Kleene's amazing second recursion theorem

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CSL, 11 September, 2009

To commemorate 100 years from Stephen Cole Kleene's birth

The work of Stephen Cole Kleene (very roughly)

$<$ 1952	**Foundations of recursion theory** Normal Form Theorem ... Arithmetical hierarchy ...
1952	**Introduction to metamathematics**
$>$ 1952	**Second and higher order definability** (1955) On the form ... constructive ordinals, II Arithmetical predicates and function quantifiers Hierarchies of number theoretic predicates 1959+ Recursion in higher types 1965: Function realizability (book with Vesley) ...

- Steele Prize for the three 1955 articles
- 1938: On notation for ordinal numbers
  (6 pages, The Second Recursion Theorem)
- All work after 1955 uses SRT

The Second Recursion Theorem (SRT), 1938. Fix $\mathbb{V} \subseteq \mathbb{N}$, and suppose $\varphi^n : \mathbb{N}^{1+n} \rightharpoonup \mathbb{V}$ is recursive and such that with

$$\{e\}(\vec{x}) = \varphi^n_e(\vec{x}) = \varphi^n(e, \vec{x}) \quad (\vec{x} = (x_1, \dots, x_n) \in \mathbb{N}^n):$$

(1) Every recursive $f : \mathbb{N}^n \rightharpoonup \mathbb{V}$ is φ^n_e for some e.

(2) For all m, n, there is a recursive $S = S^m_n : \mathbb{N}^{m+1} \to \mathbb{N}$ such that

$$\{S(e, \vec{y})\}(\vec{x}) = \{e\}(\vec{y}, \vec{x}) \quad (e \in \mathbb{N}, \vec{y} \in \mathbb{N}^m, \vec{x} \in \mathbb{N}^n).$$

Then, for every recursive, partial function $f(t, \vec{x})$ with values in $\mathbb{V}$, there is a number $\tilde{z}$ such that

$$\boxed{\{\tilde{z}\}(\vec{x}) = f(\tilde{z}, \vec{x})} \quad (\vec{x} \in \mathbb{N}^n).$$

Proof. Fix e_0 such that $\{e_0\}(t, \vec{x}) = f(S(t, t), \vec{x})$, and take $\tilde{z} = S(e_0, e_0)$.

$$\{\tilde{z}\}(\vec{x}) = \{S(e_0, e_0)\}(\vec{x}) = \{e_0\}(e_0, \vec{x}) = f(S(e_0, e_0), \vec{x}) = f(\tilde{z}, \vec{x})$$

- SCK: if you do not understand the difference between f and f(x) you should change fields

SRT: For every r.p.f. $f : \mathbb{N}^{1+m+n} \rightharpoonup \mathbb{V}$, there is a r.f. $\tilde{z}(\vec{y})$:

$$\boxed{\{\tilde{z}(\vec{y})\}(\vec{x}) = f(\tilde{z}(\vec{y}), \vec{y}, \vec{x})}$$

or with $m = 0$, there is a number $\tilde{z}$: $\boxed{\{\tilde{z}\}(\vec{x}) = f(\tilde{z}, \vec{x})}$

- 12,000 Google hits for "second recursion theorem"
- Mostly called "recursion theorem" (45,400 Google hits)
- Easy to generalize (because the proof is so trivial)
- Large number of deep applications in many parts of logic
  In the full paper (on my homepage) there are
  18 theorems with 13 (near complete) proofs

Outline:

(A) Self-reference

(B) Effective grounded recursion: hyperarithmetical hierarchy

(C) Effective grounded recursion: descriptive set theory

(A) Self-reproducing Turing machines

- **Turing machine with states $\{0, \ldots, K\}$ on the alphabet $\Sigma = \{a_1, \ldots, a_N\}$ with $a_1 \equiv 0, a_2 \equiv 1, a_3 \equiv ,$:**
  A finite sequence of quintuples

$$(q \quad s \quad q' \quad s' \quad m) \quad \text{coded by the } \Sigma\text{-string} \quad \overline{q}, \bar{s}, \overline{q}', s', \overline{m}$$

q : state	$\overline{q}$: q in binary
s : symbol or ␣	$\bar{s}$: symbol or nothing
q' : new state	$\overline{q}'$: q' in binary
s' : new symbol or ␣	$\bar{s}'$: symbol or nothing
m : move (left, stay, right)	$\overline{m}$: $10, 00,$ or 01

$\boxed{q_0 \ \sqcup \ q_2 \ , \ \text{right}}$ **is specified by the string** $0, , 10, , , 01$

- **Each M is (coded by) a Σ-string (no blanks, i.e. the sequence of its quintuples separated by commas)**

(A1) Thm For each $N \geq 3$ there is a TM M which started on the blank tape prints itself and quits

Proof. For each number u, let s(u) be its unique expansion in base N using the symbols of Σ for digits, and set

$$\varphi^n(e, \vec{x}) = w \iff s(e) \text{ is a Turing machine } M$$
$$\text{and if we start } M \text{ on } s(x_1)\sqcup s(x_2)\sqcup \cdots \sqcup s(x_n)$$
$$\text{then it stops with the tape starting with } s(w)\sqcup$$

- The standard assumptions hold.
  - M is tidy (with code e): if $\varphi^n_e(\vec{x}) = w$, then M stops on

    $s(x_1)\sqcup s(x_2)\sqcup \cdots \sqcup s(x_n)$ with just s(w) on the tape

  - For each n, $\text{tidy}^n(e)$ is the code of a tidy machine such that $\{\text{tidy}^n(e)\}(\vec{x}) = \{e\}\vec{x})$. ($\text{tidy}^n(e)$ recursive.)

If $\varphi_{\tilde{z}}(\,) = \text{tidy}^0(\tilde{z})$, then the machine with code $\text{tidy}^0(\tilde{z})$ is self-reproducing. □

Recursively enumerable, complete, creative

- A set $A \subseteq \mathbb{N}$ is recursively enumerable (r.e.) if for some e,

$$A = W_e = \{x : \{e\}(x)\downarrow\} = \text{ the domain of convergence of a r.p.f.}$$

- An r.e. set A is complete if for every r.e. set B, there exists a r.f. f(x) s.t.

$$x \in B \iff f(x) \in A$$

- An r.e. set A is creative if there is a r.p.f. u(e) s.t.

$$A \cap W_e = \emptyset \implies [u(e)\downarrow \;\&\, u(e) \notin A \,\&\, u(e) \notin W_e]$$

Thm (Post 1944). Every r.e. complete set is creative
Converse?

(A2) **Thm** (Myhill 1955) Every creative set is r.e.-complete

Proof. Assume that

$$A \cap W_e = \emptyset \implies [u(e)\downarrow \;\&\, u(e) \notin A \,\&\, u(e) \notin W_e]$$

and for a fixed r.e. set B choose a function $\tilde{z}(x)$ by SRT s.t.

$$\{\tilde{z}(x)\}(t) = \begin{cases} 1, & \text{if } x \in B \,\&\, u(\tilde{z}(x))\downarrow \;\&\, t = u(\tilde{z}(x)), \\ \perp & \text{(i.e., undefined), otherwise.} \end{cases}$$

(1) For all x, $u(\tilde{z}(x))\downarrow$ Otherwise $W_{\tilde{z}(x)} = \emptyset$ and so $u(\tilde{z}(x))\downarrow$.

(2) If $x \notin B$, then $W_{\tilde{z}(x)} = \emptyset$ and so $u(\tilde{z}(x)) \notin A$

(3) If $x \in B$, then $u(\tilde{z}(x)) \in A$ (which completes the proof)

Because if $x \in B$, then $W_{\tilde{z}(x)} = \{u(\tilde{z}(x))\}$ (the singleton); and

$$u(\tilde{z}(x)) \notin A \implies W_{\tilde{z}(x)} \cap A = \emptyset \implies u(\tilde{z}(x)) \notin \{u(\tilde{z}(x))\}$$

For a theory T in the language of Peano Arithmetic (PA)

$$\mathsf{Th}(\mathsf{T}) = \{\#\theta : \theta \text{ a sentence and } \mathsf{T} \vdash \theta\}.$$

Thm If T is axiomatizable, sufficiently strong and **sound**, then Th(T) is r.e.-complete (easy)

(A3) **Thm** (Myhill 1955) If T is axiomatizable, sufficiently strong and **consistent**, then Th(T) is creative, and so complete

The proof uses SRT for binary r.p.f.'s ($\mathbb{V} = \{0, 1\}$) and the coding

$$\varphi^n_e(\vec{x}) = \begin{cases} 1, & \text{if e codes a formula } \theta(v_1, \ldots, v_n) \text{ whose free variables} \\ & \quad \text{are in the list } v_1, \ldots, v_n, \text{ and } \mathsf{PA} \vdash \theta(\Delta x_1, \ldots, \Delta x_n), \\ 0, & \text{if e codes a formula } \theta(v_1, \ldots, v_n) \text{ whose free variables} \\ & \quad \text{are in the list } v_1, \ldots, v_n, \text{ and } \mathsf{PA} \vdash \neg\theta(\Delta x_1, \ldots, \Delta x_n), \\ \perp, & \text{otherwise,} \end{cases}$$

Provability logic

Axioms schemes and rules for GL, in the language with $\bot, \to, \Box$:

(GL0) **All tautologies;**

(GL1) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ **(transitivity of provability);**

(GL2) $\Box\varphi \to \Box\Box\varphi$ **(provable sentences are provably provable);**

(GL3) $\Big(\Box(\Box\varphi \to \varphi)\Big) \to \Box\varphi$ **(Löb's Theorem).**

(R1) $\varphi \to \psi, \varphi \implies \psi$ **(Modus Ponens); and**

(R2) $\varphi \implies \Box\varphi$ **(Necessitation).**

Interpretation π : GL-formulas $\to$ PA-sentences:

$$\pi(\bot) \equiv 0 = 1, \quad \pi(\varphi \to \psi) \equiv (\pi(\varphi) \to \pi(\psi)),$$
$$\pi(\Box\varphi) \equiv (\exists u)\mathrm{Proof}_{\mathrm{PA}}(\ulcorner \pi(\varphi) \urcorner, u).$$

(A4) Thm (Solovay 1976) For every GL-formula φ,

$$\mathrm{GL} \vdash \varphi \iff \text{for every } \pi, \quad \mathrm{PA} \vdash \pi(\varphi).$$

- **Also GL is decidable, ...**

(B) Constructive ordinals

An $\mathfrak{r}$-system is a pair $(S, |\ \ |_S)$ such that $|\ \ | : S \to$ Ordinals:

(ON1) For a recursive, partial $K(x)$, $x \in S \implies K(x)\downarrow$ and:

$$|x|_S = 0 \iff K(x) = 0,$$
$$|x|_S \text{ is a successor ordinal} \iff K(x) = 1,$$
$$|x|_S \text{ is a limit ordinal} \iff K(x) = 2.$$

(ON2) For a recursive $P(x)$,

$$[[x \in S \,\&\, K(x) = 1] \implies |x|_S = |P(x)|_S + 1.$$

(ON3) For a recursive $Q(x, t)$,

$$[x \in S \,\&\, K(x) = 2]$$
$$\implies (\forall t)[|Q(x,t)|_S < |Q(x,t{+}1)|_S] \,\&\, |x|_S = \lim_t |Q(x,t)|_S$$

- **An ordinal ξ is constructive if $\xi = |x|_S$ for some x is some $\mathfrak{r}$-system $(S, |\ \ |_S)$**

The $\mathfrak{r}$-system $(\mathbf{S_1}, |\ |)$

The numerals: $0_O = 1, (t+1)_O = t_O^* = 2^{t_O}$
$e_t = \varphi_e(t_O)$

- $(S_1, |\ |)$ is the smallest set of pairs $(a, |a|)$ such that
  - $1 \in S_1$, $|1| = 0$
  - If $a \in S_1$, then $a^* = 2^a \in S_1$, and $|a^*| = |a| + 1$
  - If for all t, $e_t\downarrow, e_t \in S_1, |e_t| < |e_{t+1}|$,
    then $3 \cdot 5^e \in S_1$ and $|3 \cdot 5^e| = \lim_t |e_t|$

(B1) Thm (Kleene 1938) For every $\mathfrak{r}$-system $(S, |\ |_S)$, there is a recursive function $f(a)$ such that

$$a \in S \implies [f(a) \in S_1 \;\&\; |a|_S = |f(a)|]$$

In particular, every constructive ordinal gets a notation in S_1

(B1) Thm (Kleene 1938) For every $\mathfrak{r}$-system $(S, |\ |_S)$, there is a recursive function f(a) such that

$$a \in S \implies [f(a) \in S_1 \,\&\, |a|_S = |f(a)|] \qquad (*)$$

Proof. Choose a number e_0 such that

$$\{S(e_0, z, x)\}(t_O) = \boxed{\{e_0\}(z, x, t_O) = \{z\}(Q(x, t))},$$

and choose by SRT a number $\tilde{z}$ such that (at least when $a \in S$)

$$\varphi_{\tilde{z}}(a) = \begin{cases} 1, & \text{if } |a|_S = 0, \\ \varphi_{\tilde{z}}(b)^*, & \text{if } |a|_S = |b|_S + 1, \\ 3 \cdot 5^{S(e_0, \tilde{z}, x)}, & \text{otherwise} \end{cases}$$

Set $f(a) = \varphi_{\tilde{z}}(a)$ and prove (*) by induction on $|a|_S$.

- Effective grounded recursion

(The only way in which Kleene applied SRT, as far as I know)

Constructive and recursive ordinals

(B2) Thm (Markwald 1955) An ordinal ξ is constructive if and only if it is finite or the order type of a recursive wellordering of $\mathbb{N}$

$$\omega_1^{\mathsf{CK}} = \text{the least non-constructive ordinal}$$

- **Baire space $\mathcal{N} = (\mathbb{N} \to \mathbb{N})$**
- **$\overline{\alpha}(t) = \langle \alpha(0), \alpha(1), \ldots, \alpha(t-1) \rangle$ (sequence code of the first t values)**
- **A relation $P(\vec{x})$ is Π^1_1 iff**
  $P(\vec{x}) \iff (\forall \alpha)(\exists t) R(\vec{x}, \overline{\alpha}(t)) \quad (R(\vec{x}, u) \text{ recursive})$
- **$R(\vec{x})$ is Δ^1_1 iff both $R(\vec{x})$ and $\neg R(\vec{x})$ are Π^1_1.**

Thm (Spector 1955) An ordinal ξ is constructive if and only if it is finite or the order type of a Δ^1_1 wellordering of $\mathbb{N}$

The hyperarithmetical hierarchy (Mostowski, Davis, Kleene)

With each $a \in S_1$ we associate the set $H_a \subseteq \mathbb{N}$:

- $H_1 = \mathbb{N}$.
- $H_{2^b} = H'_b$ (= the jump of H_b).
- If $a = 3 \cdot 5^e$, then $x \in H_a \iff (x)_0 \in H_{e_{(x)_1}}$.
- A set A is arithmetical if and only if it is recursive in some H_a with finite $|a|$
- If $|a| = \omega$, then H_a is Turing equivalent to the set of (codes of) true sentences of arithmetic
- A set $A \subseteq \mathbb{N}$ is hyperarithmetical (HYP) if it is recursive in some H_a

(B3) Thm (Kleene 1955) HYP = Δ^1_1

Spector's Uniqueness Theorem

(B4) Thm (Spector 1955) There is a recursive function $\mathbf{u(a, b)}$ such that if $\mathbf{a, b \in S_1}$ and $\mathbf{|a| \leq |b|}$, then $\mathbf{H_a}$ is recursive in $\mathbf{H_b}$ with code $\mathbf{u(a, b)}$

In particular, if $\mathbf{|a| = |b| = \xi < \omega_1^{CK}}$, then $\mathbf{H_a}$ and $\mathbf{H_b}$ have the same degree of unsolvability $\mathbf{d_\xi}$, and

$$\eta < \xi \implies \mathbf{d}_\eta < \mathbf{d}_\xi$$

(C) Descriptive Set Theory — classical and effective

- **Polish space $\mathcal{X}$: separable, complete metric space**
- **Presentation of $\mathcal{X}$: (S, P, Q) where**
  - $S = \{r_0, r_, \ldots\}$ is dense in $\mathcal{X}$
  - $P(i, j, m, k) \iff d(r_i, r_j) \leq \frac{m}{k+1}$
  - $Q(i, j, m, k) \iff d(r_i, r_j) < \frac{m}{k+1}$
- **(S, P, Q) is recursive if P, Q are recursive**
- **Examples: $\mathcal{N}$, $\mathbb{R}$ (the real numbers), $\mathbb{N}$**
- $B_s = \{x \in \mathcal{X} : d(r_{(s)_0}, x) < \frac{(s)_1}{(s)_2+1}\}$
- **$G \subseteq \mathcal{X}$ is open if for some $\varepsilon \in \mathcal{N}$ (a code of G),**

$$x \in G \iff (\exists s)[x \in B_s \,\&\, \varepsilon(s) = 0]$$

- **$G \subseteq \mathcal{X}$ is effectively open if it has a recursive code**
- **F is effectively closed if $F^c = \mathcal{X} \setminus F$ is effectively open, etc.**

The Suslin - Kleene Theorem

In a Polish space $\mathcal{X}$, with $A \subseteq \mathcal{X}$:

- **A is Borel if it belongs to the smallest σ-field of $\mathcal{X}$ which contains the open sets**
- **A is analytic ($\boldsymbol{\Sigma}^1_1$) if**

$$x \in A \iff (\exists \alpha) R(x, \alpha) \quad (R \subseteq \mathcal{X} \times \mathcal{N}, \text{ closed})$$

- **A is $\boldsymbol{\Delta}^1_1$ if both A and A^c are analytic**

Suslin's Theorem (1917): A is $\boldsymbol{\Delta}^1_1 \iff$ A is Borel

All these classes of sets are naturally coded in $\mathcal{N}$

(C1) Thm (The Suslin-Kleene Theorem)
There are recursive functions $u(\varepsilon), v(\varepsilon)$ such that

$$\text{A is Borel with code } \varepsilon \implies \text{A is } \boldsymbol{\Delta}^1_1 \text{ with code } u(\varepsilon)$$
$$\text{A is } \boldsymbol{\Delta}^1_1 \text{ with code } \varepsilon \implies \text{A is Borel with code } v(\varepsilon)$$

Recursion on $\mathcal{N}$

(C1) Thm (The Suslin-Kleene Theorem)
There are recursive functions $u(\varepsilon), v(\varepsilon)$ such that

$$\text{A is Borel with code } \varepsilon \implies \text{A is } \mathbf{\Delta}^1_1 \text{ with code } u(\varepsilon)$$
$$\text{A is } \mathbf{\Delta}^1_1 \text{ with code } \varepsilon \implies \text{A is Borel with code } v(\varepsilon)$$

- **The S-K Theorem implies easily both Suslin's Theorem and Kleene's $\Delta^1_1 = \text{HYP}$ on $\mathbb{N}$**
- **Its precise statement presupposes a recursion theory on $\mathcal{N}$**
- **It is proved by "effectivizing" a classical proof of the Suslin Theorem (in Kuratowski's book) using SRT for recursion on $\mathcal{N}$**

The axiom of determinacy

- **AD : every game A $\subseteq \mathcal{N}$ is determined**
- **AD is inconsistent with the Axiom of Choice**
- **AD implies an almost-complete structure theory for L($\mathbb{R}$)**
- **Thm (Martin, Steel, Woodin $\sim$ 1987) If sufficiently strong axioms of infinity (large cardinal axioms) are true, then AD is true in the inner model L($\mathbb{R}$)**

(C2) Thm (The Coding Lemma, in ZF + AD, ynm 1970)
If κ is a cardinal number and there exists a surjection $\pi : \mathcal{N} \twoheadrightarrow \kappa$, then there exists a surjection $\pi^* : \mathcal{N} \twoheadrightarrow \mathcal{P}(\kappa)$

(Uses SRT for a recursion theory associated with the given π)

Thm (In ZF + AD, Jackson 1970)

the smallest weakly inaccessible cardinal number

$$= \sup\{\text{rank}(\precsim) : \precsim \text{ is a well-founded relation on } \mathcal{N}, \text{ Kleene-hyperanalytic in some } \alpha\}$$