Intuitionism and effective descriptive set theory

Joan R. Moschovakis and Yiannis N. Moschovakis
Occidental College and University of California, Los Angeles

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The motivation for this paper and this talk

The Law of Excluded Middle (LEM): $\boxed{P \vee \neg P}$

The Axiom of Choice (AC) (put center stage in Zermelo [1904, 1908])

$$\boxed{(\forall x \in A)(\exists y \in B)P(x, y) \implies (\exists f : A \to B)(\forall x \in A)P(x, f(x))}$$

The "founding documents" for Intuitionism and Descriptive Set Theory:

- Brouwer [1907] (his Thesis), Brouwer [1908] (rejection of LEM)
- (Borel, Baire), Lebesgue [1905], Lusin [1917], Suslin [1917]
- The rejection of (unrestricted) AC and "constructive talk" by Borel and company suggests a strong connection between their ideas and Brouwer's and they have been called "semi-" and "pre-" intuitionists

★ In fact, *there are no significant influences in either direction or similarity in their aims and mathematical results* (Michel [2008])

★ . . . but there is a robust connection between intuitionistic analysis and effective DST developed after Kleene's work in the 1950s

Logic vs. mathematics

- In today's (classical) mathematics there is a sharp separation between
  – logic, which is (single- or many-sorted) first-order logic, and
  – mathematical assumptions, typically some fragment of ZFC
- In early 20th century there was no such separation (cf. Frege):
  – Brouwer [1907]: *logic depends upon mathematics*

  – Borel and company mainly worry about AC which they understand primarily as a logical principle, sometimes seeming to consider the possibility of making infinitely many choices *in the course of a proof*

  Lebesgue to Borel in the *Five letters* [1905]:
  To make an infinity of choices cannot be to write down or to name the elements chosen, one by one; life is too short.

★ *In DST,* LEM *was used freely from the get-go and the constructive bent shows only in the choice of mathematical assumptions*

★ *We keep logic strictly separate from mathematical axioms*

Constructions vs. definitions

- In constructive mathematics (of all flavors) a proof of $(\exists x)P(x)$ is expected to yield a construction of some object x which can be proved to have the property P, whatever "constructions" are—and they are often taken (explicitly or implicitly) to be primitives

- Lebesgue [1905]

  – doubts the general conception of a function $f : \mathbb{R}^n \to \mathbb{R}$ as an *arbitrary correspondence*;

  – notes that mathematicians are most interested in functions which are analytically representable (*definable*);

  – and argues that, if there are real functions which are not analytically representable, then

  it is important to study the common properties [of those which are]

- There was vigorous discussion on what definitions are and whether they are necessary and/or sufficient for existence

★ Today: *DST is the study of definability over the continuum*

What this paper and this talk are about

• Our main (very limited) aim: to explain and apply the connection between intuitionistic analysis and effective descriptive set theory
Briefly, we

- outline the basic notions and methods of effective DST, and
- derive a few, very basic facts about Borel and analytic sets on Baire space $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$ using
  - intuitionistic logic,
  - standard, familiar definitions, and
  - the mathematical axioms in a conservative extension $\mathbf{B}^*$ of the Kleene Basic System $\mathbf{B}$ for analysis which are intuitionistically acceptable and classically sound

• The relation between constructivity and effective definability:

– Constructive proofs yield effective results (as they should); but also
– effective methods can be used to give constructive proofs

• Intuitionistic descriptive set theory developed by Wim Veldman and his collaborators is obviously relevant (but not in this talk)

The intuitionistic systems **B** and $\mathbf{B}^*$

- Kleene formalized intuitionistic analysis in a two-sorted first-order language with variables

$$i, j, k, l, \ldots \text{ over } \mathbb{N} \text{ and } \alpha, \beta, \ldots \text{ over } \mathcal{N} = (\mathbb{N} \to \mathbb{N})$$

(B1) The Peano axioms, with induction for all formulas

(B2) AC_1^0, the Countable Axiom of Choice over $\mathcal{N}$

$$(\forall i)(\exists \alpha) R(i, \alpha) \implies (\exists \delta)(\forall i) R(i, \lambda t \delta(\langle i, t \rangle))$$

with $\langle i, t \rangle$ a recursive code of the pair (i, t), e.g., $2^{i+1} \cdot 3^{t+1}$

(B3) Proof by (backward or) Bar Induction on a grounded tree on $\mathbb{N}$

- **B** = **(B1)** - **(B3)** is classically sound and intuitionistically acceptable
- $\mathbf{B}^*$ is a *conservative extension* of **B** with (pointset) variables over every product space $X = X_1 \times \cdots \times X_n$ with $X_i = \mathbb{N}$ or $X_i = \mathcal{N}$

$\Rightarrow$ *Recursion theory on these spaces* can be developed in $\mathbf{B}^*$

- Most of *classical analysis* can be developed in $\mathbf{B}^* + \text{LEM}$

The Kleene Calculus for (partial) continuity and recursion

- We assume recursive codings of tuples from $\mathbb{N}$, $\langle k_0, \ldots, k_{t-1} \rangle$

$$\boxed{\{\varepsilon\}^{\mathcal{N},\mathbb{N}}(\alpha) = w \iff (\exists t)[(\forall i < t)\varepsilon(\overline{\alpha}(i)) = 0 \ \&\ \varepsilon(\overline{\alpha}(t)) = w + 1]}$$

with $\overline{\alpha}(i) = \langle \alpha(0), \ldots, \alpha(i \dot{-} 1) \rangle$ This is a partial function on $\mathcal{N}^2$ to $\mathbb{N}$

- Extend naturally to partial functions $\boxed{(\varepsilon, x) \mapsto \{\varepsilon\}^{X,W}(x) \in W}$

★ A partial function $f : X \rightharpoonup W$ is continuous with code $\varepsilon \in \mathcal{N}$, if

$$(*) \qquad f(x)\downarrow \implies f(x) = \{\varepsilon\}^{X,W}(x)$$

and recursive if $(*)$ holds with a recursive $\varepsilon \in \mathcal{N}$

⇒ *The partial functions $(\varepsilon, x) \mapsto \{\varepsilon\}^{X,W}(x)$ are recursive*

⇒ *For suitable recursive (total) $S = S^{Y,X,W} : \mathcal{N} \times Y \to \mathcal{N}$.*

$$\{\varepsilon\}(y, x) = \{S(\varepsilon, y)\}(x) \quad (\rightharpoonup W)$$

⇒ 2nd Recursion Theorem *For every recursive $f : \mathcal{N} \times X \rightharpoonup W$, there is a recursive $\widetilde{\varepsilon}$ such that* $\boxed{f(\widetilde{\varepsilon}, x)\downarrow \implies f(\widetilde{\varepsilon}, x) = \{\widetilde{\varepsilon}\}(x)}$

Coded sets and uniformity

- Coded (in $\mathcal{N}$) set: a pair $\mathbf{A} = (A, c^{\mathbf{A}})$ of a set and a partial surjection

$$c^{\mathbf{A}} : \mathcal{N} \twoheadrightarrow A;$$

α codes $a \in A$ if $c^{\mathbf{A}}(\alpha) = a$; $C^{\mathbf{A}} = \{\alpha \mid c^{\mathbf{A}}(\alpha)\downarrow\}$ (the code-set of $\mathbf{A}$)

★ For any two coded sets $\mathbf{A}, \mathbf{B}$, a $\forall$-$\exists$ proposition

$$(\forall P \in \mathbf{A})(\exists Q \in \mathbf{B}) R(P, Q)$$

holds uniformly, if for some recursive partial function $\mathbf{u} : \mathcal{N} \rightharpoonup \mathcal{N}$,

$$\alpha \in C^{\mathbf{A}} \implies \Big(\mathbf{u}(\alpha)\downarrow \ \& \ \mathbf{u}(\alpha) \in C^{\mathbf{B}} \ \& \ R(c^{\mathbf{A}}(\alpha), c^{\mathbf{B}}(\mathbf{u}(\alpha)))\Big) \quad (\alpha \in \mathcal{N})$$

- *If $f : \mathcal{N} \rightharpoonup \mathcal{N}$ is recursive and $f(\alpha)\downarrow$, then $f(\alpha)$ is recursive in α*

★ *The theory of coded sets provides an "axiomatization" of the theory of definability (as used in DST) which sidesteps the need to specify ahead of time what it means to define a mathematical object*

The coded pointclasses of open and closed sets, $\underset{\sim}{\mathbf{\Sigma}}^0_1$, $\underset{\sim}{\mathbf{\Pi}}^0_1$

- A coded pointclass $\underset{\sim}{\mathbf{\Gamma}}$ assigns to each X a coded subset $\underset{\sim}{\mathbf{\Gamma}} \restriction X$ of $\mathcal{P}(X)$
- A pointset $P \subseteq X$ is open ($\underset{\sim}{\mathbf{\Sigma}}^0_1$) with code ε if

$$x \in P \iff \{\varepsilon\}^{X,\mathbb{N}}(x)\downarrow$$

$\Rightarrow$ *$\underset{\sim}{\mathbf{\Sigma}}^0_1$ is uniformly closed under (total) continuous substitutions*

$$R(x) \iff P(f_1(x), \dots, f_n(x)),$$

conjunction &, disjunction $\vee$, bounded number quantification $\exists^{\leq}, \forall^{\leq}$ and existential number quantification $\exists^{\mathbb{N}}$

- $Q \subseteq X$ is closed ($\underset{\sim}{\mathbf{\Pi}}^0_1$) with code ε if

$$x \in Q \iff \{\varepsilon\}^{X,\mathbb{N}}(x)\uparrow$$

$\Rightarrow$ *$\underset{\sim}{\mathbf{\Pi}}^0_1$ is uniformly closed under continuous substitutions, &, $\forall^{\leq}$ and $\forall^{\mathbb{N}}$*

$\Rightarrow$ $\mathbf{B}^* + \text{LEM} \vdash (\forall P \subseteq X)[P \in \underset{\sim}{\mathbf{\Sigma}}^0_1 \iff (X \setminus P) \in \underset{\sim}{\mathbf{\Pi}}^0_1]$

$\Rightarrow$ $\boxed{\mathbf{B}^* \nvdash (\forall P \subseteq \mathbb{N})[P \in \underset{\sim}{\mathbf{\Sigma}}^0_1 \iff (\mathbb{N} \setminus P) \in \underset{\sim}{\mathbf{\Pi}}^0_1]}$

Parametrized pointclasses

- The effective part of a coded pointclass $\underset{\sim}{\mathbf{\Gamma}}$ is defined by

$$P \in \Gamma \iff P \in \underset{\sim}{\mathbf{\Gamma}} \text{ with a recursive code} \quad (P \subseteq X)$$

e.g., Σ^0_1 is the pointclass of effectively open (semirecursive) pointsets

- $\underset{\sim}{\mathbf{\Gamma}}$ is parametrized if it is uniformly closed under continuous substitutions and for each X there is set $G^X \subseteq \mathcal{N} \times X$ in Γ which is universal for $\underset{\sim}{\mathbf{\Gamma}} \restriction X$, i.e., such that

(U1) $G^X \in \Gamma$, the effective part of $\underset{\sim}{\mathbf{\Gamma}}$
(U2) For every $P \subseteq X$,

$$P \in \underset{\sim}{\mathbf{\Gamma}} \text{ with code } \varepsilon \iff P = G^X_\varepsilon = \{x \in X \mid G^X(\varepsilon, x)\}$$

(U3) for every Y and every $Q \subseteq Y \times X$ in Γ, there is a recursive $S^Q : Y \to \mathcal{N}$ such that $Q(y,x) \iff G^X(S^Q(y), x)$

★ *If $\underset{\sim}{\mathbf{\Gamma}}$ is parametrized and closed under an operation Φ, then $\underset{\sim}{\mathbf{\Gamma}}$ is uniformly closed under Φ*, e.g., $\exists^{\mathbb{N}} \underset{\sim}{\mathbf{\Gamma}} \subseteq \underset{\sim}{\mathbf{\Gamma}} \implies \bigcup_i G^X_{\{\varepsilon\}(i)} = G^X_{\mathbf{u}(\varepsilon)}$

$\Rightarrow$ $\underset{\sim}{\mathbf{\Sigma}}^0_1$ *and* $\underset{\sim}{\mathbf{\Pi}}^0_1$ *are parametrized*

Finite order Borel, analytic, co-analytic, projective pointsets

- By induction on k starting with $\underset{\sim}{\mathbf{\Sigma}}^0_1, \underset{\sim}{\mathbf{\Pi}}^0_1$:

$$\underset{\sim}{\mathbf{\Sigma}}^0_{k+1} = \exists^{\mathbb{N}} \underset{\sim}{\mathbf{\Pi}}^0_k, \quad \underset{\sim}{\mathbf{\Pi}}^0_{k+1} = \forall^{\mathbb{N}} \underset{\sim}{\mathbf{\Sigma}}^0_k$$

- The analytic and co-analytic pointsets

$$\boxed{\underset{\sim}{\mathbf{\Sigma}}^1_1 = \exists^{\mathcal{N}} \underset{\sim}{\mathbf{\Pi}}^0_1}, \quad \boxed{\underset{\sim}{\mathbf{\Pi}}^1_1 = \forall^{\mathcal{N}} \underset{\sim}{\mathbf{\Sigma}}^0_1}$$

and, inductively, the projective pointsets

$$\underset{\sim}{\mathbf{\Sigma}}^1_{k+1} = \exists^{\mathcal{N}} \underset{\sim}{\mathbf{\Pi}}^1_k, \quad \underset{\sim}{\mathbf{\Pi}}^1_{k+1} = \forall^{\mathcal{N}} \underset{\sim}{\mathbf{\Sigma}}^1_k$$

★ *These are all coded, parametrized pointclasses with the expected (uniform) closure properties*

⇒ *The pointclass $\underset{\sim}{\mathbf{\Sigma}}^1_1$ of analytic sets is uniformly closed under $\&, \vee$, countable unions and intersections and $\exists^{\mathcal{N}}$*

⇒ *The pointclass $\underset{\sim}{\mathbf{\Pi}}^1_1$ of co-analytic sets is uniformly closed under $\&$, countable intersections and $\forall^{\mathcal{N}}$*

⋆ (Positive) inductive definitions and the Borel sets

• $\mathfrak{B}$ should be the least pointclass which contains $\underset{\sim}{\mathbf{\Sigma}}^0_1 \cup \underset{\sim}{\mathbf{\Pi}}^0_1$ and is closed under countable unions and intersections. Its (natural) inductive definition is justified in $\mathbf{B}^*$ using the following theorem:

⇒ *For any two continuous functions $g_0 : \mathcal{N} \to \mathbb{N}$ and $g_1 : \mathcal{N} \times \mathbb{N} \to \mathcal{N}$, there is a unique set $I \subseteq \mathcal{N}$ which satisfies:*

(I1) $(\forall\alpha)\Big(\Big[g_0(\alpha) = 0 \vee [g_0(\alpha) \neq 0 \ \&\ (\forall i)[g_1(\alpha, i) \in I]]\Big] \implies \alpha \in I\Big)$

(I2) *Proof by induction on I: if $P \subseteq \mathcal{N}$ satisfies* (I1) *with $I := P$, i.e.,*

$$(\forall\alpha)\Big(\Big[g_0(\alpha) = 0 \vee [g_0(\alpha) \neq 0 \ \&\ (\forall i)[g_1(\alpha, i) \in P]]\Big] \implies \alpha \in P\Big),$$

then $I \subseteq P$.

Moreover, I is $\underset{\sim}{\mathbf{\Pi}}^1_1$ and a $\underset{\sim}{\mathbf{\Pi}}^1_1$-code for it can be recursively computed from codes of g_0 and g_1, and so if these are recursive, then I is Π^1_1

• The proof uses bar induction

The coded pointclass $\mathfrak{B}$ of Borel sets

⇒ *For every space X and every $G^X \subseteq \mathcal{N} \times X$ which is universal for $\utilde{\mathbf{\Sigma}}^1_1$ at X, there is a recursive partial function $\mathbf{u}^X : \mathcal{N} \rightharpoonup \mathcal{N}$ such that*:

(1) *The set* $\mathrm{BC} = \{\mathbf{u}^X(\alpha)\downarrow\}$ *is* Π^1_1 (and independent of X)

(2) *The pointclass of* Borel sets *defined by*

$$P \in \mathfrak{B} \text{ with code } \alpha \in \mathrm{BC} \iff P = \{x \mid G^X(\mathbf{u}^X(\alpha), x)\} \quad (P \subseteq X)$$

contains (uniformly) $\utilde{\mathbf{\Sigma}}^0_1 \cup \utilde{\mathbf{\Pi}}^0_1$, *it is uniformly closed under continuous substitutions, countable unions and countable intersections, and it is the least coded pointclass with these properties*

⇒ *Every Borel set is uniformly analytic*

⇒ (LEM) $\mathfrak{B}$ *is uniformly closed under negation*

(but this cannot be proved in $\mathbf{B}^*$)

- The simplest proofs of these facts use the 2nd Recursion Theorem

Markov's Principle

(MP) $\qquad (\forall\alpha)\Big(\neg(\forall i)[\alpha(i)=0] \implies (\exists i)[\alpha(i)\neq 0]\Big)$

- True classically and in Russian *constructive* (or *recursive*) analysis
- Rejected by Brouwer and not provable in $\mathbf{B}^*$

★ Theorems of $\mathbf{B}^* + \mathrm{MP}$ not provable in $\mathbf{B}^*$:

⇒ (1) *For all* $P \subseteq X$, $P \in \utilde{\boldsymbol{\Sigma}}^0_1 \iff (X \setminus P) \in \utilde{\boldsymbol{\Pi}}^0_1$

⇒ (2) Hierarchy Theorem for finite order Borel sets:

$$\utilde{\boldsymbol{\Sigma}}^0_1{\restriction}\mathcal{N} \subsetneq \utilde{\boldsymbol{\Sigma}}^0_2{\restriction}\mathcal{N} \subsetneq \utilde{\boldsymbol{\Sigma}}^0_3{\restriction}\mathcal{N} \subsetneq \cdots$$

(Needs only DNS_0: $(\forall t)\neg\neg\varphi(t,\alpha) \implies \neg\neg(\forall t)\varphi(t,\alpha)$, $\varphi(t,\alpha)$ arithmetical)

⇒ (3) Strong analytic separation: *Uniformly, for all analytic* $A, B \subseteq X$, *with* $A \cap B = \emptyset \Leftrightarrow \neg(\exists x)[x \in A \;\&\; x \in B]$,

$$A \cap B = \emptyset \implies (\exists C \subseteq X)[C \in \mathfrak{B} \;\&\; A \subseteq C \;\&\; C \cap B = \emptyset]$$

⇒ (4) Half of the Suslin-Kleene Theorem: *Uniformly, for all* $A \subseteq X$, *if both* A *and its complement* $(X \setminus A)$ *are analytic, then* A *is Borel*

Classically sound semi-constructive theories

• A formally intuitionistic, classically sound theory T in the language of $\mathbf{B}^*$ is semi-constructive if for every $\varphi(x, \beta)$ with no pointset variables and only x and β free,

$$(*) \quad T \vdash (\forall x)(\exists \beta)\varphi(x, \beta)$$

$$\implies T \vdash (\exists \varepsilon)[\mathrm{GR}(\varepsilon) \ \& \ (\forall x)(\exists \beta)[\{\varepsilon\}^{X,\mathcal{N}}(x) = \beta \ \& \ \varphi(x, \beta)]],$$

where $\mathrm{GR}(\varepsilon)$ expresses the relation "ε *is recursive*"

★ *If* $\mathbf{B}^* \subseteq T \subseteq \mathbf{B}^* \cup \{\mathrm{DNS}_0, \mathrm{MP}, \mathrm{DC}_\mathcal{N}\}$, *then* T *is semi-constructive* where $\mathrm{DC}_\mathcal{N} :\equiv (\forall \alpha)(\exists \beta)R(\alpha, \beta) \implies (\exists \delta)(\forall t)R((\delta)_t, (\delta)_{t+1})$ (a consequence of Kleene's formalized q-realizability theory for $\mathbf{B}$)

★ *If* T *is semi-constructive and* $\varphi(x, \beta)$ *as in* $(*)$ *defines (classically) a relation* $P(x, \beta)$, *then*

$$T \vdash (\forall x)(\exists \beta)\varphi(x, \beta) \implies (\forall x)(\exists \beta)P(x, \beta) \text{ holds uniformly}$$

• Question. How much of Descriptive Set Theory can be developed in semi-constructive theories, preferably theories as above for which there is some constructive justification?