ERRORS IN rcc.tex

This is a dynamic file listing the errors in rcc.tex and their corrections as they are found.

#1 (07/10/2015). Equation (1A-5) is wrong (and I do not remember how it got there). The correct inequality is

$$\mathrm{leaves}(\mathcal{T}) \le \mathrm{degree}(\mathcal{T})^{\mathrm{depth}(\mathcal{T})}$$

where $\mathrm{leaves}(\mathcal{T})$ is the number of leaves of $\mathcal{T}$.

As a consequence, Problem x1A.1 is also wrong; it is not referred to anywhere in the ms and is probably the remnant of some other material which needed (something like) it and has been removed long ago. The problem will be replaced by the following:

x1.1. **Problem.** Suppose $\mathcal{T}$ is a finite tree.
(1) Prove (1A-5) (the correct one).
(2) Prove that if $\mathrm{degree}(\mathcal{T}) = 1$, then $\mathrm{size}(\mathcal{T}) = \mathrm{depth}(\mathcal{T}) + 1$.
(3) Prove that if degree ≥ 2, then

$$\mathrm{size}(\mathcal{T}) \le \mathrm{degree}^{\mathrm{depth}(\mathcal{T})+1} - 1 < \mathrm{degree}(\mathcal{T})^{\mathrm{depth}(\mathcal{T})+1}.$$

#2. The wrong equation (1A-5) is used twice in the ms.

The first use of it is in the hint to Problem x1D.7*, and in this case it is only the correct (new) version of it that is needed, so there is no need to change this part. (Except that Problem x1D.7* will almost certainly be reformulated to something more general in the final version).

The second use of the wrong inequality is in the "proof" of Part (c) of Theorem 3A.3, which is a significant result in the study of complexity inequalities. There is another (older) proof of Part (c) which is now asked for by Problem x3A.7*, and the final version will have this proof instead of the appeal to the wrong (1A-5). This proof is as follows:

Correct proof of Part (c) of Theorem 3A.3:

We may assume $\ell \ge 2$, since Problem x3A.6 gives the (stronger) result when $\ell = 1$. For $\ell \ge 2$, we show the result by induction on $D(M)$.

Case 1. $D(M) = 0$. In this case M is tt, ff or a parameter $x \in A$, so that $L^p(M) = L^s(M) = 0$.

Case 2, $M \equiv \phi(M_1, \dots, M_n)$. The induction hypothesis gives us the result for $M_1, \dots, M_n$, and we compute:

$$\begin{aligned} L^s(M) &= L^s(M_1) + \cdots + L^s(M_n) + 1 \\ &\le (\ell+1)^{L^p(M_1)} + \cdots + (\ell+1)^{L^p(M_n)} + 1 \\ &\le \ell(\ell+1)^A + 1 \qquad\qquad (A = \max\{L^p(M_1), \dots, L^p(M_n)\}) \\ &\le \ell(\ell+1)^A + (\ell+1)^A \\ &= (\ell+1)^{A+1} = (\ell+1)^{L^p(M)}. \end{aligned}$$

Case 3, $M \equiv \text{if } M_0 \text{ then } M_1 \text{ else } M_2$. Assume, for definiteness that $\overline{M}_0 = \mathrm{tt}$, and the induction hypothesis for M_0 and M_1 and compute as in Case 2:

$$\begin{aligned} L^s(M) &= L^s(M_0) + L^s(M_1) + 1 \\ &\le (\ell+1)^{L^p(M_0)} + (\ell+1)^{L^p(M_1)} + 1 \\ &\le 2(\ell+1)^A + 1 \qquad\qquad (A = \max\{L^p(M_0), L^p(M_1)\}) \\ &\le \ell(\ell+1)^A + 1 \qquad (\text{because } 2 \le \ell) \\ &\le (\ell+1)^{A+1} = (\ell+1)^{L^p(M)}. \end{aligned}$$

Case 4, $M \equiv p(M_1, \dots, M_n)$. Now

$$L^s(M) = L^s(M_1) + \cdots + L^s(M_n) + L^s(E_p(\overline{M}_1, \dots, \overline{M}_n)) + 1.$$

If $L^p(M_1) = \cdots = L^p(M_n) = 0$, then each M_i is a constant, so their sequential logical complexities are also $= 0$, and then, using the induction hypothesis:

$$\begin{aligned} &L^s(M) = L^s(E_p(\overline{M}_1, \dots, \overline{M}_n)) + 1 \\ &\le (\ell+1)^{L^p(E_p(\overline{M}_1, \dots, \overline{M}_n))} + 1 \le (\ell+1)^{L^p(E_p(\overline{M}_1, \dots, \overline{M}_n))+1} = (\ell+1)^{L^p(M)}. \end{aligned}$$

In the opposite case, setting

$$A = \max\{L^p(M_1), \dots, L^p(M_n)\} \ge 1, \quad B = L^p(E_p(\overline{M}_1, \dots, \overline{M}_n)),$$

we can compute as above:

$$\begin{aligned} L^s(M) &= L^s(M_1) + \cdots + L^s(M_n) + L^s(E_i(\overline{M}_1, \dots, \overline{M}_n)) + 1 \\ &\le (\ell+1)^{L^p(M_1)} + \cdots + (\ell+1)^{L^p(M_n)} + (\ell+1)^B + 1 \\ &\le \ell(\ell+1)^A + (\ell+1)^{B+1}, \end{aligned}$$

and it sufficed to prove that for all $A \ge 1$ and all $B \in \mathbb{N}$,

$$\ell(\ell+1)^A + (\ell+1)^{B+1} \le (\ell+1)^{A+B+1};$$

but this inequality is equivalent to

$$\frac{\ell}{(\ell+1)^{B+1}} + \frac{1}{(\ell+1)^A} \le 1,$$

which is obvious for $A \ge 1$.

#3 (07/23/15). Problem x2D.4 is very difficult to do at this stage (even with hints, which are not given) and will be removed. It will be replaced by the introduction (at some later point in the manuscript) of the appropriate references for the solutions of x1C.13* − x1C.15*; for example, the solution of Problem x1C.15* is part of Problem x5D.1.

#4 (07/24/15). The inequalities in Problem x3A.8 are not correct as they are stated; the problem will be restated *to ask for some $r > 1$ such that*

$$l^s_E(x) \geq r^{l^p_E(x)}, \quad c^s_E(x) \geq r^{c^p_E(x)}.$$

(The proof embedded in the current version proves the second of these inequalities with $r = \sqrt{2}$.)

#5 (08/05/15). The sentence following Lemma 3B.2 is false and should be deleted and the Lemma should be changed to read that $\overline{M}$ *is either a truth value or a parameter which occurs in* M. The change does not affect the later references to this Lemma. (This is a remnant of a previous version of the Notes which had slightly different definitions of the basic notions, and it is likely that there are more inessential errors of the same kind.)

#6 (08/05/15). In Problem x4F.1, replace both occurrences of $\mathrm{depth}_R(\mathbf{A}, \vec{x})$ by $\mathrm{calls}_R(\mathbf{A}, \vec{x})$.

#7 (08/13/15). In Problem x1D.4*, the last displayed equation should be

$$b(m) = \begin{cases} \log m + 1, & \text{if } m \text{ is a power of 2,} \\ \lceil \log m \rceil, & \text{otherwise,} \end{cases}$$

(rather than $\lfloor \log m \rfloor$). This affects mildly the formulas in the next two problems, but not the comment following them, that the binary-insert-sort algorithm is asymptotically no worse than the merge-sort as far as the number of comparisons required.

#8 (08/19/15). The proof of Lemma 9C.2 in based on a correct idea, but it is written so badly that it does not make sense. A correct writeup by Tyler Arant of a somewhat simpler argument is posted at http://www.math.ucla.edu/~ynm/9C2.pdf.

#9 (08/19/15). The proof of Theorem 9D.1 from Lemma 9D.2 includes a nonsensical reference to (non-trivial) automorphisms of $\mathbb{R}$. Here is a correct version of the argument:

Proof of Theorem 9D.1 from Lemma 9D.2. Fix $n \geq 2$ and for $\mathbf{R}$ first, suppose $a_0, a_1, \ldots, a_n, b \in \mathbb{R}$ are algebraically independent so that

$$a_0 + a_1 b + \cdots + a_n b^n \neq 0.$$

Choose positive, algebraically independent $-z, x_1, \ldots, x_n, y$ in $\mathbf{R}$ (so that $z < 0$) and let

$$\rho : \mathbb{Q}(a_0, s_1, \ldots, a_n, b) \rightarrowtail\!\!\!\!\!\twoheadrightarrow \mathbb{Q}(-z, x_0, \ldots, x_n, y)$$

be the relabelling isomorphism taking $a_0 \mapsto -z, a_1 \mapsto x_1, \ldots, a_n \mapsto x_n, b \mapsto y$. If $\mathbf{U} \subseteq_p \mathbf{R}$ is generated by $a_0, a_1, \ldots, a_n, b$, then $\mathbf{U} \subseteq_p \mathbb{Q}(a_0, s_1, \ldots, a_n, b)$; and if it is finite and has fewer than $n-1$ additions and subtractions, then the image structure $\rho[\mathbf{U}] \subseteq_p \mathbb{Q}(-z, x_0, \ldots, x_n, y) \subseteq_p \mathbf{R}$ is generated by $-z, x_1, \ldots, x_n$ and also has no more than $(n-1)$ additions and subtractions. Now Lemma 9D.2

supplies us with a partial ring homomorphism $\pi : \mathbf{R} \rightharpoonup \mathbf{R}$ which is total and injective on $\rho[\mathbf{U}]$ and satisfies

$$\pi(-z) = \pi(x_1)\pi(y)^1 + \cdots + \pi(x_n)\pi(y)^n.$$

The composition $\sigma = \pi \circ \rho : \mathbf{F} \rightharpoonup \mathbf{R}$ is total and injective on $\mathbf{U}$ and satisfies

$$\begin{aligned} &\sigma(a_0) + \sigma(a_1)\sigma(b)^1 + \cdots \sigma(a_n)\sigma(b)^n \\ &\qquad = \pi(-z) + \pi(x_1)\pi(y)^1 + \cdots + \pi(x_n)\pi(y)^n \\ &\qquad\qquad = -\pi(z) + \pi(x_1)\pi(y)^1 + \cdots + \pi(x_n)\pi(y)^n = 0 \end{aligned}$$

so its restriction $\sigma \upharpoonright \mathbf{U} \rightarrowtail \mathbf{R}$ is an embedding of $\mathbf{U}$ into $\mathbf{R}$ which changes the truth value of the nullity relation. From this, the Homomorphism Test (Lemma 4F.2) yields Theorem 9D.1 with $\mathbf{F} = \mathbf{R}$.

The same argument works for $\mathbf{C}$, if we notice that the proof of Lemma 9D.2 also works for $\mathbf{C}$, since the embeddings into $\mathbf{R}$ constructed in it can also be viewed as embeddings into $\mathbb{C}$. $\dashv$