Effective Descriptive Set Theory

what it is about

Lecture 1, Recursion in Polish spaces

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The child of two fields

- Classical descriptive set theory, 1895 –

Borel, Baire, Hadamard, Lebesgue 1905, Lusin, Suslin, Novikov, ...
Definability theory on the continuum at first represented by

$$\mathcal{R} = \text{ the real numbers}, \quad \mathcal{N} = \text{ Baire space } = (\mathbb{N} \to \mathbb{N})$$

with $\mathbb{N} = \{0, 1, \ldots\}$, later studied on Polish spaces

- Hyperarithmetical computability on $\mathbb{N}$, 1950 –

Martin Davis, Mostowski, Kleene 1955, Spector, ...

Common motivation (after Lebesgue):
⋆ *Constructively defined sets and functions should have special properties that distinguish them from arbitrary ones*

- **Effective descriptive set theory** (EDST): a common extension, on recursive Polish spaces, with applications to both (and other fields)

Outline

Lecture 1. Recursion in Polish spaces

Lecture 2. Effective Borel, analytic and co-analytic pointsets

Lecture 3. Structure theory for pointclasses

• Primary sources for the lectures (posted on my homepage):
Descriptive set theory, ynm, 1980, 2nd edition 2009
Classical descriptive set theory as a refinement of effective descriptive set theory, ynm, 2010
Kleene's amazing second recursion theorem, ynm, 2010

Notes on effective descriptive set theory, Gregoriades and ynm (in preparation)

• I will try to give an elementary introduction to some of the fundamental notions, ideas and methods of proof specific to EDST not to cover a large part of the field, recent results or applications

• There are several proofs on the slides that I will skip in the lectures

EDST as a recursion theory: what comes first?

– In classical recursion (computability) theory on $\mathbb{N}$ and $\mathcal{N}$,
we typically define

first the recursive partial functions $f : \mathbb{N}^n \times \mathcal{N}^k \rightharpoonup \mathbb{N}$

next the semirecursive (r.e.) relations $P \subseteq \mathbb{N}^n \times \mathcal{N}^k$
(the domains of convergence of recursive partial functions)

and then the arithmetical and analytical relations, etc

– In Polish recursion theory we must reverse the order: define

first the semirecursive relations (pointsets) $P \subseteq \mathcal{X}$

next the locally recursive partial functions $f : \mathcal{X} \rightharpoonup \mathcal{Y}$
(whose domains of convergence are arbitrary)

and then the arithmetical and analytical relations, etc

(and we must define recursive Polish spaces, which include $\mathbb{N}, \mathcal{N}, \mathcal{R}$)

- Emil Post followed this second order of definitions for recursion on $\mathbb{N}$

⋆ Recursively presented Polish metric spaces

- Fix a recursive enumeration $q_0, q_1, \ldots$ of the rational numbers $\mathbb{Q}$,

  i.e., such that $k \mapsto \mathrm{sign}(q_k), \mathrm{num}(q_k), \mathrm{den}(q_k)$ are recursive

Def A recursive presentation of a Polish (= separable, complete) metric space $(\mathcal{X}, d)$ is a sequence $\mathbf{r} = (r_0, r_1, \ldots)$ of points which is dense in $\mathcal{X}$ and such that the following two relations are recursive:

$$\boxed{P^{\mathbf{r}}(i,j,k) \iff d(r_i, r_j) \le q_k, \quad Q^{\mathbf{r}}(i,j,k) \iff d(r_i, r_j) < q_k}$$

- Recursively presented Polish metric space: $(\mathcal{X}, d, \mathbf{r})$

$\Rightarrow$ *The relations $P^{\mathbf{r}}, Q^{\mathbf{r}}$ determine $(\mathcal{X}, d, \mathbf{r})$ up to isometry*

- Relativization: For any $\varepsilon \in \mathcal{N}$, $\mathbf{r}$ is an ε-recursive presentation of $(\mathcal{X}, d)$ if the relations $P^{\mathbf{r}}, Q^{\mathbf{r}}$ are recursive in ε

$\Rightarrow$ *Every Polish metric space has an ε-recursive presentation, for some $\varepsilon \in \mathcal{N}$* (Used to apply results of EDST to all Polish metric spaces)

Examples (with natural metrics and presentations)

$\Rightarrow$ $\{0, \dots, m\}$ and $\mathbb{N}$ with $d(n,k) = 1$ for $n \neq k$,
$\mathcal{R}$, Baire space $\mathcal{N}$, Cantor space $\mathcal{C} = (\mathbb{N} \to \{0,1\}) \subset \mathcal{N}$

$\Rightarrow$ Products $\mathcal{X} \times \mathcal{Y}$, $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \dots$ of recursively presented metric spaces (with either of the standard product metrics)

$\Rightarrow$ $C[0,1] = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$ (with the sup norm)

- ... All "popular" Polish metric spaces have recursive presentations (mostly immediately from their definitions)

- A Polish metric space $(\mathcal{U}, d_{\mathcal{U}})$ is Urysohn (universal) if

  for every finite metric space $(X \cup \{y\}, d)$
  and every isometric embedding $f : X \rightarrowtail \mathcal{U}$,
  there is an isomeric embedding $f^* : X \cup \{y\} \rightarrowtail \mathcal{U}$ *which extends* f

Theorem (Urysohn) *Up to isometry, there is exactly one Urysohn space*

$\Rightarrow$ *The Urysohn space has a recursive presentation*

⋆ Open ($\boldsymbol{\Sigma}^0_1$) and semirecursive (Σ^0_1) pointsets

- Coding of open balls (neighborhoods): for given $(\mathcal{X}, d, \mathbf{r})$, put

$$N_s = N_s(\mathcal{X}) = \{x \in \mathcal{X} : d(x, r_{(s)_0}) < q_{(s)_1}\} \quad (s \in \mathbb{N}),$$

where $s \mapsto ((s)_0, (s)_1)$ is a recursive surjection of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$

Def A set $G \subseteq \mathcal{X}$ is open (in $\boldsymbol{\Sigma}^0_1(\mathcal{X})$) if for some $\varepsilon \in \mathcal{N}$,

$(*)$ $$G = \bigcup_s N_{\varepsilon(s)};$$

it is semirecursive (in $\Sigma^0_1(\mathcal{X})$) if $(*)$ holds with a recursive $\varepsilon : \mathbb{N} \to \mathbb{N}$

Σ^0_1-Normal Form *A pointset $P \subseteq \mathcal{X} \times \mathcal{Y}$ is in $\Sigma^0_1(\mathcal{X} \times \mathcal{Y})$ if and only if*

$$\boxed{P(x,y) \iff (\exists s,t)[x \in N_s(\mathcal{X})\ \&\ y \in N_t(\mathcal{Y})\ \&\ P^*(s,t)]}$$

with a semirecursive $P^* \subseteq \mathbb{N}^2$ (and similarly for $\mathcal{X}, \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \ldots$)

$\Rightarrow$ The family $\Sigma^0_1(\mathcal{X} \times \mathcal{Y})$ depends on $(\mathcal{X}, d_{\mathcal{X}}, \mathbf{r}_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{r}_{\mathcal{Y}})$
(but not on which of the standard metrics we choose for $\mathcal{X} \times \mathcal{Y}$)

Closure properties of Σ^0_1

$\Rightarrow$ $\emptyset, \mathcal{X}$ *are in* $\Sigma^0_1(\mathcal{X})$

$\Rightarrow$ *The basic nbhd relation* $\{(x,s) : x \in N_s(\mathcal{X})\}$ *is in* $\Sigma^0_1(\mathbb{N} \times \mathcal{X})$

$\Rightarrow$ Σ^0_1 *is closed under* $\&, \vee$ *and* $\exists^{\mathbb{N}}$, $P(x) \iff (\exists t \in \mathbb{N}) Q(x,t)$

Def $f : \mathcal{X} \to \mathcal{Y}$ is recursive if the pointset $\{(x,s) : f(x) \in N_s(\mathcal{Y})\}$ is Σ^0_1

$\Rightarrow$ $(x,y) \mapsto x, \quad \alpha \mapsto \alpha^* = \lambda t \alpha(t+1), \quad (e,\alpha) \mapsto \langle e \rangle \hat{} \alpha$
$(\alpha, i) \mapsto (\alpha)_i = (\lambda t)\alpha(\langle i, t \rangle),$

are recursive, and so is $x \mapsto (f(x), g(x))$, *if* f *and* g *are*

$\Rightarrow$ Σ^0_1 *is closed under substitution of recursive functions*

Proof. If $Q(y) \iff (\exists s)[y \in N_s \;\&\; R^*(s)]$,
then $Q(f(x)) \iff (\exists s)[f(x) \in N_s \;\&\; R^*(s)]$

$\Rightarrow$ *The composition* $x \mapsto g(h(x))$ *of recursive functions is recursive*

⋆ Recursive Polish spaces

- A Polish space is a pair $(\mathcal{X}, \mathcal{T})$ such that for some d,

(P1) $(\mathcal{X}, d)$ is a Polish (separable, complete) metric space, and
(P2) $\mathcal{T} = \boldsymbol{\Sigma}^0_1(\mathcal{X})$ = the open subsets of $(\mathcal{X}, d)$

– What is the "recursive topology" on $(\mathcal{X}, d, \mathbf{r})$ with recursive $\mathbf{r}$?
(hard to formulate the appropriate properties for $\Sigma^0_1(\mathcal{X})$)

Def A recursive Polish space is a pair $(\mathcal{X}, \mathcal{F})$ such that for some $(d, \mathbf{r})$,

(RP1) $(\mathcal{X}, d, \mathbf{r})$ is a recursively presented Polish metric space, and
(RP2) $\mathcal{F} = \Sigma^0_1(\mathbb{N} \times \mathcal{X})$ (which depends only on $(\mathcal{X}, d, \mathbf{r})$)

– $\mathcal{F} = \mathcal{F}(\mathcal{X})$ is the frame of $(\mathcal{X}, \mathcal{F})$, its recursive topology, and

– if (RP1), (RP2) hold, then $(d, \mathbf{r})$ is a compatible pair of $(\mathcal{X}, \mathcal{F})$

$\Rightarrow$ *If $(d_1, \mathbf{r}_1)$, $(d_2, \mathbf{r}_2)$ are compatible pairs of $(\mathcal{X}, \mathcal{F})$, then*

$$\Sigma^0_1(\mathcal{X}, d_1, \mathbf{r}_1) = \Sigma^0_1(\mathcal{X}, d_2, \mathbf{r}_2) =_{\text{def}} \Sigma^0_1(\mathcal{X})$$

- Strong closure properties: e.g., $\mathcal{X} \mapsto \prod_{i \in \mathbb{N}} \mathcal{X}$, $\boxed{\mathcal{X} \mapsto \mathcal{X}^{<\omega}}$

Pointsets and pointclasses (in recursive Polish spaces)

- A pointset is any subset $P \subseteq \mathcal{X}$ of a recursive Polish space (formally a pair $(P, \mathcal{X})$)

- A pointclass is any collection Γ of pointsets, e.g., $\Sigma^0_1, \boldsymbol{\Sigma^0_1}$, and for any $\mathcal{X}$, we set

$$\Gamma(\mathcal{X}) = \{P \subseteq \mathcal{X} : P \in \Gamma\} = \text{ the subsets of } \mathcal{X} \text{ which are (in) } \Gamma$$

- The points of Γ: For $x \in \mathcal{X}$, $\boxed{x \in \Gamma \iff \{s : x \in N_s(\mathcal{X})\} \in \Gamma}$

  x is recursive $\iff x \in \Sigma^0_1$ $\quad (\alpha \in \Sigma^0_1 \iff \alpha$ is Turing computable$)$

- The arithmetical pointclasses are defined inductively from Σ^0_1,

$$\Pi^0_k = \neg\Sigma^0_k, \quad \Sigma^0_{k+1} = \exists^{\mathbb{N}}\Pi^0_k \quad (k \geq 1)$$

$\Pi^0_1(\mathcal{X})$: $P(x) \iff \neg Q(x)$ for some $Q \in \Sigma^0_1(\mathcal{X})$,

$\Sigma^0_2(\mathcal{X})$: $P(x) \iff (\exists t \in \mathbb{N})Q(x,t)$ for some $Q \in \Pi^0_1(\mathcal{X} \times \mathbb{N})$

$\Pi^0_2(\mathcal{X})$: $P(x) \iff \neg(\exists t \in \mathbb{N})Q_1(x,t) \iff (\forall t \in \mathbb{N})Q(x,t)$
for some $Q_1 \in \Pi^0_1(\mathcal{X} \times \mathbb{N})$ and some $Q = \neg Q_1 \in \Sigma^0_1(\mathcal{X} \times \mathbb{N})$

Partial functions

- A partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is a (total) function $f : D_f \to \mathcal{Y}$, where $D_f \subseteq \mathcal{X}$ is the domain of convergence of f, and we write

$$f(x)\downarrow \iff x \in D_f, \quad f(x)\uparrow \iff x \notin D_f$$
$$f(x) = g(x) \iff [f(x)\uparrow \ \& \ g(x)\uparrow] \vee (\exists w)[f(x) = w \ \& \ g(y) = w]$$
$$f \sqsubseteq g \iff (\forall x)[f(x)\downarrow \implies f(x) = g(x)]$$

– Partial functions compose strictly, i.e.,

$$\begin{aligned} &g(h_1(x), \dots, h_m(x)) = w \\ &\qquad \iff (\exists y_1, \dots, y_m)[h_1(x) = y_1 \ \& \ \cdots \ \& \ h_m(x) = y_m \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \& \ g(y_1, \dots, y_m) = w] \end{aligned}$$

⋆ Locally recursive partial functions, I

Def A pointset $P \subseteq \mathcal{X} \times \mathbb{N}$ computes a partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ (where it converges) with respect to a compatible pair $(d, \mathbf{r})$ for $\mathcal{X}$, if

$$f(x)\downarrow \implies \Big(\inf\{\mathrm{radius}(N_s) : P(x,s)\} = 0 \\ \& \ \bigcap\{N_s(\mathcal{Y}) : P(x,s)\} = \{f(x)\}\Big)$$

Def $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is locally recursive if it is computed by some P in Σ^0_1

Theorem *The following are equivalent for $f : \mathcal{X} \rightharpoonup \mathcal{Y}$:*

(1) *f is locally recursive*

(2) *For some $Q \in \Sigma^0_1(\mathcal{X} \times \mathbb{N})$,*

$$\boxed{f(x)\downarrow \implies (\forall s)\Big(f(x) \in N_s(\mathcal{Y}) \iff Q(x,s)\Big)}$$

(3) *For every $Q \in \Sigma^0_1(\mathcal{Y} \times \mathcal{Z})$ there is a $P \in \Sigma^0_1(\mathcal{X} \times \mathcal{Z})$ such that*

$$f(x)\downarrow \implies [P(x,z) \iff Q(f(x),z)]$$

⋆ Locally recursive partial functions, II

- The key characterization of local recursiveness is (2),

Theorem $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ *is locally recursive if for some* $Q \in \Sigma^0_1(\mathcal{X} \times \mathbb{N})$

$$f(x)\downarrow \implies (\forall s)\Big(f(x) \in N_s(\mathcal{Y}) \iff Q(x,s)\Big)$$

$\Rightarrow$ $\boxed{\textit{If } x \textit{ is recursive and } f(x)\downarrow, \textit{ then the point } f(x) \textit{ is recursive}}$

$\Rightarrow$ *If* $f : \mathcal{X} \to \mathcal{Y}$ *is total, then* f *is locally recursive if it is recursive* (by any of the old definitions)

$\Rightarrow$ *The composition* $x \mapsto g(h(x))$ *of locally recursive partial functions is locally recursive*

Theorem (Recursion and continuity) *A function* $f : \mathcal{X} \to \mathcal{Y}$ *is continuous if and only if there is a locally recursive* $f^* : \mathcal{N} \times \mathcal{X} \rightharpoonup \mathcal{Y}$ *and some* $\varepsilon \in \mathcal{N}$ *so that* $\boxed{f(x) = f^*(\varepsilon, x) \quad (x \in \mathcal{X})}$

- *It is not always possible to insure that* $f^* : \mathcal{N} \times \mathcal{X} \rightharpoonup \mathcal{Y}$ *is total*

The Refined Surjection Theorem

Theorem (Classical) *For every Polish space $\mathcal{X}$, there is a continuous function $\pi : \mathcal{N} \to \mathcal{X}$ and a closed set $F \subseteq \mathcal{N}$ such that π is one-to-one on F, $\pi[F] = \mathcal{X}$, and the inverse $\pi^{-1} : \mathcal{X} \rightarrowtail\!\!\!\!\twoheadrightarrow F$ is Borel measurable*

Theorem (Effective) *For every recursive Polish space $\mathcal{X}$, there is a total recursive function $\pi : \mathcal{N} \to \mathcal{X}$ and a Π^0_1 set $F \subseteq \mathcal{N}$ such that π is one-to-one on F, $\pi[F] = \mathcal{X}$ and the inverse $\pi^{-1} : \mathcal{X} \rightarrowtail\!\!\!\!\twoheadrightarrow F$ is Σ^0_2-recursive, i.e., the pointset $\{(x, s) : \pi^{-1}(x) \in N_s(\mathcal{N}) \cap F\}$ is Σ^0_2*

- Proof is by a direct, effective construction

- The theorem makes it possible in many cases to prove results for $\mathcal{N}$ and then "transfer" them to every space

Extending the domain of convergence

Theorem (Classical) *Suppose $\mathcal{X}, \mathcal{Y}$ are Polish spaces, $A \subset \mathcal{X}$, and $f : A \to \mathcal{Y}$ is continuous* (with the induced topology on A)*; then there is a set A^* such that*

(1) $A \subseteq A^* \subseteq \mathcal{X}$;
(2) *A^* is a G_δ-set, i.e., $A^* = \bigcap_{n \in \mathbb{N}} A_n$ with each A_n open;*
(3) *A is dense in A^*; and*
(4) *there is an extension of f to some continuous $\Phi : A^* \to \mathcal{Y}$*

Theorem (Effective) *Every locally recursive partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ has a locally recursive extension $\Phi : \mathcal{X} \rightharpoonup \mathcal{Y}$ whose domain of convergence $\{x : \Phi(x)\downarrow\}$ is Π^0_2*

- The classical result follows from the relativized version of the effective theorem, taking $A^* = \{x : \Phi(x)\downarrow\} \cap \mathrm{closure}(A)$

- The effective result cannot be improved to insure that $\{x : f(x)\downarrow\}$ is dense in $\{x : \Phi(x)\downarrow\}$, because closure$(A)$ need not be Π^0_2

Proof of the Extension Theorem for local recursion

Theorem *Every locally recursive partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ has a locally recursive extension $\Phi : \mathcal{X} \rightharpoonup \mathcal{Y}$ whose domain of convergence $\{x : \Phi(x)\downarrow\}$ is Π^0_2*

Fix $\mathcal{X}, \mathcal{Y}$ and for any $P \in \Sigma^0_1(\mathcal{X} \times \mathbb{N})$ define $\Phi = \Phi_P^{\mathcal{X} \rightharpoonup \mathcal{Y}} : \mathcal{X} \rightharpoonup \mathcal{Y}$ by

$$\begin{aligned} \Phi(x)\downarrow \iff & \inf\{\text{radius}(N_s) : P(x,s)\} = 0 \\ & \& \bigcap\{N_t(\mathcal{Y}) : P(x,t)\} \text{ is a singleton,} \\ \Phi(x) = & \text{ the unique } y \text{ in } \bigcap\{N_t(\mathcal{Y}) : P(x,t)\} \end{aligned}$$

$\Rightarrow$ Φ *is locally recursive, as it is computed by P*

$\Rightarrow$ *For any $f : \mathcal{X} \rightharpoonup \mathcal{Y}$,* $\boxed{P \text{ computes } f \iff f \sqsubseteq \Phi}$

$\Rightarrow$ $\{x : \Phi(x)\downarrow\}$ is Π^0_2, because

$$\begin{aligned} \Phi(x)\downarrow \iff & (\forall s,t)[[P(x,s) \;\&\; P(x,t)] \implies N_s \cap N_t \neq \emptyset] \\ & \& (\forall n)(\exists s)[P(x,s) \;\&\; \text{radius}(N_s) < 2^{-n}] \end{aligned}$$

Is it recursion or just "effective continuity"?

Theorem (Primitive recursion) *If g and h are locally recursive on the appropriate spaces and $f : \mathbb{N} \times \mathcal{X} \rightharpoonup \mathcal{W}$ is defined by*

$$f(0, x) = g(x),$$
$$f(t+1, x) = h(f(t, x), t, x),$$

then f is also locally recursive

• The usual proofs for $\mathbb{N}$ (via Dedekind's analysis of recursive definition) or the attempt to show directly that f is effectively continuous are not easy to carry out

• We develop an alternative approach which also works for nested, double, . . . , recursion as well as effective transfinite recursion.

• *It is a very general, fundamental tool of EDST*

$\star$ Parametrized pointclasses

Def A pointclass Γ is parametrized if it is closed under (total) recursive substitutions and for every $\mathcal{X}$, there is some $H \in \Gamma(\mathbb{N} \times \mathcal{X})$ which enumerates $\Gamma(\mathcal{X})$, i.e.,

$$\boxed{P \in \Gamma(\mathcal{X}) \iff (\exists e)[P = H_e = \{x : H(e,x)\}]}$$

$\Rightarrow$ *For every $\mathcal{X}$ and $k \geq 1$, $\Sigma^0_k(\mathcal{X}), \Pi^0_k(\mathcal{X})$ are parametrized*

Def A pointset $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ is a (good) parametrization of $\Gamma(\mathcal{X})$ (in $\mathcal{N}$), if for every $P \in \Gamma(\mathcal{N} \times \mathcal{X})$, there is a total recursive $\mathsf{S}^P : \mathcal{N} \to \mathcal{N}$ such that $\boxed{P(\alpha, x) \iff G(\mathsf{S}^P(\alpha), x)}$

Theorem A *If Γ is parametrized, then every $\Gamma(\mathcal{X})$ has a parametrization*

Theorem B *If Γ is closed under recursive substitutions and $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ is a parametrization of $\Gamma(\mathcal{X})$, then*

$$\boxed{P \in \Gamma(\mathcal{X}) \iff (\exists \text{ recursive } \varepsilon \in \mathcal{N})[P = G_\varepsilon = \{x : G(\varepsilon, x)\}]}$$

- We think of ε as a code (name) of P (relative to G)

Proofs of Theorems A and B on the preceding slide

Theorem A *If* Γ *is parametrized, then every* $\Gamma(\mathcal{X})$ *has a parametrization*

Proof The hypothesis gives us some $H \in \Gamma(\mathbb{N} \times (\mathcal{N} \times \mathcal{X}))$ such that

$$P \in \Gamma(\mathcal{N} \times \mathcal{X}) \iff (\exists e)[P = H_e = \{(\alpha, x) : H(e, (\alpha, x))\}]$$

Put $\boxed{G(\alpha, x) \iff H(\alpha(0), (\alpha^*, x))}$; if $P = H_e$, then $P(\alpha, x) \iff H(e, (\alpha, x)) \iff G(\langle e \rangle \hat{}\, \alpha, x)$ and the required conclusion holds with $\mathsf{S}^P(\alpha) = \langle e \rangle \hat{}\, \alpha$

Theorem B *If* Γ *is closed under recursive substitutions and* $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ *is a parametrization of* $\Gamma(\mathcal{X})$*, then*

$$P \in \Gamma(\mathcal{X}) \iff (\exists \text{ recursive } \varepsilon \in \mathcal{N})[P = G_\varepsilon = \{x : G(\varepsilon, x)\}]$$

Proof For the non-trivial ($\Rightarrow$) direction, let $Q(\alpha, x) \iff P(x)$ and take $\varepsilon = \mathsf{S}^Q((\lambda t)0)$

⋆ The 2nd Recursion Theorem

Theorem (2nd RT) *If* Γ *is parametrized,* G *parametrizes* $\Gamma(\mathcal{X})$ *and* $P \in \Gamma(\mathcal{N} \times \mathcal{X})$*, then there is a recursive* $\widetilde{\varepsilon} \in \mathcal{N}$ *such that*

$$(*) \qquad P(\widetilde{\varepsilon}, x) \iff G(\widetilde{\varepsilon}, x)$$

Proof. Let $\boxed{\alpha \mapsto ((\alpha)_0, (\alpha)_1)}$ be a recursive surjection of $\mathcal{N}$ onto $\mathcal{N} \times \mathcal{N}$ with inverse $\boxed{(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle}$,
let $H \in \Gamma(\mathcal{N} \times (\mathcal{N} \times \mathcal{X}))$ parametrize $\Gamma(\mathcal{N} \times \mathcal{X})$, set

$$Q(\alpha, x) \iff H((\alpha)_0, ((\alpha)_1, x))$$

and let S^Q be recursive such that $\boxed{Q(\alpha, x) \iff G(\mathsf{S}^Q(\alpha), x)}$

Now $P(\mathsf{S}^Q(\alpha), x) \iff H(\varepsilon_0, (\alpha, x))$ (with a recursive ε_0)
$\iff Q(\langle \varepsilon_0, \alpha \rangle, x) \iff G(\mathsf{S}^Q(\langle \varepsilon_0, \alpha \rangle), x)$
and $(*)$ holds with $\widetilde{\varepsilon} = \mathsf{S}^Q(\langle \varepsilon_0, \varepsilon_0 \rangle)$

⋆ The Kleene calculus for local recursion

• For any two spaces $\mathcal{X}, \mathcal{Y}$, let $G \subseteq \mathcal{N} \times (\mathcal{X} \times \mathbb{N})$ be a parametrization of $\Sigma^0_1(\mathcal{X} \times \mathbb{N})$, let $G^*((\varepsilon, x), s) \iff G(\varepsilon, (x, s))$

$$\text{and set } \{\varepsilon\}(x) = \{\varepsilon\}^{\mathcal{X} \rightharpoonup \mathcal{Y}}(x) = \Phi_{G^*}(\varepsilon, x)$$

by the construction in the proof of the Extension Theorem

$\Rightarrow$ *The partial function* $(\varepsilon, x) \mapsto \{\varepsilon\}^{\mathcal{X} \rightharpoonup \mathcal{Y}}(x)$ *is locally recursive*

$\Rightarrow$ $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ *is locally recursive if and only if there is a recursive* $\varepsilon \in \mathcal{N}$ *such that* $\boxed{f(x)\downarrow \implies f(x) = \{\varepsilon\}(x)}$

S-Theorem *If* $f : \mathcal{N} \times \mathcal{X} \rightharpoonup \mathcal{Y}$ *is locally recursive, then there is a total, recursive* $\mathsf{S}^f : \mathcal{N} \to \mathcal{N}$ *such that* $\boxed{f(\alpha, x)\downarrow \implies [f(\alpha, x) = \{\mathsf{S}^f(\alpha)\}(x)]}$

Theorem (2nd RT for partial functions) *For every locally recursive* $f : \mathcal{N} \times \mathcal{X} \rightharpoonup \mathcal{Y}$, *there is a recursive* $\widetilde{\varepsilon} \in \mathcal{N}$ *such that*

$$f(\widetilde{\varepsilon}, x)\downarrow \implies \Big(\{\widetilde{\varepsilon}\}(x) = f(\widetilde{\varepsilon}, x)\Big)$$

Primitive recursion preserves local recursiveness

Theorem (Primitive recursion) *If g and h are locally recursive on the appropriate spaces and $f : \mathbb{N} \times \mathcal{X} \rightharpoonup \mathcal{W}$ is defined by*

$$f(0, x) = g(x),$$
$$f(t+1, x) = h(f(t, x), t, x),$$

then f is also locally recursive

Proof. By the 2nd RT (for partial functions), there is a a recursive $\widetilde{\varepsilon} \in \mathcal{N}$ such that (when the partial function on the right converges)

$$\{\widetilde{\varepsilon}\}(t, x) = \begin{cases} g(x), & \text{if } t = 0, \\ h(\{\widetilde{\varepsilon}\}(t-1, x), t-1, x) & \text{otherwise} \end{cases}$$

Proof that $\boxed{f(t, x)\downarrow \implies f(t, x) = \{\widetilde{\varepsilon}\}(t, x)}$ is by an easy induction on t