Effective Descriptive Set Theory
what it is about

Lecture 1, Recursion in Polish spaces

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The child of two fields

  Definability theory on the continuum at first represented by

\[ \mathcal{R} = \text{the real numbers}, \quad \mathcal{N} = \text{Baire space} = (\mathbb{N} \to \mathbb{N}) \]

with \( \mathbb{N} = \{0, 1, \ldots\} \), later studied on Polish spaces

• Hyperarithmetical computability on \( \mathbb{N} \), 1950 – Martin Davis, Mostowski, Kleene 1955, Spector, . . .

Common motivation (after Lebesgue):
★ Constructively defined sets and functions should have special properties that distinguish them from arbitrary ones

• Effective descriptive set theory (EDST): a common extension, on recursive Polish spaces, with applications to both (and other fields)
Outline

Lecture 1. Recursion in Polish spaces
Lecture 2. Effective Borel, analytic and co-analytic pointsets
Lecture 3. Structure theory for pointclasses

• Primary sources for the lectures (posted on my homepage):
  * Classical descriptive set theory as a refinement of effective descriptive set theory, ynm, 2010
  * Kleene’s amazing second recursion theorem, ynm, 2010
  * Notes on effective descriptive set theory, Gregoriades and ynm (in preparation)

• I will try to give an elementary introduction to some of the fundamental notions, ideas and methods of proof specific to EDST not to cover a large part of the field, recent results or applications

• There are several proofs on the slides that I will skip in the lectures
EDST as a recursion theory: what comes first?

- In classical recursion (computability) theory on \( \mathbb{N} \) and \( \mathcal{N} \), we typically define
  
  **first** the recursive partial functions \( f : \mathbb{N}^n \times \mathcal{N}^k \rightarrow \mathbb{N} \)
  
  **next** the semirecursive (r.e.) relations \( P \subseteq \mathbb{N}^n \times \mathcal{N}^k \)
  
  (the domains of convergence of recursive partial functions)
  
  **and then** the arithmetical and analytical relations, etc

- In Polish recursion theory we must reverse the order: define
  
  **first** the semirecursive relations (pointsets) \( P \subseteq \mathcal{X} \)
  
  **next** the locally recursive partial functions \( f : \mathcal{X} \rightarrow \mathcal{Y} \)
  
  (whose domains of convergence are arbitrary)
  
  **and then** the arithmetical and analytical relations, etc

  (and we must define recursive Polish spaces, which include \( \mathbb{N}, \mathcal{N}, \mathcal{R} \))

- Emil Post followed this second order of definitions for recursion on \( \mathbb{N} \)
Recursively presented Polish metric spaces

- Fix a recursive enumeration \( q_0, q_1, \ldots \) of the rational numbers \( \mathbb{Q} \), i.e., such that \( k \mapsto \text{sign}(q_k), \text{num}(q_k), \text{den}(q_k) \) are recursive.

**Def** A recursive presentation of a Polish (= separable, complete) metric space \((\mathcal{X}, d)\) is a sequence \( r = (r_0, r_1, \ldots) \) of points which is dense in \( \mathcal{X} \) and such that the following two relations are recursive:

\[
P^r(i, j, k) \iff d(r_i, r_j) \leq q_k, \quad Q^r(i, j, k) \iff d(r_i, r_j) < q_k
\]

- Recursively presented Polish metric space: \((\mathcal{X}, d, r)\)

\[ \Rightarrow \text{The relations } P^r, Q^r \text{ determine } (\mathcal{X}, d, r) \text{ up to isometry} \]

- Relativization: For any \( \varepsilon \in \mathbb{N} \), \( r \) is an \( \varepsilon \)-recursive presentation of \((\mathcal{X}, d)\) if the relations \( P^r, Q^r \) are recursive in \( \varepsilon \)

\[ \Rightarrow \text{Every Polish metric space has an } \varepsilon \text{-recursive presentation, for some } \varepsilon \in \mathbb{N} \text{ (Used to apply results of EDST to all Polish metric spaces)} \]
Examples (with natural metrics and presentations)

⇒ \{0, \ldots, m\} and \mathbb{N} with \(d(n, k) = 1\) for \(n \neq k\),

\(\mathcal{R}\), Baire space \(\mathcal{N}\), Cantor space \(C = (\mathbb{N} \to \{0, 1\}) \subset \mathcal{N}\)

⇒ Products \(\mathcal{X} \times \mathcal{Y}, \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \ldots\) of recursively presented metric spaces (with either of the standard product metrics)

⇒ \(C[0, 1] = \{f : [0, 1] \to \mathbb{R} : f\ \text{is continuous}\}\) (with the sup norm)

• ... All “popular” Polish metric spaces have recursive presentations (mostly immediately from their definitions)

• A Polish metric space \((\mathcal{U}, d_\mathcal{U})\) is Urysohn (universal) if

\begin{align*}
\text{for every finite metric space } & (X \cup \{y\}, d) \\
\text{and every isometric embedding } & f : X \hookrightarrow \mathcal{U}, \\
\text{there is an isometric embedding } & f^* : X \cup \{y\} \hookrightarrow \mathcal{U} \ \text{which extends } f
\end{align*}

**Theorem (Urysohn)** Up to isometry, there is exactly one Urysohn space

⇒ The Urysohn space has a recursive presentation
Open ($\Sigma^0_1$) and semirecursive ($\Sigma^0_1$) pointsets

- Coding of open balls (neighborhoods): for given ($\mathcal{X}, d, r$), put
  \[ N_s = N_s(\mathcal{X}) = \{ x \in \mathcal{X} : d(x, r(s)_0) < q(s)_1 \} \quad (s \in \mathbb{N}) , \]
  where $s \mapsto ((s)_0, (s)_1)$ is a recursive surjection of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$

**Def** A set $G \subseteq \mathcal{X}$ is open (in $\Sigma^0_1(\mathcal{X})$) if for some $\varepsilon \in \mathcal{N}$,

\[(\ast) \quad G = \bigcup_s N_{\varepsilon(s)} ; \]

it is semirecursive (in $\Sigma^0_1(\mathcal{X})$) if (\ast) holds with a recursive $\varepsilon : \mathbb{N} \rightarrow \mathbb{N}$

**$\Sigma^0_1$-Normal Form** A pointset $P \subseteq \mathcal{X} \times \mathcal{Y}$ is in $\Sigma^0_1(\mathcal{X} \times \mathcal{Y})$ if and only if

\[
P(x, y) \iff (\exists s, t)[ x \in N_s(\mathcal{X}) \& y \in N_t(\mathcal{Y}) \& P^*(s, t)]
\]

with a semirecursive $P^* \subseteq \mathbb{N}^2$ (and similarly for $\mathcal{X}, \mathcal{X} \times \mathcal{Y} \times \mathcal{Z}, \ldots$)

\[
\Rightarrow \text{The family } \Sigma^0_1(\mathcal{X} \times \mathcal{Y}) \text{ depends on } (\mathcal{X}, d_\mathcal{X}, r_\mathcal{X}) \text{ and } (\mathcal{Y}, d_\mathcal{Y}, r_\mathcal{Y})
\]

(but not on which of the standard metrics we choose for $\mathcal{X} \times \mathcal{Y}$)
Closure properties of $\Sigma^0_1$

$\Rightarrow$ $\emptyset, \mathcal{X}$ are in $\Sigma^0_1(\mathcal{X})$

$\Rightarrow$ The basic nbhd relation $\{(x, s) : x \in N_s(\mathcal{X})\}$ is in $\Sigma^0_1(\mathbb{N} \times \mathcal{X})$

$\Rightarrow$ $\Sigma^0_1$ is closed under $\&$, $\vee$ and $\exists^\mathbb{N}$, $P(x) \iff (\exists t \in \mathbb{N})Q(x, t)$

**Def** $f : \mathcal{X} \to \mathcal{Y}$ is recursive if the pointset $\{(x, s) : f(x) \in N_s(\mathcal{Y})\}$ is $\Sigma^0_1$

$\Rightarrow$ $(x, y) \mapsto x$, $\alpha \mapsto \alpha^* = \lambda t \alpha(t + 1)$, $(e, \alpha) \mapsto \langle e \rangle^\alpha$

$(\alpha, i) \mapsto (\alpha)_i = (\lambda t)\alpha(\langle i, t \rangle)$, are recursive, and so is $x \mapsto (f(x), g(x))$, if $f$ and $g$ are recursive.

$\Rightarrow$ $\Sigma^0_1$ is closed under substitution of recursive functions

Proof. If $Q(y) \iff (\exists s)[y \in N_s \& R^*(s)]$, then $Q(f(x)) \iff (\exists s)[f(x) \in N_s \& R^*(s)]$

$\Rightarrow$ The composition $x \mapsto g(h(x))$ of recursive functions is recursive
Recursive Polish spaces

- A Polish space is a pair \((\mathcal{X}, \mathcal{T})\) such that for some \(d\),
  
  (P1) \((\mathcal{X}, d)\) is a Polish (separable, complete) metric space, and
  
  (P2) \(\mathcal{T} = \Sigma^0_1(\mathcal{X}) = \) the open subsets of \((\mathcal{X}, d)\)

- What is the “recursive topology” on \((\mathcal{X}, d, r)\) with recursive \(r\)?
  
  (hard to formulate the appropriate properties for \(\Sigma^0_1(\mathcal{X})\))

**Def** A recursive Polish space is a pair \((\mathcal{X}, \mathcal{F})\) such that for some \((d, r)\),

(RP1) \((\mathcal{X}, d, r)\) is a recursively presented Polish metric space, and

(RP2) \(\mathcal{F} = \Sigma^0_1(\mathbb{N} \times \mathcal{X})\) (which depends only on \((\mathcal{X}, d, r)\))

- \(\mathcal{F} = \mathcal{F}(\mathcal{X})\) is the frame of \((\mathcal{X}, \mathcal{F})\), its recursive topology, and

- if (RP1), (RP2) hold, then \((d, r)\) is a compatible pair of \((\mathcal{X}, \mathcal{F})\)

⇒ If \((d_1, r_1), (d_2, r_2)\) are compatible pairs of \((\mathcal{X}, \mathcal{F})\), then

\[ \Sigma^0_1(\mathcal{X}, d_1, r_1) = \Sigma^0_1(\mathcal{X}, d_2, r_2) = \text{def} \Sigma^0_1(\mathcal{X}) \]

- **Strong closure properties**: e.g., \(\mathcal{X} \mapsto \prod_{i \in \mathbb{N}} \mathcal{X}, \mathcal{X} \mapsto \mathcal{X}^{<\omega} \)
Pointsets and pointclasses (in recursive Polish spaces)

• A pointset is any subset $P \subseteq X$ of a recursive Polish space (formally a pair $(P, X)$)

• A pointclass is any collection $\Gamma$ of pointsets, e.g., $\Sigma^0_1, \Sigma^0_1$, and for any $X$, we set

$$\Gamma(X) = \{ P \subseteq X : P \in \Gamma \} = \text{the subsets of } X \text{ which are (in) } \Gamma$$

• The points of $\Gamma$: For $x \in X$, $x \in \Gamma \iff \{ s : x \in N_s(X) \} \in \Gamma$

$x$ is recursive $\iff x \in \Sigma^0_1$ ($\alpha \in \Sigma^0_1 \iff \alpha$ is Turing computable)

• The arithmetical pointclasses are defined inductively from $\Sigma^0_1$,

$$\Pi^0_k = \neg \Sigma^0_k, \quad \Sigma^0_{k+1} = \exists \mathbb{N} \Pi^0_k \quad (k \geq 1)$$

$\Pi^0_1(X) : P(x) \iff \neg Q(x)$ for some $Q \in \Sigma^0_1(X)$,

$\Sigma^0_2(X) : P(x) \iff (\exists t \in \mathbb{N})Q(x, t)$ for some $Q \in \Pi^0_1(X \times \mathbb{N})$

$\Pi^0_2(X) : P(x) \iff \neg (\exists t \in \mathbb{N})Q_1(x, t) \iff (\forall t \in \mathbb{N})Q(x, t)$

for some $Q_1 \in \Pi^0_1(X \times \mathbb{N})$ and some $Q = \neg Q_1 \in \Sigma^0_1(X \times \mathbb{N})$
Partial functions

- A partial function \( f : \mathcal{X} \to \mathcal{Y} \) is a (total) function \( f : D_f \to \mathcal{Y} \), where \( D_f \subseteq \mathcal{X} \) is the domain of convergence of \( f \), and we write

\[
\begin{align*}
  f(x) \downarrow & \iff x \in D_f, \quad f(x) \uparrow \iff x \notin D_f \\
  f(x) = g(x) & \iff [f(x) \uparrow \land g(x) \uparrow] \lor (\exists w)[f(x) = w \land g(y) = w] \\
  f \sqsubseteq g & \iff (\forall x)[f(x) \downarrow \implies f(x) = g(x)]
\end{align*}
\]

- Partial functions compose strictly, i.e.,

\[
\begin{align*}
g(h_1(x), \ldots, h_m(x)) = w & \iff (\exists y_1, \ldots, y_m)[h_1(x) = y_1 \land \cdots \land h_m(x) = y_m \\
& \land g(y_1, \ldots, y_m) = w]
\end{align*}
\]
Locally recursive partial functions, I

**Def** A pointset $P \subseteq \mathcal{X} \times \mathbb{N}$ computes a partial function $f : \mathcal{X} \rightarrow \mathcal{Y}$ (where it converges) with respect to a compatible pair $(d, r)$ for $\mathcal{X}$, if

$$f(x) \downarrow \implies \left( \inf \{ \text{radius}(N_s) : P(x, s) \} = 0 \right.$$

$$\text{and } \cap \{ N_s(\mathcal{Y}) : P(x, s) \} = \{ f(x) \} \right)$$

**Def** $f : \mathcal{X} \rightarrow \mathcal{Y}$ is **locally recursive** if it is computed by some $P$ in $\Sigma^0_1$.

**Theorem** *The following are equivalent for $f : \mathcal{X} \rightarrow \mathcal{Y}$:*

1. $f$ is locally recursive
2. For some $Q \in \Sigma^0_1(\mathcal{X} \times \mathbb{N})$,
   
   $$f(x) \downarrow \implies (\forall s) \left( f(x) \in N_s(\mathcal{Y}) \iff Q(x, s) \right)$$

3. For every $Q \in \Sigma^0_1(\mathcal{Y} \times \mathcal{Z})$ there is a $P \in \Sigma^0_1(\mathcal{X} \times \mathcal{Z})$ such that
   
   $$f(x) \downarrow \implies [P(x, z) \iff Q(f(x), z)]$$
Locally recursive partial functions, II

- The key characterization of local recursiveness is (2),

**Theorem** \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is locally recursive if for some \( Q \in \Sigma_1^0(\mathcal{X} \times \mathbb{N}) \)

\[
f(x) \downarrow \implies (\forall s) \left( f(x) \in N_s(\mathcal{Y}) \iff Q(x, s) \right)
\]

⇒ If \( x \) is recursive and \( f(x) \downarrow \), then the point \( f(x) \) is recursive

⇒ If \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is total, then \( f \) is locally recursive if it is recursive (by any of the old definitions)

⇒ The composition \( x \mapsto g(h(x)) \) of locally recursive partial functions is locally recursive

**Theorem (Recursion and continuity)** A function \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is continuous if and only if there is a locally recursive \( f^* : \mathbb{N} \times \mathcal{X} \rightarrow \mathcal{Y} \) and some \( \varepsilon \in \mathbb{N} \) so that

\[
f(x) = f^*(\varepsilon, x) \quad (x \in \mathcal{X})
\]

- It is not always possible to insure that \( f^* : \mathbb{N} \times \mathcal{X} \rightarrow \mathcal{Y} \) is total
The Refined Surjection Theorem

Theorem (Classical) For every Polish space $\mathcal{X}$, there is a continuous function $\pi : \mathcal{N} \to \mathcal{X}$ and a closed set $F \subseteq \mathcal{N}$ such that $\pi$ is one-to-one on $F$, $\pi[F] = \mathcal{X}$, and the inverse $\pi^{-1} : \mathcal{X} \rightarrow F$ is Borel measurable.

Theorem (Effective) For every recursive Polish space $\mathcal{X}$, there is a total recursive function $\pi : \mathcal{N} \to \mathcal{X}$ and a $\Pi^0_1$ set $F \subseteq \mathcal{N}$ such that $\pi$ is one-to-one on $F$, $\pi[F] = \mathcal{X}$ and the inverse $\pi^{-1} : \mathcal{X} \rightarrow F$ is $\Sigma^0_2$-recursive, i.e., the pointset $\{(x, s) : \pi^{-1}(x) \in N_s(\mathcal{N}) \cap F\}$ is $\Sigma^0_2$.

- Proof is by a direct, effective construction.
- The theorem makes it possible in many cases to prove results for $\mathcal{N}$ and then “transfer” them to every space.
Extending the domain of convergence

Theorem (Classical) Suppose $\mathcal{X}, \mathcal{Y}$ are Polish spaces, $A \subset \mathcal{X}$, and $f : A \rightarrow \mathcal{Y}$ is continuous (with the induced topology on $A$); then there is a set $A^*$ such that

1. $A \subseteq A^* \subseteq \mathcal{X}$;
2. $A^*$ is a $G_\delta$-set, i.e., $A^* = \bigcap_{n \in \mathbb{N}} A_n$ with each $A_n$ open;
3. $A$ is dense in $A^*$; and
4. there is an extension of $f$ to some continuous $\Phi : A^* \rightarrow \mathcal{Y}$

Theorem (Effective) Every locally recursive partial function $f : \mathcal{X} \rightarrow \mathcal{Y}$ has a locally recursive extension $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ whose domain of convergence $\{x : \Phi(x) \downarrow\}$ is $\Pi^0_2$

• The classical result follows from the relativized version of the effective theorem, taking $A^* = \{x : \Phi(x) \downarrow\} \cap \text{closure}(A)$

• The effective result cannot be improved to insure that $\{x : f(x) \downarrow\}$ is dense in $\{x : \Phi(x) \downarrow\}$, because $\text{closure}(A)$ need not be $\Pi^0_2$
Proof of the Extension Theorem for local recursion

**Theorem** Every locally recursive partial function $f : \mathcal{X} \rightarrow \mathcal{Y}$ has a locally recursive extension $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ whose domain of convergence $\{x : \Phi(x) \downarrow\}$ is $\Pi^0_2$

Fix $\mathcal{X}, \mathcal{Y}$ and for any $P \in \Sigma^0_1(\mathcal{X} \times \mathbb{N})$ define $\Phi = \Phi^\mathcal{X}_P : \mathcal{X} \rightarrow \mathcal{Y}$ by

\[
\Phi(x) \downarrow \iff \inf\{\text{radius}(N_s) : P(x, s)\} = 0
\]

\[
\& \bigcap\{N_t(\mathcal{Y}) : P(x, t)\} \text{ is a singleton},
\]

\[
\Phi(x) = \text{the unique } y \text{ in } \bigcap\{N_t(\mathcal{Y}) : P(x, t)\}
\]

$\Rightarrow \Phi$ is locally recursive, as it is computed by $P$

$\Rightarrow$ For any $f : \mathcal{X} \rightarrow \mathcal{Y}$, $P$ computes $f \iff f \sqsubseteq \Phi$

$\Rightarrow$ $\{x : \Phi(x) \downarrow\}$ is $\Pi^0_2$, because

\[
\Phi(x) \downarrow \iff (\forall s, t)[[P(x, s) \& P(x, t)] \Rightarrow N_s \cap N_t \neq \emptyset]
\]

\[
\& (\forall n)(\exists s)[P(x, s) \& \text{radius}(N_s) < 2^{-n}]
\]
Is it recursion or just “effective continuity”? 

**Theorem (Primitive recursion)** If $g$ and $h$ are locally recursive on the appropriate spaces and $f : \mathbb{N} \times X \to W$ is defined by

\[
\begin{align*}
    f(0, x) &= g(x), \\
    f(t + 1, x) &= h(f(t, x), t, x),
\end{align*}
\]

then $f$ is also locally recursive

- The usual proofs for $\mathbb{N}$ (via Dedekind’s analysis of recursive definition) or the attempt to show directly that $f$ is effectively continuous are not easy to carry out

- We develop an alternative approach which also works for nested, double, . . . , recursion as well as effective transfinite recursion.

- *It is a very general, fundamental tool of EDST*
**Parametrized pointclasses**

**Def** A pointclass $\Gamma$ is **parametrized** if it is closed under (total) recursive substitutions and for every $\mathcal{X}$, there is some $H \in \Gamma(\mathbb{N} \times \mathcal{X})$ which enumerates $\Gamma(\mathcal{X})$, i.e.,

$$P \in \Gamma(\mathcal{X}) \iff (\exists e)[P = H_e = \{x : H(e, x)\}]$$

$\Rightarrow$ For every $\mathcal{X}$ and $k \geq 1$, $\Sigma^0_k(\mathcal{X}), \Pi^0_k(\mathcal{X})$ are parametrized

**Def** A pointset $G \in \Gamma(\mathbb{N} \times \mathcal{X})$ is a (good) **parametrization** of $\Gamma(\mathcal{X})$ (in $\mathbb{N}$), if for every $P \in \Gamma(\mathbb{N} \times \mathcal{X})$, there is a total recursive $S^P : \mathbb{N} \to \mathbb{N}$ such that $P(\alpha, x) \iff G(S^P(\alpha), x)$

**Theorem A** If $\Gamma$ is parametrized, then every $\Gamma(\mathcal{X})$ has a parametrization

**Theorem B** If $\Gamma$ is closed under recursive substitutions and $G \in \Gamma(\mathbb{N} \times \mathcal{X})$ is a parametrization of $\Gamma(\mathcal{X})$, then

$$P \in \Gamma(\mathcal{X}) \iff (\exists \text{recursive } \varepsilon \in \mathbb{N})[P = G_\varepsilon = \{x : G(\varepsilon, x)\}]$$

• We think of $\varepsilon$ as a code (name) of $P$ (relative to $G$)
Proofs of Theorems A and B on the preceding slide

**Theorem A** If $\Gamma$ is parametrized, then every $\Gamma(\mathcal{X})$ has a parametrization

*Proof* The hypothesis gives us some $H \in \Gamma(\mathbb{N} \times (\mathcal{N} \times \mathcal{X}))$ such that

$$P \in \Gamma(\mathcal{N} \times \mathcal{X}) \iff (\exists e)[P = H_e = \{(\alpha, x) : H(e, (\alpha, x))\}]$$

Put $G(\alpha, x) \iff H(\alpha(0), (\alpha^*, x))$; if $P = H_e$, then

$$P(\alpha, x) \iff H(e, (\alpha, x)) \iff G(\langle e \rangle \hat{\alpha}, x)$$

and the required conclusion holds with $S^P(\alpha) = \langle e \rangle \hat{\alpha}$

**Theorem B** If $\Gamma$ is closed under recursive substitutions and $G \in \Gamma(\mathcal{N} \times \mathcal{X})$ is a parametrization of $\Gamma(\mathcal{X})$, then

$$P \in \Gamma(\mathcal{X}) \iff (\exists \text{recursive } \varepsilon \in \mathcal{N})[P = G_\varepsilon = \{x : G(\varepsilon, x)\}]$$

*Proof* For the non-trivial ($\Rightarrow$) direction, let $Q(\alpha, x) \iff P(x)$ and take $\varepsilon = S^Q((\lambda t)0)$
Theorem (2nd RT) If $\Gamma$ is parametrized, $G$ parametrizes $\Gamma(\mathcal{X})$ and $P \in \Gamma(\mathcal{N} \times \mathcal{X})$, then there is a recursive $\bar{\epsilon} \in \mathcal{N}$ such that

\begin{equation}
(*) \quad P(\bar{\epsilon}, x) \iff G(\bar{\epsilon}, x)
\end{equation}

Proof. Let $\alpha \mapsto ((\alpha)_0, (\alpha)_1)$ be a recursive surjection of $\mathcal{N}$ onto $\mathcal{N} \times \mathcal{N}$ with inverse $(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle$, let $H \in \Gamma(\mathcal{N} \times (\mathcal{N} \times \mathcal{X}))$ parametrize $\Gamma(\mathcal{N} \times \mathcal{X})$, set

$$Q(\alpha, x) \iff H((\alpha)_0, ((\alpha)_1, x))$$

and let $S^Q$ be recursive such that

$$Q(\alpha, x) \iff G(S^Q(\alpha), x)$$

Now $P(S^Q(\alpha), x) \iff H(\epsilon_0, (\alpha, x))$ (with a recursive $\epsilon_0$)

$$\iff Q(\langle \epsilon_0, \alpha \rangle, x) \iff G(S^Q(\langle \epsilon_0, \alpha \rangle), x)$$

and (*) holds with $\bar{\epsilon} = S^Q(\langle \epsilon_0, \epsilon_0 \rangle)$
The Kleene calculus for local recursion

For any two spaces $\mathcal{X}, \mathcal{Y}$, let $G \subseteq \mathcal{N} \times (\mathcal{X} \times \mathbb{N})$ be a parametrization of $\Sigma^0_1(\mathcal{X} \times \mathbb{N})$, let $G^*((\varepsilon, x), s) \iff G(\varepsilon, (x, s))$

and set $\{\varepsilon\}(x) = \{\varepsilon\}^{\mathcal{X}\twoheadrightarrow\mathcal{Y}}(x) = \Phi_{G^*}(\varepsilon, x)$

by the construction in the proof of the Extension Theorem

⇒ The partial function $(\varepsilon, x) \mapsto \{\varepsilon\}^{\mathcal{X}\twoheadrightarrow\mathcal{Y}}(x)$ is locally recursive

⇒ $f : \mathcal{X} \rightarrow \mathcal{Y}$ is locally recursive if and only if there is a recursive $\varepsilon \in \mathcal{N}$ such that $f(x) \downarrow \implies f(x) = \{\varepsilon\}(x)$

S-Theorem If $f : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y}$ is locally recursive, then there is a total, recursive $S f : \mathcal{N} \rightarrow \mathcal{N}$ such that $f(\alpha, x) \downarrow \implies [f(\alpha, x) = \{S f(\alpha)\}(x)]$

Theorem (2nd RT for partial functions) For every locally recursive $f : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y}$, there is a recursive $\tilde{\varepsilon} \in \mathcal{N}$ such that $f(\tilde{\varepsilon}, x) \downarrow \implies \left(\{\tilde{\varepsilon}\}(x) = f(\tilde{\varepsilon}, x)\right)$
Primitive recursion preserves local recursiveness

Theorem (Primitive recursion) If $g$ and $h$ are locally recursive on the appropriate spaces and $f : \mathbb{N} \times X \to W$ is defined by

$$f(0, x) = g(x),$$
$$f(t + 1, x) = h(f(t, x), t, x),$$

then $f$ is also locally recursive.

Proof. By the 2nd RT (for partial functions), there is a a recursive $\bar{\varepsilon} \in \mathcal{N}$ such that (when the partial function on the right converges)

$$\{\bar{\varepsilon}\}(t, x) = \begin{cases} g(x), & \text{if } t = 0, \\ h(\{\bar{\varepsilon}\}(t - 1, x), t - 1, x) & \text{otherwise} \end{cases}$$

Proof that $\left\lfloor f(t, x) \right\rfloor \implies f(t, x) = \{\bar{\varepsilon}\}(t, x)$ is by an easy induction on $t$.