N} \rightarrow \{0, 1\}) \subset \mathcal{N}$
- \Rightarrow Products $\mathcal{X} \times \mathcal{Y}$, $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \dots$ of recursively presented metric spaces (with either of the standard product metrics)
- $\Rightarrow C[0, 1] = \{f : [0, 1] \rightarrow \mathbb{R} : f \text{ is continuous}\}$ (with the sup norm)
- ... All “popular” Polish metric spaces have recursive presentations (mostly immediately from their definitions)
- A Polish metric space $(\mathcal{U}, d_{\mathcal{U}})$ is **Urysohn** (universal) if
*for every finite metric space $(X \cup \{y\}, d)$
and every isometric embedding $f : X \rightarrow \mathcal{U}$,
there is an isometric embedding $f^* : X \cup \{y\} \rightarrow \mathcal{U}$ which extends f*

Theorem (Urysohn) *Up to isometry, there is exactly one Urysohn space*

\Rightarrow *The Urysohn space has a recursive presentation*

★ Open (Σ_1^0) and semirecursive (Σ_1^0) pointsets

- **Coding** of open balls (neighborhoods): for given $(\mathcal{X}, d, \mathbf{r})$, put

$$N_s = N_s(\mathcal{X}) = \{x \in \mathcal{X} : d(x, r_{(s)_0}) < q_{(s)_1}\} \quad (s \in \mathbb{N}),$$

where $s \mapsto ((s)_0, (s)_1)$ is a recursive surjection of \mathbb{N} onto $\mathbb{N} \times \mathbb{N}$

Def A set $G \subseteq \mathcal{X}$ is **open** (in $\Sigma_1^0(\mathcal{X})$) if for some $\varepsilon \in \mathcal{N}$,

$$(*) \quad G = \bigcup_s N_{\varepsilon(s)};$$

it is **semirecursive** (in $\Sigma_1^0(\mathcal{X})$) if $(*)$ holds with a recursive $\varepsilon : \mathbb{N} \rightarrow \mathbb{N}$

Σ_1^0 -Normal Form A pointset $P \subseteq \mathcal{X} \times \mathcal{Y}$ is in $\Sigma_1^0(\mathcal{X} \times \mathcal{Y})$ if and only if

$$P(x, y) \iff (\exists s, t)[x \in N_s(\mathcal{X}) \ \& \ y \in N_t(\mathcal{Y}) \ \& \ P^*(s, t)]$$

with a semirecursive $P^* \subseteq \mathbb{N}^2$ (and similarly for $\mathcal{X}, \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \dots$)

\Rightarrow The family $\Sigma_1^0(\mathcal{X} \times \mathcal{Y})$ depends on $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{r}_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{r}_{\mathcal{Y}})$
(but not on which of the standard metrics we choose for $\mathcal{X} \times \mathcal{Y}$)

Closure properties of Σ_1^0

$\Rightarrow \emptyset, \mathcal{X}$ are in $\Sigma_1^0(\mathcal{X})$

\Rightarrow The **basic nbhd relation** $\{(x, s) : x \in N_s(\mathcal{X})\}$ is in $\Sigma_1^0(\mathbb{N} \times \mathcal{X})$

$\Rightarrow \Sigma_1^0$ is closed under $\&, \vee$ and $\exists^{\mathbb{N}}, P(x) \iff (\exists t \in \mathbb{N})Q(x, t)$

Def $f : \mathcal{X} \rightarrow \mathcal{Y}$ is **recursive** if the pointset $\{(x, s) : f(x) \in N_s(\mathcal{Y})\}$ is Σ_1^0

$\Rightarrow (x, y) \mapsto x, \quad \alpha \mapsto \alpha^* = \lambda t \alpha(t+1), \quad (e, \alpha) \mapsto \langle e \rangle^\wedge \alpha$
 $(\alpha, i) \mapsto (\alpha)_i = (\lambda t) \alpha(\langle i, t \rangle),$

are recursive, and so is $x \mapsto (f(x), g(x))$, if f and g are

$\Rightarrow \Sigma_1^0$ is closed under substitution of recursive functions

Proof. If $Q(y) \iff (\exists s)[y \in N_s \& R^*(s)],$

then $Q(f(x)) \iff (\exists s)[f(x) \in N_s \& R^*(s)]$

\Rightarrow The composition $x \mapsto g(h(x))$ of recursive functions is recursive

★ Recursive Polish spaces

- A **Polish space** is a pair $(\mathcal{X}, \mathcal{T})$ such that for some d ,

(P1) (\mathcal{X}, d) is a Polish (separable, complete) metric space, and

(P2) $\mathcal{T} = \Sigma_1^0(\mathcal{X}) =$ the open subsets of (\mathcal{X}, d)

- What is the “recursive topology” on $(\mathcal{X}, d, \mathbf{r})$ with recursive \mathbf{r} ?
(hard to formulate the appropriate properties for $\Sigma_1^0(\mathcal{X})$)

Def A **recursive Polish space** is a pair $(\mathcal{X}, \mathcal{F})$ such that for some (d, \mathbf{r}) ,

(RP1) $(\mathcal{X}, d, \mathbf{r})$ is a recursively presented Polish metric space, and

(RP2) $\mathcal{F} = \Sigma_1^0(\mathbb{N} \times \mathcal{X})$ (which depends only on $(\mathcal{X}, d, \mathbf{r})$)

- $\mathcal{F} = \mathcal{F}(\mathcal{X})$ is the **frame** of $(\mathcal{X}, \mathcal{F})$, its **recursive topology**, and
- if (RP1), (RP2) hold, then (d, \mathbf{r}) is a **compatible pair** of $(\mathcal{X}, \mathcal{F})$

\Rightarrow If $(d_1, \mathbf{r}_1), (d_2, \mathbf{r}_2)$ are compatible pairs of $(\mathcal{X}, \mathcal{F})$, then

$$\Sigma_1^0(\mathcal{X}, d_1, \mathbf{r}_1) = \Sigma_1^0(\mathcal{X}, d_2, \mathbf{r}_2) =_{\text{def}} \Sigma_1^0(\mathcal{X})$$

- **Strong closure properties:** e.g., $\mathcal{X} \mapsto \prod_{i \in \mathbb{N}} \mathcal{X}$, $\boxed{\mathcal{X} \mapsto \mathcal{X}^{<\omega}}$

Pointsets and pointclasses (in recursive Polish spaces)

- A **pointset** is any subset $P \subseteq \mathcal{X}$ of a recursive Polish space
(formally a pair (P, \mathcal{X}))
- A **pointclass** is any collection Γ of pointsets, e.g., Σ_1^0, Σ_1^0 ,
and for any \mathcal{X} , we set

$\Gamma(\mathcal{X}) = \{P \subseteq \mathcal{X} : P \in \Gamma\} =$ the subsets of \mathcal{X} which **are** (in) Γ

- The **points of** Γ : For $x \in \mathcal{X}$, $\boxed{x \in \Gamma \iff \{s : x \in N_s(\mathcal{X})\} \in \Gamma}$
 x is **recursive** $\iff x \in \Sigma_1^0$ ($\alpha \in \Sigma_1^0 \iff \alpha$ is Turing computable)
- The **arithmetical pointclasses** are defined inductively from Σ_1^0 ,

$$\Pi_k^0 = \neg \Sigma_k^0, \quad \Sigma_{k+1}^0 = \exists^{\mathbb{N}} \Pi_k^0 \quad (k \geq 1)$$

$$\Pi_1^0(\mathcal{X}) : P(x) \iff \neg Q(x) \text{ for some } Q \in \Sigma_1^0(\mathcal{X}),$$

$$\Sigma_2^0(\mathcal{X}) : P(x) \iff (\exists t \in \mathbb{N}) Q(x, t) \text{ for some } Q \in \Pi_1^0(\mathcal{X} \times \mathbb{N})$$

$$\Pi_2^0(\mathcal{X}) : P(x) \iff \neg(\exists t \in \mathbb{N}) Q_1(x, t) \iff (\forall t \in \mathbb{N}) Q(x, t) \\ \text{for some } Q_1 \in \Pi_1^0(\mathcal{X} \times \mathbb{N}) \text{ and some } Q = \neg Q_1 \in \Sigma_1^0(\mathcal{X} \times \mathbb{N})$$

Partial functions

- A **partial function** $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a (total) function $f : D_f \rightarrow \mathcal{Y}$, where $D_f \subseteq \mathcal{X}$ is the **domain of convergence** of f , and we write

$$f(x) \downarrow \iff x \in D_f, \quad f(x) \uparrow \iff x \notin D_f$$

$$f(x) = g(x) \iff [f(x) \uparrow \ \& \ g(x) \uparrow] \vee (\exists w)[f(x) = w \ \& \ g(x) = w]$$

$$f \sqsubseteq g \iff (\forall x)[f(x) \downarrow \implies f(x) = g(x)]$$

- Partial functions **compose strictly**, i.e.,

$$g(h_1(x), \dots, h_m(x)) = w$$

$$\iff (\exists y_1, \dots, y_m)[h_1(x) = y_1 \ \& \ \dots \ \& \ h_m(x) = y_m \\ \& \ g(y_1, \dots, y_m) = w]$$

★ Locally recursive partial functions, I

Def A pointset $P \subseteq \mathcal{X} \times \mathbb{N}$ **computes** a partial function $f : \mathcal{X} \rightarrow \mathcal{Y}$ (where it converges) with respect to a compatible pair (d, \mathbf{r}) for \mathcal{X} , if

$$f(x) \downarrow \implies \left(\inf \{ \text{radius}(N_s) : P(x, s) \} = 0 \right. \\ \left. \& \bigcap \{ N_s(\mathcal{Y}) : P(x, s) \} = \{ f(x) \} \right)$$

Def $f : \mathcal{X} \rightarrow \mathcal{Y}$ is **locally recursive** if it is computed by some P in Σ_1^0

Theorem *The following are equivalent for $f : \mathcal{X} \rightarrow \mathcal{Y}$:*

- (1) f is locally recursive
- (2) For some $Q \in \Sigma_1^0(\mathcal{X} \times \mathbb{N})$,

$$f(x) \downarrow \implies (\forall s) \left(f(x) \in N_s(\mathcal{Y}) \iff Q(x, s) \right)$$

- (3) For every $Q \in \Sigma_1^0(\mathcal{Y} \times \mathcal{Z})$ there is a $P \in \Sigma_1^0(\mathcal{X} \times \mathcal{Z})$ such that

$$f(x) \downarrow \implies [P(x, z) \iff Q(f(x), z)]$$

★ Locally recursive partial functions, II

- The key characterization of local recursiveness is (2),

Theorem $f : \mathcal{X} \rightarrow \mathcal{Y}$ is locally recursive if for some $Q \in \Sigma_1^0(\mathcal{X} \times \mathbb{N})$

$$f(x) \downarrow \implies (\forall s) \left(f(x) \in N_s(\mathcal{Y}) \iff Q(x, s) \right)$$

\Rightarrow *If x is recursive and $f(x) \downarrow$, then the point $f(x)$ is recursive*

\Rightarrow *If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is total, then f is locally recursive if it is recursive (by any of the old definitions)*

\Rightarrow *The composition $x \mapsto g(h(x))$ of locally recursive partial functions is locally recursive*

Theorem (Recursion and continuity) *A function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous if and only if there is a locally recursive $f^* : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y}$ and some $\varepsilon \in \mathcal{N}$ so that* $f(x) = f^*(\varepsilon, x) \quad (x \in \mathcal{X})$

- *It is not always possible to insure that $f^* : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y}$ is total*

The Refined Surjection Theorem

Theorem (Classical) For every Polish space \mathcal{X} , there is a *continuous function* $\pi : \mathcal{N} \rightarrow \mathcal{X}$ and a *closed* set $F \subseteq \mathcal{N}$ such that π is one-to-one on F , $\pi[F] = \mathcal{X}$, and the inverse $\pi^{-1} : \mathcal{X} \rightarrow F$ is *Borel measurable*

Theorem (Effective) For every recursive Polish space \mathcal{X} , there is a total *recursive function* $\pi : \mathcal{N} \rightarrow \mathcal{X}$ and a Π_1^0 set $F \subseteq \mathcal{N}$ such that π is one-to-one on F , $\pi[F] = \mathcal{X}$ and the inverse $\pi^{-1} : \mathcal{X} \rightarrow F$ is Σ_2^0 -*recursive*, i.e., the pointset $\{(x, s) : \pi^{-1}(x) \in N_s(\mathcal{N}) \cap F\}$ is Σ_2^0

- Proof is by a direct, effective construction
- The theorem makes it possible in many cases to prove results for \mathcal{N} and then “transfer” them to every space

Extending the domain of convergence

Theorem (Classical) Suppose \mathcal{X}, \mathcal{Y} are Polish spaces, $A \subseteq \mathcal{X}$, and $f : A \rightarrow \mathcal{Y}$ is continuous (with the induced topology on A); then there is a set A^* such that

- (1) $A \subseteq A^* \subseteq \mathcal{X}$;
- (2) A^* is a G_δ -set, i.e., $A^* = \bigcap_{n \in \mathbb{N}} A_n$ with each A_n open;
- (3) A is dense in A^* ; and
- (4) there is an extension of f to some continuous $\Phi : A^* \rightarrow \mathcal{Y}$

Theorem (Effective) Every locally recursive partial function $f : \mathcal{X} \rightarrow \mathcal{Y}$ has a locally recursive extension $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ whose domain of convergence $\{x : \Phi(x) \downarrow\}$ is Π_2^0

- The classical result follows from the relativized version of the effective theorem, taking $A^* = \{x : \Phi(x) \downarrow\} \cap \text{closure}(A)$
- The effective result cannot be improved to insure that $\{x : f(x) \downarrow\}$ is dense in $\{x : \Phi(x) \downarrow\}$, because $\text{closure}(A)$ need not be Π_2^0

Proof of the Extension Theorem for local recursion

Theorem Every locally recursive partial function $f : \mathcal{X} \rightarrow \mathcal{Y}$ has a locally recursive extension $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ whose domain of convergence $\{x : \Phi(x) \downarrow\}$ is Π_2^0

Fix \mathcal{X}, \mathcal{Y} and for any $P \in \Sigma_1^0(\mathcal{X} \times \mathbb{N})$ define $\Phi = \Phi_P^{\mathcal{X} \rightarrow \mathcal{Y}} : \mathcal{X} \rightarrow \mathcal{Y}$ by

$$\begin{aligned}\Phi(x) \downarrow &\iff \inf\{\text{radius}(N_s) : P(x, s)\} = 0 \\ &\quad \& \bigcap \{N_t(\mathcal{Y}) : P(x, t)\} \text{ is a singleton,} \\ \Phi(x) &= \text{the unique } y \text{ in } \bigcap \{N_t(\mathcal{Y}) : P(x, t)\}\end{aligned}$$

$\Rightarrow \Phi$ is locally recursive, as it is computed by P

\Rightarrow For any $f : \mathcal{X} \rightarrow \mathcal{Y}$, $\boxed{P \text{ computes } f \iff f \sqsubseteq \Phi}$

$\Rightarrow \{x : \Phi(x) \downarrow\}$ is Π_2^0 , because

$$\begin{aligned}\Phi(x) \downarrow &\iff (\forall s, t)[[P(x, s) \& P(x, t)] \implies N_s \cap N_t \neq \emptyset] \\ &\quad \& (\forall n)(\exists s)[P(x, s) \& \text{radius}(N_s) < 2^{-n}]\end{aligned}$$

Is it recursion or just “effective continuity”?

Theorem (Primitive recursion) *If g and h are locally recursive on the appropriate spaces and $f : \mathbb{N} \times \mathcal{X} \rightarrow \mathcal{W}$ is defined by*

$$\begin{aligned}f(0, x) &= g(x), \\f(t + 1, x) &= h(f(t, x), t, x),\end{aligned}$$

then f is also locally recursive

- The usual proofs for \mathbb{N} (via Dedekind's analysis of recursive definition) or the attempt to show directly that f is effectively continuous are not easy to carry out
- We develop an alternative approach which also works for **nested, double, . . . , recursion** as well as **effective transfinite recursion**.
- *It is a very general, fundamental tool of EDST*

★ Parametrized pointclasses

Def A pointclass Γ is **parametrized** if it is closed under (total) recursive substitutions and for every \mathcal{X} , there is some $H \in \Gamma(\mathbb{N} \times \mathcal{X})$ which enumerates $\Gamma(\mathcal{X})$, i.e.,

$$P \in \Gamma(\mathcal{X}) \iff (\exists e)[P = H_e = \{x : H(e, x)\}]$$

\Rightarrow For every \mathcal{X} and $k \geq 1$, $\Sigma_k^0(\mathcal{X})$, $\Pi_k^0(\mathcal{X})$ are parametrized

Def A pointset $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ is a (good) **parametrization** of $\Gamma(\mathcal{X})$ (in \mathcal{N}), if for every $P \in \Gamma(\mathcal{N} \times \mathcal{X})$, there is a total recursive

$S^P : \mathcal{N} \rightarrow \mathcal{N}$ such that

$$P(\alpha, x) \iff G(S^P(\alpha), x)$$

Theorem A If Γ is parametrized, then every $\Gamma(\mathcal{X})$ has a parametrization

Theorem B If Γ is closed under recursive substitutions and $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ is a parametrization of $\Gamma(\mathcal{X})$, then

$$P \in \Gamma(\mathcal{X}) \iff (\exists \text{ recursive } \varepsilon \in \mathcal{N})[P = G_\varepsilon = \{x : G(\varepsilon, x)\}]$$

- We think of ε as a **code** (name) **of** P (relative to G)

Proofs of Theorems A and B on the preceding slide

Theorem A *If Γ is parametrized, then every $\Gamma(\mathcal{X})$ has a parametrization*

Proof The hypothesis gives us some $H \in \Gamma(\mathbb{N} \times (\mathcal{N} \times \mathcal{X}))$ such that

$$P \in \Gamma(\mathcal{N} \times \mathcal{X}) \iff (\exists e)[P = H_e = \{(\alpha, x) : H(e, (\alpha, x))\}]$$

Put $\boxed{G(\alpha, x) \iff H(\alpha(0), (\alpha^*, x))}$; if $P = H_e$, then

$P(\alpha, x) \iff H(e, (\alpha, x)) \iff G(\langle e \rangle^\wedge \alpha, x)$ and the required conclusion holds with $S^P(\alpha) = \langle e \rangle^\wedge \alpha$

Theorem B *If Γ is closed under recursive substitutions and $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ is a parametrization of $\Gamma(\mathcal{X})$, then*

$$P \in \Gamma(\mathcal{X}) \iff (\exists \text{ recursive } \varepsilon \in \mathcal{N})[P = G_\varepsilon = \{x : G(\varepsilon, x)\}]$$

Proof For the non-trivial (\Rightarrow) direction, let $Q(\alpha, x) \iff P(x)$ and take $\varepsilon = S^Q((\lambda t)0)$

★ The 2nd Recursion Theorem

Theorem (2nd RT) *If Γ is parametrized, G parametrizes $\Gamma(\mathcal{X})$ and $P \in \Gamma(\mathcal{N} \times \mathcal{X})$, then there is a recursive $\tilde{\varepsilon} \in \mathcal{N}$ such that*

$$(*) \quad P(\tilde{\varepsilon}, x) \iff G(\tilde{\varepsilon}, x)$$

Proof. Let $\boxed{\alpha \mapsto ((\alpha)_0, (\alpha)_1)}$ be a recursive surjection of \mathcal{N} onto $\mathcal{N} \times \mathcal{N}$ with inverse $\boxed{(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle}$,
let $H \in \Gamma(\mathcal{N} \times (\mathcal{N} \times \mathcal{X}))$ parametrize $\Gamma(\mathcal{N} \times \mathcal{X})$, set

$$Q(\alpha, x) \iff H((\alpha)_0, ((\alpha)_1, x))$$

and let S^Q be recursive such that $\boxed{Q(\alpha, x) \iff G(S^Q(\alpha), x)}$

Now

$$\begin{aligned} P(S^Q(\langle \alpha, \alpha \rangle), x) &\iff H(\varepsilon_0, (\alpha, x)) \quad (\text{with a recursive } \varepsilon_0) \quad (\text{corr}) \\ &\iff Q(\langle \varepsilon_0, \alpha \rangle, x) \iff G(S^Q(\langle \varepsilon_0, \alpha \rangle), x) \end{aligned}$$

and $(*)$ holds with $\tilde{\varepsilon} = S^Q(\langle \varepsilon_0, \varepsilon_0 \rangle)$

★ The Kleene calculus for local recursion

- For any two spaces \mathcal{X}, \mathcal{Y} , let $G \subseteq \mathcal{N} \times (\mathcal{X} \times \mathbb{N})$ be a parametrization of $\Sigma_1^0(\mathcal{X} \times \mathbb{N})$, let $G^*((\varepsilon, x), s) \iff G(\varepsilon, (x, s))$

$$\text{and set } \{\varepsilon\}(x) = \{\varepsilon\}^{\mathcal{X} \rightarrow \mathcal{Y}}(x) = \Phi_{G^*}(\varepsilon, x)$$

by the construction in the proof of the Extension Theorem

\Rightarrow The partial function $(\varepsilon, x) \mapsto \{\varepsilon\}^{\mathcal{X} \rightarrow \mathcal{Y}}(x)$ is locally recursive

$\Rightarrow f : \mathcal{X} \rightarrow \mathcal{Y}$ is locally recursive if and only if there is a recursive

$\varepsilon \in \mathcal{N}$ such that $f(x) \downarrow \implies f(x) = \{\varepsilon\}(x)$

S-Theorem If $f : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y}$ is locally recursive, then there is a total, recursive $S^f : \mathcal{N} \rightarrow \mathcal{N}$ such that $f(\alpha, x) \downarrow \implies [f(\alpha, x) = \{S^f(\alpha)\}(x)]$

Theorem (2nd RT for partial functions) For every locally recursive $f : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y}$, there is a recursive $\tilde{\varepsilon} \in \mathcal{N}$ such that

$$f(\tilde{\varepsilon}, x) \downarrow \implies \left(\{\tilde{\varepsilon}\}(x) = f(\tilde{\varepsilon}, x) \right)$$

Primitive recursion preserves local recursiveness

Theorem (Primitive recursion) *If g and h are locally recursive on the appropriate spaces and $f : \mathbb{N} \times \mathcal{X} \rightarrow \mathcal{W}$ is defined by*

$$\begin{aligned}f(0, x) &= g(x), \\f(t + 1, x) &= h(f(t, x), t, x),\end{aligned}$$

then f is also locally recursive

Proof. By the 2nd RT (for partial functions), there is a recursive $\tilde{\varepsilon} \in \mathcal{N}$ such that (when the partial function on the right converges)

$$\{\tilde{\varepsilon}\}(t, x) = \begin{cases} g(x), & \text{if } t = 0, \\ h(\{\tilde{\varepsilon}\}(t - 1, x), t - 1, x) & \text{otherwise} \end{cases}$$

Proof that $\boxed{f(t, x) \downarrow \implies f(t, x) = \{\tilde{\varepsilon}\}(t, x)}$ is by an easy induction on t