A survey of the origins and development of Descriptive Set Theory

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Outline

- I will try to give an overview of Descriptive Set Theory (DST) as it developed (mostly) during the last century;
  it will be primarily a historical talk, but I will also emphasize the substantial foundational aspects of DST

Part A. The beginnings, Lebesgue[1905] – Kondo[1938]

Interlude: Gödel[1938] and Cohen[1963]

Part B. Enters Recursion Theory, Kleene[1943]–

Part C. Going beyond ZFC, Scott[1961]–

Part D. Enter games, Blackwell[1967]–

Epilogue

Asking the basic question (Lebesgue[1905])

On the functions which are analytically representable

- The context in 1905 is the loss of confidence in Cantor's (naive) theory of arbitrary sets caused by the paradoxes
- Lebesgue mistrusts the definition of function $f : X \to Y$ as an arbitrary correspondence (1870's, Cantor, Dedekind) and adds

  if there are real functions $f : \mathbb{R}^n \to \mathbb{R}$ which are not analytically definable, then it is worth identifying those which are definable and study their characteristic properties

- Def. A collection $\mathcal{S} \subseteq \mathcal{P}(\mathbb{R}^n)$ is a σ-field in $\mathbb{R}^n$ if
  (B1) $\mathcal{S}$ contains all open balls
  (B2) $\mathcal{S}$ is closed under countable unions, $\bigcup_{i \in \mathbb{N}} A_i$
  (B2) $\mathcal{S}$ is closed under countable intersections, $\bigcap_{i \in \mathbb{N}} A_i$
- Def. $\mathbf{B}(\mathbb{R}^n) :=$ the smallest σ-field in $\mathbb{R}^n$ (the Borel subsets of $\mathbb{R}^n$)
- Def. $f : \mathbb{R}^n \to \mathbb{R}$ is Borel (measurable) if for every open interval $(a, b) \subset \mathbb{R}$, $f^{-1}(a, b) \in \mathbf{B}(\mathbb{R}^n)$

Regularity properties of Borel subsets of $\mathbb{R}^n$

. . . not true of all sets if we assume the Axiom of Choice AC

- *Every Borel set is Lebesgue measurable* (Lebesgue, a basic fact of his theory of integration)
- $\mathbf{B}(\mathbb{R}^n)$ *is closed under complementation* (Lebesgue, easy, gives a characterization of $\mathbf{B}(\mathbb{R}^n)$ as *the least collection of subsets of* $\mathbb{R}^n$ *which contains all open sets and is closed under countable unions and complementation*
- *Property P* $(\mathbf{B})$: *Every uncountable Borel set has a non-empty perfect subset—and so it is equinumerous with* $\mathbb{R}$ (Alexandroff[1916], Hausdorff[1916], difficult)
- *Property Baire* $(\mathbf{B})$: *For every Borel set* $A \subseteq \mathbb{R}^n$*, there is an open set* A^* *such that the symmetric difference* $A \triangle A^* = (A \setminus A^*) \cup (A^* \setminus A)$ *is meager—topologically trivial* (by the Baire Category Theorem)

. . . and there are many more results of this type

Legesgue's seminal error and Suslin's seminal correction

- The wrong theorem in Lebesgue[1905]: *If $A \subseteq (\mathbb{R}^n \times \mathbb{R})$ is Borel, then so is its projection to $\mathbb{R}^n$*, where

$$\mathrm{proj}(A) = \{x \in \mathbb{R}^n \mid (\exists y)A(x,y)\}$$

- Corollary: *Borel equations which have unique solutions have Borel solutions*, i.e., *For every Borel function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,*

$$(\forall x)(\exists! y)[f(x,y) = 0] \implies (\exists \text{ Borel } g : \mathbb{R}^n \to \mathbb{R})(\forall x)[f(x,g(x)) = 0]$$

- The error was noticed in 1917 by the 23-year old Mikhail Suslin (a student of Luzin, in Moscow) who first set
- Def. A set $A \subseteq \mathbb{R}^n$ is analytic if $A = \mathrm{proj}(B)$ for some Borel $B \subseteq \mathbb{R}^{n+1}$ and then proved the Corollary above and (among many other things):
  - *There exist analytic sets which are not Borel*
  - (Suslin's Thm): $\boxed{A \in \mathbf{B}(\mathbb{R}^n) \iff \text{both } A \text{ and } \mathbb{R}^n \setminus A \text{ are analytic}}$
  - *Every uncountable analytic set has a nonempty perfect subset*

The projective hierarchy (Luzin[1925], Sierpinski[1925])

- Def. $\utilde{\boldsymbol{\Sigma}}^1_1(X)$ = the analytic subsets of $X = \mathbb{R}^n$; set by induction,

$$\utilde{\boldsymbol{\Pi}}^1_n(X) = C(\utilde{\boldsymbol{\Sigma}}^1_n(X)) = \{X \setminus A \mid A \in \utilde{\boldsymbol{\Sigma}}^1_n(X)\},$$
$$\utilde{\boldsymbol{\Sigma}}^1_{n+1}(X) = \mathrm{proj}(\utilde{\boldsymbol{\Pi}}^1_n(X \times \mathbb{R})) = \{\mathrm{proj}(A) \mid A \in \utilde{\boldsymbol{\Pi}}^1_n(X \times \mathbb{R})\}$$

and also $\utilde{\boldsymbol{\Delta}}^1_n(X) = \utilde{\boldsymbol{\Sigma}}^1_n(X) \cap \utilde{\boldsymbol{\Pi}}^1_n(X)$.

▶ $A \subseteq X$ is projective if it belongs to one of these families of sets, and

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$$\begin{array}{ccccccccc}
 & & \utilde{\boldsymbol{\Sigma}}^1_1 & & & & \boxed{\utilde{\boldsymbol{\Sigma}}^1_2} & & \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
\mathbf{B} = \utilde{\boldsymbol{\Delta}}^1_1 & & & & \utilde{\boldsymbol{\Delta}}^1_2 & & & & \utilde{\boldsymbol{\Delta}}^1_3 \quad \cdots \\
 & \subsetneq & & \subsetneq & & \subsetneq & & \subsetneq & \\
 & & \utilde{\boldsymbol{\Pi}}^1_1 & & & & \utilde{\boldsymbol{\Pi}}^1_2 & & \cdots
\end{array}$$

The Uniformization Theorem

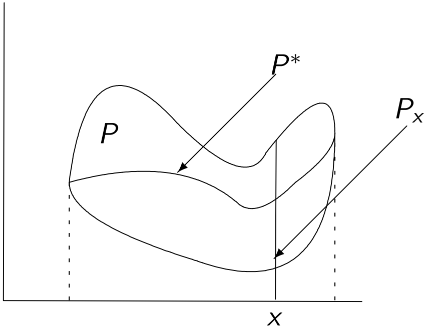


Figure: P^* uniformizes P.

- The Uniformization Theorem (Kondo[1938]). *Every* $P \in \underset{\sim}{\boldsymbol{\Sigma}}^1_2(X \times \mathbb{R})$ *can be uniformized by some* $P^* \in \underset{\sim}{\boldsymbol{\Sigma}}^1_2(X \times \mathbb{R})$*, i.e.,*

$$P^* \subseteq P \;\&\; (\forall x)[(\exists y)P(x,y) \implies (\exists! y)P^*(x,y)]$$

Interlude: Gödel and Cohen

ZFC : *Zermelo-Fraenkel set theory with the Axiom of Choice*
CH: *Every uncountable* $A \subseteq \mathbb{R}$ *is equinumerous with* $\mathbb{R}$

- *The following propositions are consistent with* ZFC (Gödel[1938] using the universe L of constructible sets)
  (1) CH;
  (2) *some uncountable* $\underset{\sim}{\mathbf{\Sigma}}^1_2$ *set* $A \subset \mathbb{R}$ *is not Lebesgue measurable; does not have a non-empty perfect subset; and does not have the property of Baire*
  (3) *For* $k \geq 2$*, every* $\underset{\sim}{\mathbf{\Sigma}}^1_k$ *set* $A \subseteq \mathbb{R}^{n+1}$ *can be uniformized by a* $\underset{\sim}{\mathbf{\Sigma}}^1_k$ *set* $A^* \subseteq \mathbb{R}^{n+1}$
- CH *is independent of* ZFC (Cohen[1963], using forcing)

• Following Gödel[1938] and Cohen[1963], an immense body of consistency and independence results about ZFC has been created, showing in particular that *no interesting property of* $\underset{\sim}{\mathbf{\Sigma}}^1_k$ *with* $k \geq 3$ *can be decided in* ZFC

Arithmetical relations on $\mathbb{N} = \{0, 1, \ldots\}$

- Def (Kleene[1943]). A relation $P \subseteq \mathbb{N}^n$ is arithmetical if

$$P(x) \iff (\exists y_1)(\forall y_2)(\exists y_3) \cdots (Q_n y_n) R(x, y_1, \ldots, y_n)$$

  where $R(x, y_1, \ldots, y_n)$ is recursive (Turing computable), e.g.,

$$R(x),\ (\exists y_1) R(x, y_1),\ (\exists y_1)(\forall y_2) R(x, y_1, y_2), \ldots$$

- These fall into a hierarchy which measures the complexity (degree of undecidability) of all relations on $\mathbb{N}$ which are first-order definable in the standard model $\mathbf{N} = (\mathbb{N}, 0, 1, +, \cdot)$ of Peano arithmetic
- The "simplest" non-arithmetical relation on $\mathbb{N}$ is the truth set of $\mathbf{N}$,

$$\text{Truth}(\mathbf{N}) = \Big\{ e \in \mathbb{N} \mid e \text{ is the code (Gödel number) of a sentence } \theta \text{ which is true in } \mathbf{N} \Big\}$$

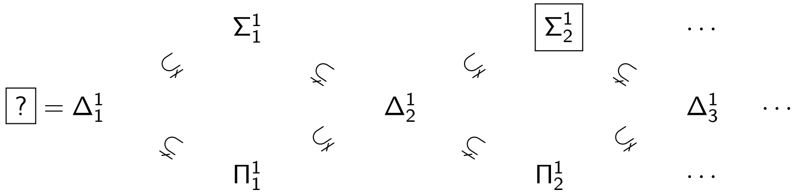
- With three articles published in 1955, Kleene initiated a deep study of the relations on $\mathbb{N}$ which are 2^{nd}-order definable in $\mathbf{N}$

The hierarchy of 2^{nd}-order definable relations on $\mathbb{N}$

- $\mathcal{N} =$ the set of all infinite sequences $\alpha : \mathbb{N} \to \mathbb{N}$ (Baire space) and
  $\boxed{\mathbf{N}^{(2)} = (\mathbb{N}, \mathcal{N}, 0, 1, +, \cdot, (s, \alpha) \mapsto \alpha(s))} =$ 2^{nd}-order arithmetic,
- Let $X = X_1 \times \cdots \times X_n$ where each X_i is $\mathbb{N}$ or $\mathcal{N}$, e.g., $\mathbb{N}, \mathcal{N} \times \mathbb{N}$, etc.
- Def. A set $A \subseteq X$ is Σ^1_1 if $A = \mathrm{proj}(B)$ for some arithmetical $B \subseteq X \times \mathcal{N}$ (definable without quantifiers over $\mathcal{N}$); and then, by induction on n,

$$\Pi^1_n(X) = C(\Sigma^1_n(X)) = \{X \setminus A \mid A \in \Sigma^1_n(X)\},$$
$$\Sigma^1_{n+1}(X) = \mathrm{proj}(\Pi^1_n(X \times \mathcal{N})) = \{\mathrm{proj}(A) \mid A \in \Pi^1_n(X \times \mathcal{N})\}$$

and also $\Delta^1_n(X) = \Sigma^1_n(X) \cap \Pi^1_n(X)$.



The hyperarithmetical sets and Kleene's Theorem

- Def. A coded family of subsets of $\mathbb{N}$ is a pair $(\pi, \mathcal{S})$ such that
  (C1) $\mathcal{S} \subset \mathcal{P}(\mathbb{N})$
  (C2) The coding π is a surjection $\pi : I \twoheadrightarrow \mathcal{S}$ of some $I \subseteq \mathbb{N}$ onto $\mathcal{S}$
- $A \in \mathcal{S}$ with code $i \iff [i \in I \ \& \ \pi(i) = A]$
- *Notation*: $\varphi_e : \mathbb{N} \rightharpoonup \mathbb{N}$ is the recursive partial function with code e
- Def. An effective σ-field in $\mathbb{N}$ is a coded family $(\pi, \mathcal{S})$ such that for suitable recursive partial functions c_1, c_2, c_3,
  (H1) c_1 is total, and every singleton $\{n\}$ is in $\mathcal{S}$ with code $c_1(n)$;
  (H2) if φ_e is total and for every i, $\varphi_e(i)$ is a code of some $A_i \in \mathcal{S}$, then $c_2(e)\downarrow$ and is a code of $\bigcup_{i \in \mathbb{N}} A_i$;
  (H3) if φ_e is total and for every i, $\varphi_e(i)$ is a code of some $A_i \in \mathcal{S}$, then $c_2(e)\downarrow$ and is a code of $\bigcap_{i \in \mathbb{N}} A_i$

► (Essentially Kleene[1955]). *There is an effective σ-field $(\pi_{\text{HYP}}, \text{HYP})$ such that for every effective σ-field $(\pi, \mathcal{S})$,* $\boxed{\text{HYP} \subseteq \mathcal{S}}$

► (Kleene's Theorem, 1955). $\boxed{\Delta^1_1 = \text{HYP}}$

Effective Descriptive Set Theory

$$\begin{array}{ccccccccc}
 & & \utilde{\mathbf{\Sigma}}^1_1 & & & & \boxed{\utilde{\mathbf{\Sigma}}^1_2} & & \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
\boxed{\mathbf{B}} = \utilde{\mathbf{\Delta}}^1_1 & & & & \utilde{\mathbf{\Delta}}^1_2 & & & & \utilde{\mathbf{\Delta}}^1_3 \quad \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
(\mathbb{R}^n) & & \utilde{\mathbf{\Pi}}^1_1 & & & & \utilde{\mathbf{\Pi}}^1_2 & & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
 & & \Sigma^1_1 & & & & \boxed{\Sigma^1_2} & & \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
\boxed{\mathsf{HYP}} = \Delta^1_1 & & & & \Delta^1_2 & & & & \Delta^1_3 \quad \cdots \\
 & \subsetneq & & \supsetneq & & \subsetneq & & \supsetneq & \\
(\mathbb{N}^n \times \mathcal{N}^k) & & \Pi^1_1 & & & & \Pi^1_2 & & \cdots
\end{array}$$

- First conceived as analogies (Addison, Mostowski, 1959), these two theories merged in Effective DST, on recursive Polish spaces
- Codings—mostly in $\mathcal{N}$; structural results about families of sets in addition to regularity results about sets; use of effective methods to prove "classical" results (about projective sets)

Going beyond ZFC: MC

- ZFC does not answer many natural questions about sets, so we can:

  (1): Treat it like any other axiomatic theory, e.g., the *theory of rings*, and study various models of it with interesting properties,or
  (2): look for strengthening ZFC by adding axioms which are plausibly true of the universe of sets as we understand it

- The most promising candidates for (2) were large cardinal axioms, which postulate the existence of sets which are

- Def. MC : *there exists a measurable cardinal*

▶ (Scott[1961]). *If* MC, *then there exists a set which is not constructible*

▶ (Silver[1973], Rowbottom[1971]). *If* MC, *then there are only countably many constructible real numbers*

▶ (Solovay[1969]) *If* MC, *then every* $\utilde{\boldsymbol{\Sigma}}^1_2$ *set* $A \subseteq \mathbb{R}$ *is Lebesgue measurable, has the property of Baire, and either it is countable or has a non-empty perfect subset*

- MC does not prove anything important about sets in $\utilde{\boldsymbol{\Sigma}}^1_k$ for $k \geq 3$

Games (patient) people play

- Blackwell[1967] gives a new proof of a basic property of analytic sets using a simple result about games, and many who learn of it see immediately that it leads to a powerful new axiom which answers many open problems about projective sets
- Given a set $A \subseteq \mathcal{N}$, consider the infinite game of perfect information $G(A)$ which is played on $\mathbb{N}$ as follows:

$$\begin{array}{lccccccccc} \text{I}: & a_0 & & a_2 & & a_4 & & \cdots & \\ \text{II}: & & a_1 & & a_3 & & a_5 & & \cdots \end{array}$$

- At the end of time, a sequence $\alpha = (a_0, a_1, a_2, \ldots)$ has been defined
- I wins if $\alpha \in A$, otherwise II wins
- Def. A is determined if either player I or player II has a winning strategy

▶ (Gayle-Stewart[1953]). *Every open set is determined*

▶ (Gayle-Stewart[1953], using AC). *Some $A \subset \mathcal{N}$ is not determined*

Projective Determinacy

- Axiom of Projective Determinacy (PD) *Every projective set is determined*

- PD is not known to be inconsistent with ZFC

▶ (Mycielski-Steinhaus[1962], Mycielski[1964,1966]) (PD) *Every projective set is Lebesgue measurable and has property P and the property of Baire*

- PD *solves a great number of problems for projective sets which cannot be decided in ZFC or ZFC + MC* (Addison, Martin, ynm), Harrington, Hjorth, Kechris, Louveau, Solovay, Steel, Woodin, . . . 1967 —)

▶ (PD, $n \geq 1$) (ynm[1971] *Every* $\underset{\sim}{\boldsymbol{\Sigma}}^1_{2n}$ *set can be uniformized by a* $\underset{\sim}{\boldsymbol{\Sigma}}^1_{2n}$ *set*; and *every* $\underset{\sim}{\boldsymbol{\Pi}}^1_{2n+1}$ *set can be uniformized by a* $\underset{\sim}{\boldsymbol{\Pi}}^1_{2n+1}$ *set*

- The structure of the projective hierarchy is very different under PD from that in Gödel's constructible universe L

Proving PD

- Proofs of determinacy are very difficult, and there are no intrinsic arguments why it should be true; in the beginning, the only evidence for it was extrinsic, i.e., it came from its consequences

- (Martin[1975]) *Every Borel set is determined*

- (Martin-Steel[1988]) PD *can be proved from suitable* (fairly weak) *large cardinal axioms* (Woodin cardinals)
  (This was substantially strengthened by Woodin)

EPILOGUE

- Starting with Harrington-Kechris-Louveau[1990], the most important results in DST have been applications to many areas of mathematics