Effective descriptive set theory

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Outline

(I) A bit of history (3 slides)
(II) The basic notions (7 slides)
(III) Some characteristic effective results (6 slides)
(IV) HYP isomorphism and reducibility (Gregoriades) (3 slides)

- *Descriptive set theory*, ynm, 1980, Second Edition 2009
- *Classical descriptive set theory as a refinement of effective descriptive set theory*, ynm, 2010
- *Kleene's amazing second recursion theorem*, ynm, 2010
- *Notes on effective descriptive set theory*, ynm and Vassilios Gregoriades, in preparation

(The first three are posted on `www.math.ucla.edu/~ynm`)

The arithmetical hierarchy

- **Kleene [1943]**: The arithmetical hierarchy on subsets of $\mathbb{N}$

recursive $\subsetneq \Sigma^0_1$ (rec. enumerable) $\subsetneq \Sigma^0_2 \subsetneq \cdots$ $(\neg\Sigma = \Pi,\ \Sigma \cap \Pi = \Delta)$

  - Tool for giving easy (semantic) proofs of Gödel's First Incompleteness Theorem, Tarski's Theorem on the arithmetical undefinability of arithmetical truth, etc.

- **Mostowski [1947]**: Reinvents the arithmetical hierarchy, using as a model the classical projective hierarchy on sets of real numbers

$$\text{Borel} \subsetneq \mathbf{\Sigma}^1_1 \text{ (analytic) } \subsetneq \mathbf{\Sigma}^1_2 \subsetneq \cdots \quad (\neg\mathbf{\Sigma} = \mathbf{\Pi},\ \mathbf{\Sigma} \cap \mathbf{\Pi} = \mathbf{\Delta})$$

  - He grounds the analogy on the two basic results

$$\text{Kleene: } \Delta^0_1 = \text{recursive}, \quad \text{Suslin: } \mathbf{\Delta}^1_1 = \text{Borel}$$

  - Mostowski is unaware of Kleene [1943]: the only post 1939 paper he cites is Post [1944]
    (He refers to Kleene [1943] in a Postscript added "in press" saying that it "just became available in Poland")

Mostowski's definition of HYP on $\mathbb{N}$

- **Mostowski [1951]** introduces the hyperarithmetical hierarchy
  - In modern notation, roughly, he defines for each constructive ordinal $\xi < \omega_1^{\mathrm{CK}}$ a universal set for a class P_ξ of subsets of $\mathbb{N}$
  - The analogy now is between HYP and the Borel sets of reals. M. mimics closely Lebesgue's classical definition of $\mathbf{\Sigma}^0_\xi$ sets, replacing countable unions by projection along $\mathbb{N}$ and using effective diagonalization at limit ordinals
  - There are technical difficulties with the effective version. M. does not give detailed proofs and refers to the need for

    "a rather developed technique which we do not wish to presuppose here"

To make the definition precise, one needs effective transfinite recursion on ordinal notations.
This depends on the 2nd Recursion Theorem and was introduced in the literature by Kleene [1938], not cited in this paper.

Most likely, Mostowski is referring to a version of this method

The definition of HYP on $\mathbb{N}$ by Kleene and Davis

- **Kleene 1955a**: HYP = recursive in some H_a, $a \in O$ Crucial case:

$$|a| = |b| + 1 \implies H_a = H_b' \ \text{(the jump of } H_b\text{)}$$

  - Kleene outlines the details needed to make Mostowski's definition rigorous and establishes the relation between the two definitions Roughly, for $\omega \le \xi < \omega_1^{\mathrm{CK}}$,

$$P_\xi = \text{ the class of sets recursively reducible to } H_a \ (|a| = \xi{+}1)$$

  - Kleene credits Martin Davis who introduced essentially the same definition and at the same time in his Thesis
  - He alludes to **Kleene [1955b]** in which he proves the basic result

$$\Delta^1_1 = \text{HYP} : \text{ compare to Suslin's Theorem } \boldsymbol{\Delta}^1_1 = \text{Borel}$$

- **Kleene 1950**: The "analogy" $\Sigma^0_1 \sim \boldsymbol{\Sigma}^1_1$ fails, because *there exist recursively inseparable r.e., sets Does not mention* $\Sigma^0_1 \sim \boldsymbol{\Pi}^1_1$

$$\text{r.e.} = \Sigma^0_1 \sim \Pi^1_1 \quad \text{open} = \boldsymbol{\Sigma}^0_1 \sim \boldsymbol{\Pi}^1_1 \quad \text{HYP} \sim \text{Borel}$$

- John Addison established these analogies firmly

Recursive Polish metric spaces

- A metric space $(\mathcal{X}, d)$ is Polish if it is separable and complete
- A presentation of $\mathcal{X}$ is any pair $(d, \mathbf{r})$ where $\mathbf{r} : \mathbb{N} \to \mathcal{X}$ and $\mathbf{r}[\mathbb{N}] = \{r_0, r_1, \dots\}$ is dense in $\mathcal{X}$
- A presentation $(d, \mathbf{r})$ is recursive if the relations

  $$P^{d,\mathbf{r}}(i,j,k) \iff d(r_i, r_j) \le q_k, \quad Q^{d,\mathbf{r}}(i,j,k) \iff d(r_i, r_j) < q_k,$$

  are recursive, where $q_k = \frac{(k)_0}{(k)_1 + 1}$
- Recursive Polish metric space: $(\mathcal{X}, d, \mathbf{r})$ with recursive $(d, \mathbf{r})$
- $\mathbb{N} = \{0, 1, \dots\}$ as a discrete space;
  the reals $\mathbb{R}$ and Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ with their standard metrics;
  products of recursive Polish metric spaces, etc.
- Every Polish metric space is recursive in some $\varepsilon \in \mathcal{N}$
- Computable space: assume only that $P^{d,\mathbf{r}}$ and $Q^{d,\mathbf{r}}$ are r.e.
  There is a computable metric space which does not admit a recursive presentation

Open and effectively open (Σ^0_1) pointsets

Fix a recursive Polish metric space $(\mathcal{X}, d, \{r_0, r_1, \ldots\})$

- Codes of nbhds: For each $s \in \mathbb{N}$, let

$$N_s = N(\mathcal{X}, s) = \Big\{x \in \mathcal{X} : d(x, r_{(s)_0}) < q_{(s)_1}\Big\}$$

- A pointset $G \subseteq \mathcal{X}$ is open if $G = \bigcup_n N_{\varepsilon(n)} = \bigcup_n N\big(\mathcal{X}, \varepsilon(n)\big)$ with some $\varepsilon \in \mathcal{N}$ Any such ε is a code of G
- G is semirecursive or Σ^0_1 if it has a recursive code

Lemma (Normal Form for Σ^0_1)

$P \subseteq \mathcal{X}$, $Q \subseteq \mathcal{X} \times \mathcal{Y}$ are Σ^0_1 if and only if

$$\begin{aligned} P(x) &\iff (\exists s)[x \in N(\mathcal{X}, s) \ \&\ P^*(s)] \\ Q(x, y) &\iff (\exists s)(\exists t)[x \in N(\mathcal{X}, s) \ \&\ y \in N(\mathcal{Y}, t) \ \&\ Q^*(s, t)] \end{aligned}$$

with semirecursive P^*, Q^* relations on $\mathbb{N}$

Recursive Polish spaces

Def A topological space $(\mathcal{X}, \mathcal{T})$ is Polish if there is a d such that $(\mathcal{X}, d)$ is a Polish metric space which induces $\mathcal{T}$

Def A recursive Polish space is a set $\mathcal{X}$ together with a family $\mathcal{R} = \mathcal{R}(\mathcal{X})$ of subsets of $\mathbb{N} \times \mathcal{X}$ such that for some $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and $\mathbf{r} : \mathbb{N} \to \mathcal{X}$ the following conditions hold:

(RP1) $(\mathcal{X}, d, \mathbf{r})$ is a recursive Polish metric space, and

(RP2) the frame $\mathcal{R}$ of $\mathcal{X}$ is the family of semirecursive subsets of $\mathbb{N} \times \mathcal{X}$

- Every recursive Polish metric space $(\mathcal{X}, d, \mathbf{r})$ determines a recursive Polish space $(\mathcal{X}, \mathcal{R}(\mathcal{X}))$ by setting

$$\mathcal{R}(\mathcal{X}) = \text{the family of semirecursive subsets of } \mathbb{N} \times \mathcal{X}$$

- If (RP1), (RP2) hold: $(d, \mathbf{r})$ is a compatible pair for $\mathcal{X}$
- Every Polish space is recursive in some $\varepsilon \in \mathcal{N}$
- Every (naturally defined) computable Polish space is recursive

The analogies between the classical and the effective theory

- A pointset is any subset $P \subseteq \mathcal{X}$ of a recursive Polish space, formally a pair $(P, \mathcal{X})$
- A pointclass is any collection Γ of pointsets, e.g.,

$$\Sigma^0_1 = \text{the semirecursive pointsets}, \quad \Pi^0_1 = \{\mathcal{X} \setminus P : P \in \Sigma^0_1\},$$
$$\boldsymbol{\Sigma}^0_1 = \text{the open pointsets} \supseteq \Sigma^0_1, \quad \boldsymbol{\Pi}^0_1 = \text{all closed pointsets} \supseteq \Pi^0_1$$

- The arithmetical and analytical pointclasses

$$\Pi^0_k = \neg\Sigma^0_k, \;\; \Sigma^0_{k+1} = \exists^{\mathbb{N}}\Pi^0_k, \quad \Delta^0_k = \Sigma^0_k \cap \Pi^0_k$$
$$\Sigma^1_1 = \exists^{\mathcal{N}}\Pi^0_1, \quad \Pi^1_k = \neg\Sigma^0_k, \;\; \Sigma^1_{k+1} = \exists^{\mathcal{N}}\Pi^1_k, \;\; \Delta^1_k = \Sigma^1_k \cap \Pi^1_k$$

- The finite Borel and projective pointclasses

$$\boldsymbol{\Pi}^0_k = \neg\boldsymbol{\Sigma}^0_k, \;\; \boldsymbol{\Sigma}^0_{k+1} = \exists^{\mathbb{N}}\boldsymbol{\Pi}^0_k, \;\; \boldsymbol{\Delta}^0_k = \boldsymbol{\Sigma}^0_k \cap \boldsymbol{\Pi}^0_k$$
$$\boldsymbol{\Sigma}^1_1 = \exists^{\mathcal{N}}\boldsymbol{\Pi}^0_1, \quad \boldsymbol{\Pi}^1_k = \neg\boldsymbol{\Sigma}^1_k, \;\; \boldsymbol{\Sigma}^1_{k+1} = \exists^{\mathcal{N}}\boldsymbol{\Pi}^1_k, \;\; \boldsymbol{\Delta}^1_k = \boldsymbol{\Sigma}^1_k \cap \boldsymbol{\Pi}^1_k$$

- Missing analogy: Hyperarithmetical $\sim$ Borel

From the classical to the effective – coding

- A coding (in $\mathcal{N}$) of a set $\mathcal{A}$ is any surjection $\pi : C \twoheadrightarrow \mathcal{A}$, $C \subseteq \mathcal{N}$
- The pointclasses $\boldsymbol{\Sigma}^i_k, \boldsymbol{\Pi}^i_k, \boldsymbol{\Delta}^i_k$ are all (naturally) coded, and

$$G \in \Sigma^i_k \iff G \in \boldsymbol{\Sigma}^i_k \text{ with a recursive code}$$

- The Borel pointclasses: starting with $\boldsymbol{\Sigma}^0_1 =$ the open sets, set

$$A \in \boldsymbol{\Sigma}^0_\xi \iff A = \bigcup_{i\in\omega}(\mathcal{X} \setminus A_i)$$

$$\text{with each } A_i \text{ in } \bigcup_{\eta<\xi} \boldsymbol{\Sigma}^0_\eta \quad (A \subseteq \mathcal{X}, \xi > 1)$$

$$\mathbf{B} = \bigcup_{\xi<\aleph_1} \boldsymbol{\Sigma}^0_\xi = \text{the Borel sets}$$

- $\alpha^*(i) = \alpha(i+1); \quad (\beta)_i(j) = \beta(\langle i, j\rangle)$
- Borel codes: starting with $\mathsf{K}_1 = \{\alpha \in \mathcal{N} : \alpha(0) = 0\}$, set for $\xi > 1$

$$\mathsf{K}_\xi = \mathsf{K}_1 \cup \{\alpha : \alpha(0) = 1 \ \& \ (\forall i)[(\alpha^*)_i \in \bigcup_{\eta<\xi} \mathsf{K}_\eta]\}, \quad \mathsf{K} = \bigcup_{1\le\xi<\aleph_1} \mathsf{K}_\xi$$

$$\alpha(0) = 0 : B^{\mathcal{X}}_\alpha = \bigcup_i N(\mathcal{X}, \alpha^*(i)), \quad \alpha(0) > 0 : B^{\mathcal{X}}_\alpha = \bigcup_i (\mathcal{X} \setminus B^{\mathcal{X}}_{(\alpha^*)_i})$$

Hyperarithmetical as effective Borel

For $A \subseteq \mathcal{X}$:

- Def α is a K_ξ-code of A : $\alpha \in \mathsf{K}_\xi$ & $A = B^{\mathcal{X}}_\alpha$
  - $A \in \boldsymbol{\Sigma}^0_\xi \iff A$ has a K_ξ-code
- Def α is a Borel code of A : $\alpha \in \mathsf{K}$ & $A = B^{\mathcal{X}}_\alpha$
  - A is Borel $\iff A$ has a Borel code
- Def A is HYP $\iff A$ has a recursive Borel code
- Def $f : \mathcal{X} \to \mathcal{Y}$ is HYP if $\{(x, s) : f(x) \in N(\mathcal{Y}, s)\} \in \text{HYP}$
- Def (Louveau 1980) $A \in \Sigma^0_\xi \iff A$ has a recursive K_ξ-code

- For $\mathcal{X} = \mathcal{Y} = \mathbb{N}$, these definitions of HYP agree with the classical ones
- The pointclasses Σ^0_ξ stabilize for $\xi \geq \omega_1^{\mathrm{CK}}$, and for $\xi < \omega_1^{\mathrm{CK}}$ on $\mathbb{N}$ they are (essentially) those defined by Mostowski and Kleene
- $A \subseteq \mathcal{X}$ is Borel exactly when it is HYP(α) for some $\alpha \in \mathcal{N}$
- $f : \mathcal{X} \to \mathcal{Y}$ is Borel (measurable) exactly when it is HYP(α) for some $\alpha \in \mathcal{N}$

Partial functions and potential recursion

- A partial function $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is potentially recursive if there is a Σ^0_1 pointset $P \subseteq \mathcal{X} \times \mathbb{N}$ which computes f on its domain, i.e.,

$$f(x)\downarrow \implies \Big(f(x) \in N(\mathcal{Y}, s) \iff P(x, s)\Big) \qquad (*)$$

Canonical Extension Theorem (Eff. version of classical fact)

Every potentially recursive $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ has a potentially recursive extension $\overline{f} \supseteq f$ whose domain is Π^0_2

Refined Embedding Theorem (Eff. version of classical fact)

For every recursive Polish $\mathcal{X}$, there is a (total) recursive surjection

$$\pi : \mathcal{N} \twoheadrightarrow \mathcal{X}$$

and a Π^0_1 set $A \subseteq \mathcal{N}$ such that π is injective on A and $\pi[A] = \mathcal{X}$

- *If $f : \mathcal{N} \rightharpoonup \mathcal{N}$ and $f(\alpha)\downarrow$, then $f(\alpha)$ is recursive in α*
- Every potentially recursive function is continuous on its domain; and every $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ which is continuous on its domain is potentially ε-recursive for some $\varepsilon \in \mathcal{N}$

The Suslin-Kleene Theorem

Theorem

(a) *Every Borel pointset is $\mathbf{\Delta}^1_1$, uniformly*
(b) *Every $\mathbf{\Delta}^1_1$ pointset is Borel, uniformly*

The precise version of (b): *For some potentially recursive $\mathbf{u} : \mathcal{N} \rightharpoonup \mathcal{N}$,*
if α is a $\mathbf{\Delta}^1_1$-code of some $A \subseteq \mathcal{X}$,

then $\mathbf{u}(\alpha)\downarrow$ and $\mathbf{u}(\alpha)$ is a Borel code of A

- Suslin's Theorem: $\mathbf{\Delta}^1_1 =$ Borel
- Kleene's Theorem: On $\mathbb{N}$, $\Delta^1_1 =$ HYP
- Both proofs use effective transfinite recursion and (a) is routine (b) uses the fact that a classical proof of Suslin's Theorem (in Kuratowski) is constructive (cf. Kleene's realizability theory)
- Classical version of (b): replace "potentially recursive" by "defined and continuous on a G_δ subset of $\mathcal{N}$"

- Is there a "classical" proof of the classical version of (b)?
- Does the classical version of (b) have any classical applications?

The effective Perfect Set Theorem

Theorem (Harrison)

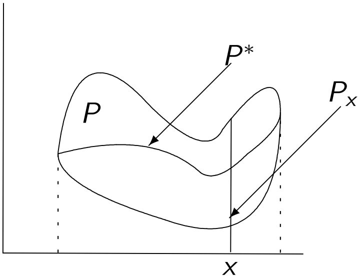
If $A \subseteq \mathcal{X}$ is Σ^1_1 and has a a member $x \in A$ which is not HYP, *then A has a perfect subset*

Corollary (Suslin)

Every uncountable $\boldsymbol{\Sigma}^1_1$ pointset has a perfect subset

- Suslin's Perfect Set Theorem followed earlier results of Hausdorff and Alexandrov for Borel sets and was very important for the classical theory: it implies that the Continuum Hypothesis holds for $\boldsymbol{\Sigma}^1_1$ (analytic) sets
- The (relativized) effective version "explains" the theorem of Suslin
- It also has many effective applications, some of them with further classical applications

The HYP Uniformization Criterion



P^* uniformizes P.

Theorem

A pointset $P \subseteq \mathcal{X} \times \mathcal{Y}$ in HYP *can be uniformized by a* HYP *set P^* if and only if for every $x \in \mathcal{X}$,*

$$(\exists y)P(x, y) \iff (\exists y \in \mathrm{HYP}(x))P(x, y)$$

- (Classical) *If every section of a Borel set $P \subseteq \mathcal{X} \times \mathcal{Y}$ is countable, then P has a Borel uniformization*

Louveau's Theorem

Theorem (Louveau 1980)

For every $\mathcal{X}$, every $P \subseteq \mathcal{X}$ and every recursive ordinal ξ

$$P \in (\mathsf{HYP} \cap \mathbf{\Sigma}^0_\xi) \iff P \in \Sigma^0_\xi(\alpha) \text{ for some } \alpha \in \mathsf{HYP}$$

- This is a basic result about the (relativized) effective hierarchies $\Sigma^0_\xi(\alpha)$ and has many classical and effective applications (including some more detailed versions of the results in the last three slides)

- The proof uses ramified versions of the Harrington-Gandy topology generated by the Σ^1_1 subsets of a recursive $\mathcal{X}$ This is a basic tool of the effective theory, also used in the next result

The Harrington-Kechris-Louveau Theorem

- For equivalence relations $E \subseteq \mathcal{X} \times \mathcal{X}$, $F \subseteq \mathcal{Y} \times \mathcal{Y}$:
  $f : \mathcal{X} \to \mathcal{Y}$ is a reduction if $x \, E \, y \iff f(x) \, F \, f(y)$

$$E \leq_{\text{HYP}} F \iff \text{there is a HYP reduction } f : \mathcal{X} \to \mathcal{Y},$$
$$E \leq_{\text{Borel}} F \iff \text{there is a Borel reduction } f : \mathcal{X} \to \mathcal{Y}$$

- $\alpha \, \Delta \, \beta \iff \alpha = \beta \quad (\alpha, \beta \in \mathcal{N})$
- $\alpha \, E_0 \, \beta \iff (\exists m)(\forall n \geq m)[\alpha(m) = \beta(m)] \quad (\alpha, \beta \in \mathcal{N})$

Dichotomy Theorem (HKL 1990)

For every HYP *equivalence relation E on a recursive Polish space $\mathcal{X}$:*

$$\underline{\textit{Either}}\ E \leq_{\text{HYP}} \Delta \quad \underline{\textit{or}} \quad E_0 \leq_{\textit{Borel}} E$$

- The relativized version with HYP replaced by Borel extends the classical Glimm-Effros Dichotomy Theorem
- It was the beginning of a rich and developing structure theory for Borel equivalence relations and graphs with many applications

Luzin's favorite characterization of the Borel sets

Theorem

A set $A \subseteq \mathcal{X}$ is HYP *if and only if A is the recursive, injective image of a* Π^0_1 *subset of* $\mathcal{N}$

Corollary (Luzin)

A set $A \subseteq \mathcal{X}$ is Borel if and only if A is the continuous, injective image of a closed subset of $\mathcal{N}$

- Luzin's proof is not difficult, so the effective version does not contribute much beyond the stronger statement However, by its proof:

Theorem (ynm 1973)

Assume $\boldsymbol{\Sigma}^1_2$*-determinacy A set $A \subseteq \mathcal{X}$ is* Δ^1_3 *if and only if A is the recursive, injective image of a* Π^1_2 *subset of* $\mathcal{N}$

- The effective theory is indispensable in the study of projective sets under strong, set theoretic hypotheses

—and it was developed partly for these applications

HYP-recursive (eff. Borel) functions and isomorphisms

- *Every uncountable Polish space $\mathcal{X}$ is Borel isomorphic with $\mathcal{N}$*
- Thm [G] *There exist uncountable recursive Polish spaces which are not* HYP*-isomorphic with $\mathcal{N}$*
- A (total) function $f : \mathcal{X} \to \mathcal{Y}$ is HYP(ε)-recursive if it is computed by a HYP(ε) relation, i.e.,

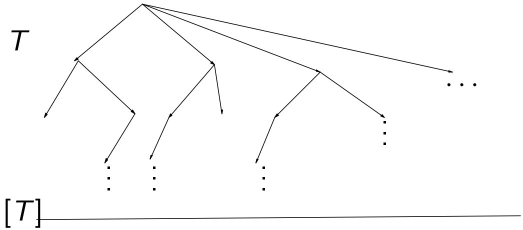
$$\{(x, s) : f(x) \in N(\mathcal{Y}, s)\} \in \text{HYP}(\varepsilon)$$

- The local space parameter For any space $\mathcal{X}$, put

$$P_{\mathcal{X}} = \{s \in \mathbb{N} : N(\mathcal{X}, s) \text{ is uncountable}\}$$

  - $P_{\mathcal{X}}$ is Σ^1_1;
    it is recursive if $\mathcal{X}$ perfect;
    and for some $\mathcal{X}$ it is Σ^1_1-complete
- *Every uncountable recursive $\mathcal{X}$ is* HYP($P_{\mathcal{X}}$)*-isomorphic with $\mathcal{N}$*

The spaces $\mathcal{N}^T$



- [G] For each recursive tree T on $\mathbb{N}$ set

$$\mathcal{N}^T = T \cup [T]$$

with the natural metric, so that $\lim_n(\alpha(0), \ldots, \alpha(n)) = \alpha$

Thm[G] *Every recursive Polish space is* HYP*-isomorphic with some* $\mathcal{N}^T$

- The structure of $\mathcal{N}^T$ reflects combinatorial properties of T

Thm[G] *If* $[T]$ *is non-empty with no* HYP *branches, then* $\mathcal{N}^T$ *is not* HYP*-isomorphic with* $\mathcal{N}$

Recursive Polish spaces under $\preceq_{\mathrm{HYP}}$

- $\mathcal{X} \preceq_{\mathrm{HYP}} \mathcal{Y}$ if there exists a HYP injection $f : \mathcal{X} \rightarrowtail \mathcal{Y}$
- $\mathcal{X} \sim_{\mathrm{HYP}} \mathcal{Y} \iff \mathcal{X} \preceq_{\mathrm{HYP}} \mathcal{Y}$ & $\mathcal{Y} \preceq_{\mathrm{HYP}} \mathcal{X}$
  $\iff \mathcal{X}$ is HYP-isomorphic with $\mathcal{Y}$

- $\mathcal{N}^T$ is a Kleene space if $[T]$ is not empty and has no HYP branches

Theorem (G)

(a) *Every Kleene space occurs in an infinite $\preceq_{\mathrm{HYP}}$-antichain of Kleene spaces*

(b) *Every Kleene space is the first element of an infinite strictly $\prec_{\mathrm{HYP}}$-increasing and an infinite strictly $\prec_{\mathrm{HYP}}$-decreasing sequence of Kleene spaces*

... many more results on the structure of the partial preorder $\preceq_{\mathrm{HYP}}$