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PREFACE

Effective descriptive set theory is the study of definable sets of natural and real numbers, or, more generally Polish (separable, complete, metrizable) spaces. It extends and refines the classical theory of Borel, analytic and projective sets in Polish spaces, and in some cases it provides new proofs and new results in the classical theory.

Our aim in this course is to introduce the basic notions, facts and applications of effective descriptive set theory: there will be no attempt to cover the subject in a comprehensive way, which cannot be done in one term. We will work in the theory

\[ ZFDC = ZF + DC = \text{Zermelo-Fraenkel set theory with Dependent Choice} \]

and will identify appeals to the full Axiom of Choice (AC) or other set theoretic hypotheses beyond ZFDC. (This is important because the theory is often interpreted in models of set theory which satisfy ZFDC but not necessarily AC, e.g., the smallest transitive class \( L(R) \) which satisfies ZF.)

The course is aimed at fairly advanced grad students in logic, who know basic recursion theory (at least through the arithmetical hierarchy; set theory (including transfinite recursion); basic analysis and (metric) topology; and certainly logic. No classical descriptive set theory will be assumed, but in some places we may refer to more advanced material from set theory.

These notes are based on and will occasionally refer to the basic monographs Moschovakis [2009], Kechris [1995] and Marker [2002], as well as on Gregoriades [2009].
PRELIMINARIES: RECURSION ON $\omega$

We summarize here the basic theory of recursive and arithmetical relations on the natural numbers $\omega = \{0, 1, \ldots\}$, primarily to set up notation. Proofs of them can be found in Moschovakis [2012] or any textbook in elementary recursion (computability) theory.

A partial function

$$f : X \rightarrow Y$$

on a set $X$ to some set $Y$ is any (ordinary, total) function

$$f : X_0 \rightarrow Y,$$

where $X_0$ is any subset of $X$. We call $X_0$ the domain of convergence of $f$, and we set

$$f(x) \downarrow \iff x \in X_0 \quad (f(x) \text{ converges or is defined})$$

$$f(x) \uparrow \iff x \in X \setminus X_0 \quad (f(x) \text{ diverges}).$$

The graph of $f : X \rightarrow W$ is the set

$$\text{Graph}(f) = \{(x, w) : f(x) = w\}.$$

It is sometimes called the representing relation of $f$.

Typically $X = X_1 \times \cdots \times X_n$ is a product space, and the composition of (substitution into) such partial functions is defined in the natural (strict) way: if $f_i : X_i \rightarrow U_i$ and $g : U_1 \times \cdots \times U_n \rightarrow Y$, then

$$g(f_1(x_1), \ldots, f_n(x_n)) = y$$

$$\iff (\exists u_1, \ldots, u_n)[f_1(x_1) = u_1 & \cdots & f_n(x_n) = u_n & g(u_1, \ldots, u_n) = y].$$

Primitive recursion. A partial function $f : \omega \times X \rightarrow Y$ is defined by primitive recursion (on $t \in \omega$) from partial functions $g, h$ (on the appropriate sets) if it satisfies the following equations, for all $t \in \omega, x \in X$:

$$f(0, x) = g(x),$$

$$f(t + 1, x) = h(f(t, x), t, x).$$
Minimalization. A partial function \( f : X \to \omega \) is defined by minimalization from \( g : \omega \times X \to \omega \) if
\[
f(x) = \mu t [g(t, x) = 0]
\]
= the least \( t \) such that \((\forall s < t)(\exists u \neq 0)g(s, x) = u + 1 \& g(t, x) = 0\).

Recursive partial functions on \( \omega \). A set \( F \) of partial functions of all numbers of arguments on \( \omega \) is closed under recursion if it satisfies the following conditions:

1. \( F \) contains the successor function \( S(x) = x + 1 \), the constant functions \( C^n_q(x_1, \ldots, x_n) = q \) and the projections \( P^n_i(x_1, \ldots, x_n) = x_i \), for all \( n \geq 1, q \in \omega \) and \( 1 \leq i \leq n \).
2. \( F \) is closed under composition, primitive recursion and minimalization.

The arithmetic subtraction and predecessor functions
\[
x - y = \begin{cases} 0, & \text{if } x \leq y, \\ x - y, & \text{otherwise} \end{cases}, \quad \text{Pd}(x) = x - 1
\]
on \( \omega \) can be defined by primitive recursion, and so they belong to every set of partial functions closed under recursion.

The class of recursive partial functions on \( \omega \) is the least set of partial functions on \( \omega \) which is closed under recursion,
\[
\text{rec} = \bigcap \{F : F \text{ is closed under recursion}\};
\]
more generally, for any total functions \( \vec{\psi} = \psi_0, \ldots, \psi_{k-1} \) on \( \omega \) of any arities,
\[
\text{rec}[\vec{\psi}] = \bigcap \{F : \psi_1, \ldots, \psi_{k-1} \in F \text{ and } F \text{ is closed under recursion}\}
\]
is the set of partial functions on \( \omega \) which are recursive in \( \vec{\psi} \) or \( \vec{\psi} \)-recursive.

A relation \( R \subseteq \omega^k \) is recursive or \( \vec{\psi} \)-recursive if its (total) characteristic function is in the relevant class, where
\[
\chi_n(\vec{x}) = \begin{cases} 1, & \text{if } R(\vec{x}), \\ 0, & \text{otherwise}. \end{cases}
\]

Relativization. The definition of the relativized class \( \text{rec}[\vec{\psi}] \) obviously makes sense for partial \( \psi_i \), but then it does not have all the properties of \( \text{rec} \) which we will list below; while for total \( \vec{\psi} \), all the further definitions and results of this preliminary chapter make sense and are true (and by the same proofs) if we systematically replace “recursive” with “\( \vec{\psi} \)-recursive” in them.

We now list a sequence of basic results about recursive partial functions on \( \omega \).
Theorem P.5 (Tuple coding). The following (total) functions and relations on $\omega$ of the indicated arities are recursive:

$p_i$ is the $i$th prime number, with $p_0 = 2$,

$$\langle x_0, \ldots, x_{n-1} \rangle = p_0^{x_0+1} p_1^{x_1+1} \cdots p_{n-1}^{x_{n-1}+1} \quad \text{(with } \langle \rangle = 1),$$

$\text{Seq}(u) \iff (\exists x_0, \ldots, x_{n-1})[w = \langle x_0, \ldots, x_{n-1} \rangle],$

$$(w)_i = \max \left( j : p_j \text{ divides } w \right) - 1, \quad (= 0 \text{ if } w = 0),$$

$$(w)_{i,j} = ((w)_i)_j,$$

$\text{lh}(w) = \max \{ i : p_i \text{ divides } w \} \quad (= 0 \text{ if } w \text{ is 1 or a prime}),$

$u \sqsubseteq v \Leftrightarrow (\forall i < \text{lh}(u))[(u)_i = (v)_i].$

Moreover, the function $(x_0, \ldots, x_{n-1}) \mapsto \langle x_0, \ldots, x_{n-1} \rangle$ is an injection of the set $\omega^{<\omega}$ of all finite sequences of natural numbers into $\omega$, with image the set Seq and such that

$\text{lh}(\langle x_0, \ldots, x_{n-1} \rangle) = n, \quad (\langle x_0, \ldots, x_{n-1} \rangle)_i = x_i \quad (i < n),$

and there is a recursive binary function $u \ast v$ such that

$$\langle x_0, \ldots, x_{n-1} \rangle \ast \langle y_0, \ldots, x_{m-1} \rangle = \langle x_0, \ldots, x_{n-1}, y_0, \ldots, x_{m-1} \rangle.$$

Theorem P.6. The class of recursive relations on $\omega$ is closed under the propositional operations $\neg, \&$, $\lor$, bounded quantification $\exists^*, \forall^*$ and the substitution of total recursive functions.

In detail, this means that if $R_1, R_2$ are recursive (of the appropriate number of arguments) and $f_1, \ldots, f_n$ are total, recursive, then $R(\bar{x})$ is also recursive if it satisfies one of the following equivalences:

$$R(\bar{x}) \iff \neg R_1(\bar{x}),$$

$$R(\bar{x}) \iff R_1(\bar{x}) \& R_2(\bar{x}),$$

$$R(\bar{x}) \iff R_1(\bar{x}) \lor R_2(\bar{x}),$$

$$R(\bar{x}) \iff R_1(f_1(\bar{x}), \ldots, f_n(\bar{x})),$$

$$R(t, \bar{x}) \iff (\exists s \leq t)R_1(s, \bar{x}),$$

$$R(t, \bar{x}) \iff (\forall s \leq t)R_1(s, \bar{x}).$$

In the sequel we will skip writing out definitions like these for various operations on relations and functions, which are intuitively simple but messy to specify precisely because their “domains” are complex.

Theorem P.7 (Kleene’s Normal Form and Enumeration). There is a recursive total function $U : \omega \to \omega$ and for each $n \geq 1$ a recursive relation
Preliminaries

$T_n \subseteq \omega^{1+n+1}$, so that a partial function $f : \omega^n \to \omega$ is recursive if and only if there is a number $e$ (a code or Gödel number of $f$) such that

$$f(x_1, \ldots, x_n) = U(\mu y T_n(e, x_1, \ldots, x_n, y)).$$

In particular, if we set for each $n \geq 1$

$$\varphi^n_0, \varphi^n_1, \ldots$$

then the sequence

$$\varphi^n_0, \varphi^n_1, \ldots$$

enumerates all $n$-ary recursive partial functions, and in such a way that the diagonal $\varphi^n(e, \vec{x}) = \varphi^n_0(\vec{x}) = \{e\}(\vec{x})$ is recursive.

Kleene’s favorite notation $\{e\}(\vec{x}) = \varphi^n_0(\vec{x})$ does not refer explicitly to the arity $n$ of the partial function with code $e$, and this makes it necessary sometimes to use the alternative $\varphi^n_0$ when we need a name for this partial function. It is, however, very useful, and we will tend to use it most of the time.

**Theorem P.8** (The $S_m^n$ Theorem). For any $m, n$, there is a recursive (total) function $S_m^n : \omega^{1+m} \to \omega$ such that

$$\{e\}(\vec{y}, \vec{x}) = \{S_m^n(e, \vec{y})\}(\vec{x}),$$

for all $\vec{y} = (y_1, \ldots, y_m), \vec{x} = (x_1, \ldots, x_n)$.

In the alternative $\varphi$-notation, the $S_m^n$ equation takes the form

$$\varphi^{m+n}(\vec{y}, \vec{x}) = \varphi^n(S_m^n(\vec{y}))(\vec{x}).$$

**Theorem P.9** (Kleene’s 2nd Recursion Theorem). For every recursive partial function $f : \omega^{1+n} \to \omega$, there is a number $e^* \in \omega$ such that

$$\{e^*\}(\vec{x}) = f(e^*, \vec{x}) \quad (\vec{x} \in \omega^n).$$

More generally (with parameters), for every $f : \omega^{1+m+n} \to \omega$, there is a recursive total $g^* : \omega^m \to \omega$ such that

$$\{g^*(\vec{y})\}(\vec{x}) = f(g^*(\vec{y}), \vec{y}, \vec{x}) \quad (\vec{y} \in \omega^m, \vec{x} \in \omega^n).$$

**Proof.** For the simpler statement, set

$$h(m, \vec{x}) = f(S_m^1(m, m), \vec{x}),$$

choose $\hat{h}$ so that $\{\hat{h}\}^{1+n}(m, x) = h(m, \vec{x})$ and take $e^* = S_n^1(\hat{h}, \hat{h})$. Now compute, by the $S_m^n$ Theorem:

$$f(e^*, \vec{x}) = f(S_n^1(\hat{h}, \hat{h}), \vec{x}) = h(\hat{h}, \vec{x}) = \{S_n^1(\hat{h}, \hat{h})\}(\vec{x}) = \{e^*\}(\vec{x}). \quad \Box$$

Informal notes, full of errors, December 8, 2012, 16:24
We put down Kleene’s trivial (if opaque) proof of the 2nd Recursion Theorem because it makes it clear that no deep facts about recursive partial functions or their normal forms are involved: all that is used is that there are enumerations—codings—of the recursive partial functions of all arities which “cohere” in such a way that the $S^m_n$ Theorem holds. The idea is powerful and we will find many applications of it in more general contexts.

The Church-Turing Thesis. This is the claim that a function $f : \omega \to \omega$ is recursive exactly when it is computable by some algorithm. It is an important principle which we will not discuss here—although some of the definitions we will give assume it implicitly, if they are to have their intended interpretation.

Semirecursive relations and r.e. sets. A relation $R \subseteq \omega^n$ is semirecursive (or $\Sigma^0_1$) if it is the domain of convergence of a recursive partial function,

$$R(\vec{x}) \iff f(\vec{x}) \downarrow$$

(with some $f \in \text{rec}$); and a set $A \subseteq \omega$ is recursively enumerable if either $A = \emptyset$ or $A = f[\omega] = \{f(0), f(1), \ldots\}$ for some total $f \in \text{rec}$.

The next result follows immediately from the definition and Theorems P.7, P.8 and P.9, but it is worth stating again for future reference:

**Corollary P.12 ($\Sigma^0_1$ good parametrization).** For each $n \geq 1$, set

$$\tilde{S}^0_n(e, \vec{x}) \iff \{e\}(\vec{x}) \downarrow$$

and let $S^m_n$ be the recursive functions of Theorem P.8.

1. An $n$-ary relation $P \subseteq \omega^n$ is semirecursive if and only if there is some $e$ such that

$$P(\vec{x}) \iff \tilde{S}^0_n(e, \vec{x}).$$

2. For all $e \in \omega$, $\vec{y} \in \omega^m$, $\vec{x} \in \omega^n$, $\tilde{S}^0_{m+n}(e, \vec{y}, \vec{x}) \iff \tilde{S}^0_n(S^m_n(e, \vec{y}), \vec{x}).$

3. For each semirecursive $P \subseteq \omega^{1+n}$, there is some $e^* \in \omega$ such that

$$P(e^*, \vec{x}) \iff \tilde{S}^0_n(e^*, \vec{x}).$$

Next we collect in one theorem some of the basic properties of semirecursive relations.

**Theorem P.13.** (1) A relation $R \subseteq \omega^n$ is semirecursive if and only if there is a recursive relation $P \subseteq \omega^{n+1}$ such that

$$R(\vec{x}) \iff (\exists t) P(\vec{x}, t).$$

(2) The class of semirecursive relations on $\omega$ is closed under the positive propositional operations $\&$, $\lor$, bounded quantification $\exists^\leq$, $\forall^\leq$, existential
quantification

\[ R(\vec{x}) \iff (\exists t) R_1(\vec{x}, t), \]

and the substitution of recursive (total or partial) functions.

(3) A relation \( R \subseteq \omega^n \) is recursive if and only if both \( R \) and its negation \( \neg R \) are semirecursive.

(4) The \( \Sigma_0 \)-Selection Lemma. If \( R \subseteq \omega^{n+1} \) is semirecursive, then there is a recursive partial function \( f : \omega^n \rightarrow \omega \) such that

(i) \( f(\vec{x}) \downarrow \iff (\exists y) R(\vec{x}, y). \)

(ii) \( (\exists y) R(\vec{x}, y) \implies P(\vec{x}, f(\vec{x})). \)

(5) A partial function \( f : \omega^n \rightarrow \omega \) is recursive if and only if its graph \( \text{Graph}(f) \subseteq \omega^n \times \omega = \omega^{n+1} \) is semirecursive.

(6) A set \( A \subseteq \omega \) is r.e. if and only if the relation of membership in \( A \)

\[ R_A(x) \iff x \in A \]

is semirecursive.

We give the simple proof of (4), not so much because the result is important—which it is—but because the “pairing” trick it uses has many applications, including (5). Choose a recursive \( P(\vec{x}, y, t) \) such that

\[ R(\vec{x}, y) \iff (\exists t) P(\vec{x}, y, t), \]

and set

\[ f(\vec{x}) = \left( \mu w P(\vec{x}, (w)_0, (w)_1) \right)_0. \]

We also state here just one of (infinitely) many interesting basic facts about r.e. sets, because it is very useful for constructing counterexamples:

**Corollary P.14** (Recursively inseparable r.e. sets). There exist two disjoint r.e. sets \( K_0, K_1 \subseteq \mathbb{N} \) which cannot be separated by a recursive set, i.e.,

\[ (K_0 \subseteq A \& A \cap K_1 = \emptyset) \implies A \text{ is not recursive.} \]

**Proof.** Take

\[ K_0 = \{ x : (\exists s)[T_1((x)_0, x, s) \& (\forall t \leq s) \neg T_1((x)_1, x, t)] \}, \]

\[ K_1 = \{ x : (\exists t)[T_1((x)_1, x, t) \& (\forall s < t) \neg T_1((x)_0, x, s)] \}. \]
The arithmetical hierarchy. The classes \( \Sigma_k^0, \Pi_k^0, \Delta_k^0 \) relations on \( \omega \) are defined recursively, as follows:

\[
\begin{align*}
\Sigma_1^0 &: \text{the semirecursive relations} \\
\Pi_1^0 &= \neg \Sigma_1^0: \text{the negations (complements) of relations in } \Sigma_1^0 \\
\Sigma_{k+1}^0 &= \exists \Pi_k^0: \text{the relations which satisfy an equivalence} \\
P(\vec{x}) &\iff (\exists y)Q(\vec{x}, y), \text{ where } Q(\vec{x}, y) \text{ is } \Pi_k^0 \\
\Delta_k^0 &= \Sigma_k^0 \cap \Pi_k^0: \text{the relations which are both } \Sigma_k^0 \text{ and } \Pi_k^0.
\end{align*}
\]

A set \( A \) is in one of these classes \( \Gamma \) if the relation \( x \in A \) is in \( \Gamma \).

Canonical forms. These classes of the arithmetical hierarchy are (obviously) characterized by the following “canonical forms”, in the sense that a given relation \( P(\vec{x}) \) is in a class \( \Gamma \) if it is equivalent with the canonical form for \( \Gamma \), with some recursive \( Q \):

\[
\begin{align*}
\Sigma_0^0 &: (\exists y)Q(\vec{x}, y) \\
\Pi_0^0 &= (\forall y)Q(\vec{x}, y) \\
\Sigma_2^0 &: (\exists y_1)(\forall y_2)Q(\vec{x}, y_1, y_2) \\
\Pi_2^0 &= (\forall y_1)(\exists y_2)Q(\vec{x}, y_1, y_2) \\
\Sigma_3^0 &: (\exists y_1)(\forall y_2)(\exists y_3)Q(\vec{x}, y_1, y_2, y_3) \\
&\vdots
\end{align*}
\]

The relations on \( \omega \) which occur in the arithmetical hierarchy are exactly those which are first-order definable in the structure \( (\omega, 0, S, +, \cdot) \) of arithmetic—hence the name.

**Theorem P.17.** (1) For each \( k \geq 1 \), the classes \( \Sigma_k^0, \Pi_k^0, \text{ and } \Delta_k^0 \) are closed for (total) recursive substitutions and for the operations \&, \lor, \exists^\leq \text{ and } \forall^\leq. \text{ In addition:}

- Each \( \Delta_k^0 \) is closed for negation \( \neg \).
- Each \( \Sigma_k^0 \) is closed for \( \exists^\leq \), existential quantification over \( \omega \).
- Each \( \Pi_k^0 \) is closed for \( \forall^\leq \), universal quantification over \( \omega \).

(2) For each \( k \geq 1 \),

\[
\Sigma_k^0 \subseteq \Delta_k^{k+1},
\]

and hence the arithmetical classes satisfy the following diagram:

\[
\begin{array}{ccccccc}
\Sigma_1^0 & \subseteq & \Sigma_2^0 & \subseteq & \Sigma_3^0 & \subseteq & \cdots \\
\Delta_1^0 & \subseteq & \Delta_2^0 & \subseteq & \Delta_3^0 & \subseteq & \cdots \\
\Pi_1^0 & \subseteq & \Pi_2^0 & \subseteq & \Pi_3^0 & \subseteq & \cdots
\end{array}
\]
More interesting is the next theorem which justifies the appellation “hierarchy” for the classes $\Sigma^0_k$, $\Pi^0_k$.

**Theorem P.18.** (1) (Enumeration for $\Sigma^0_k$) For each $k \geq 1$ and each $n \geq 1$, there is an $1 + n$-ary relation $\tilde{S}^0_{k,n}(e, \vec{x})$ in the class $\Sigma^0_k$ which enumerates all the $n$-ary, $\Sigma^0_k$ relations, i.e., $P(\vec{x})$ is $\Sigma^0_k$ if and only if for some $e$,

$$P(\vec{x}) \iff \tilde{S}^0_{k,n}(e, \vec{x}).$$

(2) (Enumeration for $\Pi^0_k$) For each $k \geq 1$ and each $n \geq 1$, there is an $n + 1$-ary relation $\tilde{P}^0_{k,n}(e, \vec{x})$ in $\Pi^0_k$ which enumerates all the $n$-ary, $\Pi^0_k$ relations, i.e., $P(\vec{x})$ is $\Pi^0_k$ if and only if, for some $e$,

$$P(\vec{x}) \iff \tilde{P}^0_{k,n}(e, \vec{x}).$$

(3) (Hierarchy Theorem) The inclusions in the Diagram of Proposition P.17 are all strict, i.e.,

$$
\begin{align*}
\Delta^0_0 \subset \Sigma^0_1 \subset \Pi^0_1 \subset \Sigma^0_2 \subset \Pi^0_2 \subset \Delta^0_2 \\
\Delta^0_3 \subset \Sigma^0_4 \subset \Pi^0_3 \subset \Delta^0_4 \subset \cdots
\end{align*}
$$

There are many interesting results which classify various relations in the arithmetical hierarchy. We state here just two of them, because they are often useful in the construction of examples:

**Proposition P.19.** The Halting Relation

$$H(e, x) \iff \{e\} \downarrow \iff (\exists y)T_1(e, x, y)$$

is $\Sigma^0_1 \setminus \Pi^0_1$; and the set

$$F = \{e : (\forall x)(\exists y)T_1(e, x, y)\}$$

of codes of total recursive functions is $\Pi^0_2 \setminus \Sigma^0_2$.

**Recursive functions on the rationals.** We fix an effective enumeration of the rational numbers, e.g.,

$$q_k = (-1)^{(k)_0} \frac{(k)_1}{(k)_2 + 1}.$$  

A partial function $f : \mathbb{N} \rightarrow \mathbb{Q}$ is recursive if

$$f(n) = q_{\tilde{f}(n)}$$

with some recursive $\tilde{f} : \mathbb{N} \rightarrow \mathbb{N}$, and similarly with functions with either natural number or rational arguments and values. For example, the ordinary arithmetic operations $+, -, \cdot, \div$ are recursive on $\mathbb{Q}$.
CHAPTER 1

RECURSION ON POLISH SPACES

Recall that a metric space is a set $X$ together with a distance function $d : X \times X \rightarrow \mathbb{R}$ such that
\[ d(x, x) = 0, \quad d(x, y) = d(y, x), \quad d(x, z) \leq d(x, y) + d(y, z) \quad (x, y, z \in X). \]
The space is separable if some countable set $\{r_0, r_1, \ldots\} \subseteq X$ intersects every open ball
\[ B(x_0, \epsilon) = \{x \in X : d(x_0, x) < \epsilon\} \quad (\epsilon > 0), \]
and complete if every sequence $\{x_n\}_{n \in \omega}$ in $X$ which is Cauchy with respect to the metric $d$ has a limit.

A topological space $X$ is Polish if its topology is generated by a (compatible) metric $d$ with which $X$ is separable and complete; in other words, if $X$ is homeomorphic with a separable, complete metric space.

In this chapter we will initiate the study of computability and definability on recursive Polish spaces, for which a compatible metric can be specified recursively. They include the natural numbers $\omega$ (as a discrete space), the reals $\mathbb{R}$, Baire space $\mathcal{N}$ and most of the important Polish spaces which have been studied in analysis and topology; in fact, “relativized” versions of the results we will prove yield “refined versions” of theorems which apply to all Polish spaces.

1A. Recursively presented metric spaces

Suppose $X$ is a separable, complete metric space with distance function $d$. A recursive presentation of $X$ is any function $r : \omega \rightarrow X$ whose image
\[ r[\omega] = \{r_0, r_1, \ldots\} \]
is dense in $X$ and such that the relations
\[ P^d_r(i, j, k) \iff d(r_i, r_j) \leq q_k, \]
\[ Q^d_r(i, j, k) \iff d(r_i, r_j) < q_k \]
are recursive, where \( \{q_k\}_k \) is the standard enumeration of the rational numbers we have fixed; and a **recursively presented metric space** is a separable, complete metric space \( X \) together with a recursive presentation of it, formally a triple \((X, d, r)\).

**Problem 1A.1.** Prove that if \( d(r_i, r_j) \) is a rational number for all \( i, j \), then \( r \) is a recursive presentation if and only if the function

\[
f_d(i, j) = d(r_i, r_j)
\]

(on \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{Q} \)) is recursive.

The more complex definition we gave covers many interesting cases where \( d(r_i, r_j) \) is not rational, e.g., the plane \( \mathbb{R} \times \mathbb{R} \), with its Euclidean metric and

\[
r_i = (q(i)_0, q(i)_1).
\]

More generally, for any \( \varepsilon : \omega \to \omega \), an \( \varepsilon \)-**recursively presented metric space** is a separable, complete metric space \( X \) together with a function \( r : \omega \to X \) such that \( r[\omega] \) is dense in \( X \) and the relations \( P^{d,r}, Q^{d,r} \) above are \( \varepsilon \)-recursive.

**Problem 1A.2.** Prove that every separable, complete metric space is \( \varepsilon \)-recursive, for some \( \varepsilon : \omega \to \omega \).

We now specify recursive presentations for some of the basic spaces with which we will be dealing, skipping most of the proofs of recursiveness which are all routine.

(1) **The natural numbers** \( \omega \), with

\[
d(i, j) = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i \neq j \end{cases}
\]

and \( r(i) = i \).

(2) **The real numbers** \( \mathbb{R} \) with the usual distance function and \( r_i = q_i \).

We will also need the **closed unit interval** \([0, 1]\) as a subspace of \( \mathbb{R} \), with the obvious enumeration of the rational numbers in it:

\[
r_i = \begin{cases} q_i, & \text{if } 0 \leq q_i \leq 1, \\ 0, & \text{otherwise.} \end{cases}
\]

(3) **Baire space** \( \mathcal{N} = \omega^\omega \). Set

\[
\overline{\alpha}(t) = \langle \alpha(0), \ldots, \alpha(t-1) \rangle \quad (\alpha \in \mathcal{N}, t \in \omega).
\]

The topology of \( \mathcal{N} \) is generated by all (clopen) basic neighborhoods of the form

\[
N_u = \{ \alpha : (\exists t)[\overline{\alpha}(t) = u] \} = \{ \alpha : \overline{\alpha}(lh(u)) = u \}
\]
(which are empty unless Seq(u)), and the set of all ultimately 0 sequences is dense. So we can set

$$r_u(i) = (u)_i,$$

and check that the following metric defines the topology:

$$d(\alpha, \beta) = \begin{cases} 
0, & \text{if } \alpha = \beta, \\
1 + \frac{1}{\mu_n[\alpha(n) \neq \beta(n)]}, & \text{if } \alpha \neq \beta.
\end{cases}$$

Problem 1A.3. Prove that this presentation of $$N$$ is recursive.

(4) The Cantor set $$C = \{0, 1\}^\omega$$. This is a subspace of $$N$$, so we use the same metric and enumerate the ultimately 0 binary sequences by

$$r_u(i) = 1 - (u)_i.$$

(5) Finite products. Suppose $$X = X_1 \times \cdots \times X_k$$ and each $$X_i$$ is a metric space with distance function $$d_i$$ and recursive presentation $$r_i$$. It is easy to check that the function

$$d((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \max\{d_1(x_1, y_1), \ldots, d_k(x_k, y_k)\}$$

is a metric on $$X$$ which generates the product topology, and the function

$$r(i) = (r_1((i)_1), \ldots, r_k((i)_k))$$

is a recursive presentation of $$X$$ with it.

Alternative, we can use the “Euclidean metric”

$$d'((x_1, \ldots, x_k), (y_1, \ldots, y_k)) = \sqrt{d_1(x_1, y_1)^2 + \cdots + d_k(x_k, y_k)^2}.$$}

Problem 1A.4. Prove that $$d$$ and $$d'$$ are complete metrics on the product space $$X_1 \times \cdots \times X_k$$, and that $$r$$ is a recursive presentation for both.

The next exercise gives us a useful tool to “remetrize” a space so that it becomes bounded:

Problem 1A.5 (Bounding the metric). Suppose $$(X, d)$$ is a metric space and set

$$\tilde{d}(x, y) = \min(d(x, y), 1).$$

Prove that $$\tilde{d}$$ is a metric on $$X$$ which generates the same topology as $$d$$, and that if $$r$$ is a recursive presentation of $$(X, d)$$, then it is also a recursive presentation of $$(X, \tilde{d})$$.

(6) Products of sequences. Suppose that each $$X_n$$ is a metric space with bounded distance function $$d_n : X \to [0, 1]$$ and recursive presentation $$r_n : \omega \to X_n$$ and that the diagonal relations

$$P(n, i, j, k) \iff P^{d_n, r_n}(i, j, k), \quad Q(n, i, j, k) \iff Q^{d_n, r_n}(i, j, k)$$
are recursive. Define on the product
\[ X = \prod_{n \in \omega} X_n = \{ x : \omega \rightarrow \bigcup_n X_n : (\forall n)[x(n) \in X_n] \} \]
the distance function
\[ d(x, y) = \sum_{n \in \omega} \frac{1}{2^{n+1}} d_n(x(n), y(n)), \]
and let \( r(i)(n) = r_n((i)_n) \) so that \( r : \omega \rightarrow X \).

**Problem 1A.6.** Prove that with this distance function, \( r \) is a recursive presentation of \( X = \prod_{n \in \omega} X_n \).

An important special case of this construction is when \( X_n = Y \) for a fixed recursive metric space \( Y \), when the uniformity hypothesis is satisfied trivially. In particular, the Hilbert cube
\[ \mathbb{H} = \prod_{n \in \omega} [0, 1] \]
is recursively presented with this metric.

**1A.7. Coding the basic nbhds.** For each recursively presented metric space \( (X, d, r) \) and each \( s \in \omega \), let
\[ (1A-1) \quad N_s = N(X, s) = \left\{ x \in X : d(x, r(s)_n) < q(s)_1 \right\}, \]
\[ \text{center}(N_s) = \text{center}(N(X, s)) = r(s)_n, \]
\[ \text{radius}(N_s) = \text{radius}(N(X, s)) = q(s)_1. \]
The sequence \( (s \mapsto N_s) \) enumerates all “rational” open balls about points in the dense set given by the presentation (including the empty one, many times, whenever \( q(s)_1 \leq 0 \)), and the other two functions give us the centers and radii of these balls, when they are not empty. Notice that the sequence \( (s \mapsto N_s) \) depends on both the metric \( d \) and the presentation \( r \) of \( X \).

**Additional problems for Section 1A**

The next three easy problems simplify computations with nbhd codings, often to the point where the claimed results become obvious.

**Problem 1A.8.** (1) Prove that the basic nbhd relation
\[ R(n, s) \iff n \in N(\omega, s) \]
on \( \omega \) is recursive (as a binary relation on \( \omega \)).

(2) Prove that there are recursive functions \( g : \omega \rightarrow \omega \) and \( h : \omega^2 \rightarrow \omega \) such that
\[ \alpha \in N(X, s) \iff q(s)_1 > 0 \land (\forall i < g(s))[\alpha(i) = h(s, i)]. \]
The closure $\overline{A}$ of any set $A \subseteq X$ is the smallest closed superset of $A$. In particular, for basic nbhds, 

$$\overline{N}_s = \{x : d(x, \text{center}(N_s)) \leq \text{radius}(N_s)\}.$$

**Problem 1A.9.** Prove that for any recursively presented metric space $X$, the following relations are recursive:

- $N_s \neq \emptyset \iff \text{radius}(N_s) > 0$,
- $N_s \subseteq i N_t \iff d(\text{center}(N_t), \text{center}(N_s)) + \text{radius}(N_s) \leq \text{radius}(N_t)$,
- $N_s \subseteq i N_t \iff d(\text{center}(N_t), \text{center}(N_s)) + \text{radius}(N_s) < \text{radius}(N_t)$,
- $\overline{N}_s \cap i \overline{N}_t = \emptyset \iff \text{radius}(N_s) + \text{radius}(N_t) < d(\text{center}(N_s), \text{center}(N_t))$.

Check that the conditions on the right imply the relations on the left without the superscript $^i$ (which stands for “intensional”). Show that the full equivalences (without the $^i$) are true in many spaces which “do not have holes”, including $\mathbb{R}^2$, but they do not hold in all recursively presented metric spaces.

**Problem 1A.10.** Prove that for every recursively presented metric space $X$, there are recursive functions $f$ and $g$ such that

- $N(X, s) \cap N(X, t) = \bigcup_n N(X, f(s, t, n))$,
- $\bigcap_{l \leq m} N(X, (u)_l) = \bigcup_n N(X, g(u, m, n))$.

**Problem 1A.11.** Suppose $(X, d)$ is a metric space and set

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq 1.$$

Prove that $\tilde{d}$ is a metric on $X$ which generates the same topology as $d$, and that if $r$ is a recursive presentation of $(X, d)$, then it is also a recursive presentation of $(X, \tilde{d})$. (This is an interesting, somewhat more difficult alternative construction to that in Problem 1A.5.)

**1A.12. Recursive real numbers.** The (lower, open) Dedekind cut of a real number $x$ is the set

$$D(x) = \{r \in \mathbb{Q} : r < x\};$$

$x$ is **recursive** if $D(x)$ is a recursive set of rational numbers, and lower semi-recursive if $D(x)$ is recursively enumerable. It can be shown (easily) that the set $\mathbb{R}^{\text{rec}}$ of recursive real numbers is an ordered field, and (less easily) that the **recursive complex numbers** $\mathbb{R}^{\text{rec}} + i\mathbb{R}^{\text{rec}}$ form an algebraically closed field.

**Problem 1A.13.** Prove that there exist lower semi-recursive real numbers which are not recursive.
Problem 1A.14. For any real number $x > 0$, consider $[0, x]$ as a subspace of $\mathbb{R}$, i.e., with the metric $d(u, v) = |u - v|$.

1. Let

$$r^x(i) = \begin{cases} q_i & \text{if } 0 \leq q_i \leq x, \\ 0, & \text{otherwise}. \end{cases}$$

Prove that $r^x$ is a recursive presentation of $[0, x]$ if and only if $x$ is a recursive real number.

2. Prove that $[0, x]$ has a recursive presentation if and only if $x$ is lower semi-recursive.

For each compact Polish space $X$, let $C[X]$ be the space of all continuous functions $f : X \to \mathbb{R}$ with the usual supnorm distance function

$$d(f, g) = \max \{|f(x) - g(x)| : x \in X\}.$$

Problem 1A.15. Prove that $C[0, 1]$ has a recursive presentation.

1B. Semirecursive ($\Sigma^0_1$) pointsets

In developing recursion theory on $\omega$, we first define the recursive partial functions and then use them to define the semirecursive sets. It is possible to reverse this order, in fact Emil Post defined r.e. sets first (using his canonical systems) and then used them to define recursive partial functions by taking (5) of Theorem P.13 as a definition. It turns out that the extension of recursion theory to recursively presented metric spaces is smoother if we follow Post’s approach, and so we start with the semirecursive sets.

For this section, a pointset is any subset of a recursively presented metric space $X$, or more formally, a pair $(X, P)$ where $P \subseteq X$. We think of pointsets both as sets and as relations on the underlying space, and we will use interchangeably the customary notations for these, i.e., for $P \subseteq X$,

$$x \in P \iff P(x).$$

We will not be studying individual pointsets so much as pointclasses (in the category of recursively presented metric spaces in this section), i.e., collections of pointsets in several spaces. We start with the most basic of these.

A pointset $G \subseteq X$ is semirecursive or $\Sigma^0_1$ if

$$G = \bigcup_n N_{\varepsilon(n)} = \bigcup_n N(X, \varepsilon(n))$$

with some recursive $\varepsilon : \omega \to \omega$. More generally, $G$ is semirecursive in $\varepsilon_0 \in \mathcal{N}$ or $\varepsilon_0$-semirecursive if (1B-1) holds with some $\varepsilon$ which is recursive in $\varepsilon_0$. 

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1B. SEMIRECURSIVE ($\Sigma^0_1$) POINTSETS

Intuitively, $G$ is $\Sigma^0_1$ if it can be written as a recursive union of open balls, so in particular every $\Sigma^0_1$ pointset is open. The definition clearly depends on the recursive presentation of $\mathcal{X}$, but (as we will see) not on the specific enumeration of the rational open balls that we chose.

**Problem 1B.1.** Prove that a set $G \subseteq \mathcal{X}$ is open if and only if it is semirecursive in some $\varepsilon_0$.

It is natural to consider the family of all $\Sigma^0_1$ subsets of $\mathcal{X}$ as a recursive topology on $\mathcal{X}$. It is not closed under arbitrary unions, however, and in fact its closure properties and its structure are difficult to formulate: it is much better to study the pointclass of $\Sigma^0_1$ pointsets—in all recursively presented metric spaces—which has very regular structure and natural closure properties, as in the next few results.

**Lemma 1B.2** (Product Lemma). (1) A pointset $P \subseteq \mathcal{X}$ is $\Sigma^0_1$ if and only if it satisfies an equivalence

$$P(x) \iff (\exists s)[x \in N(\mathcal{X}, s) \& P^*(s)]$$

with some semirecursive $P^* \subseteq \omega$.

(2) A pointset $P \subseteq \mathcal{X} \times \mathcal{Y}$ is $\Sigma^0_1$ if and only if it satisfies an equivalence

$$P(x, y) \iff (\exists s, t)[x \in N(\mathcal{X}, s) \& y \in N(\mathcal{Y}, t) \& P^*(s, t)]$$

with some semirecursive $P^* \subseteq \mathbb{N}^2$ (and similarly with more factors).

**Proof.** (1) If $P = \bigcup_n N(\mathcal{X}, \varepsilon(n))$, set $P^*(s) \iff (\exists t)[s = \varepsilon(t)]$; and if $P$ satisfies (1B-2) with

$$P^*(s) \iff (\exists t)R(s, t)$$

where $R(s, t)$ is recursive, set

$$\varepsilon(n) = \begin{cases} (n)_0, & \text{if } R((n)_0, (n)_1), \\ 0, & \text{otherwise}, \end{cases}$$

so that $\varepsilon$ is recursive and $G = \bigcup_n N(\mathcal{X}, \varepsilon(n))$ (easily, using $N(\mathcal{X}, 0) = \emptyset$).

**Problem 1B.3.** Prove (2) of the Lemma.

**Problem 1B.4.** Prove that $P \subseteq \mathcal{X} \times \omega$ is $\Sigma^0_1$ if and only if there is a semirecursive $P^* \subseteq \omega^2$ such that

$$P(x, n) \iff (\exists s)[x \in N(\mathcal{X}, s) \& P^*(s, n)].$$

We need one more, simple definition before we collect in one theorem the basic properties of $\Sigma^0_1$.

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1B.5. Trivial functions. Suppose $X_1 \times \cdots \times X_n$ and $Y_1 \times \cdots \times Y_m$ are products and each $Y_i$ is $X_j$ for some $j$. A function

$$f : X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_m$$

is trivial (or explicit) if it is a composition of projections, i.e., if for some $\pi : \{1, \ldots, m\} \to \{1, \ldots, n\}$,

$$f(x_1, \ldots, x_n) = (x_{\pi(1)}, \ldots, x_{\pi(m)}).$$

For example, the map $(x_1, x_2, x_3, x_4) \mapsto (x_2, x_1, x_1)$ of $X_1 \times X_2 \times X_3 \times X_4$ into $X_2 \times X_1 \times X_1$ is trivial.

Theorem 1B.6. (1) A pointset $G \subseteq \omega^n$ is $\Sigma^0_1$ exactly when it is $\Sigma^0_1$ as a relation on $\omega^n$.

(2) The pointclass $\Sigma^0_1$ contains $\emptyset$, every recursively presented metric space $X$, every basic nbhd $N(X, s)$ as well as every basic nbhd relation

$$\{(x, s) : x \in N(X, s)\} \subseteq X \times \omega,$$

and it is closed under substitution of trivial functions and the operations $\&$, $\lor$, $\exists \subseteq$, $\forall \subseteq$, $\exists^\omega$ and $\exists^Y$ for every recursively presented metric space $Y$.

(3) For each $X$, there is a semirecursive $G^X \subseteq \omega \times X$ which enumerates all the $\Sigma^0_1$ subsets of $X$, i.e., $P \subseteq X$ is $\Sigma^0_1$ if and only if for some $e$,

$$P(x) \iff G^X(e, x) \quad (x \in X).$$

Moreover, for each $m \in \omega$, there is a recursive function $S^{X, m} : \omega^{1+m} \to \omega$ such that (skipping the superscripts)

$$G(e, \bar{y}, x) \iff G(S(e, \bar{y}), x) \quad (\bar{y} \in \omega^m, x \in X);$$

and for each semirecursive $P \subseteq \omega \times X$, there is some $e^*$ such that

$$P(e^*, x) \iff G^X(e^*, x) \quad (x \in X).$$

Proof is mostly trivial, using Lemma 1B.2 together with Corollary P.12 and Theorem P.13, the corresponding lists of properties of $\Sigma^0_1$ relations on $\omega$. For example, for (part of) (3), we set

$$G^X(e, x) \iff (\exists s)[x \in N(X, s) \& \bar{S}^0_1(e, s)]$$

in the notation of Corollary P.12 and then compute, for any semirecursive $P \subseteq X$:

$$P(x) \iff (\exists s)[x \in N(X, s) \& P^*(s)] \quad \text{(for some $P^*$, by Lemma 1B.2)},$$

$$\iff (\exists s)[x \in N(X, s) \& \bar{S}^0_1(e, s)] \quad \text{(for some $e$, by Corollary P.12)},$$

$$\iff G^X(e, x).$$
We outline the argument for (2) in some of the cases, primarily to make precise exactly what is meant by the claimed closure properties.

Suppose that \( G_1, G_2 \subseteq X \times Y \)
\[
G_1(x) \iff (\exists s)[x \in N_s & P_1(s)], \quad G_2(x) \iff (\exists t)[x \in N_t & P_2(t)],
\]
and compute:
\[
(G_1 \lor G_2)(x) \iff G_1(x) \lor G_2(x) \iff (\exists s)[x \in N_s & (P_1(s) \lor P_2(s))],
\]
\[
(G_1 \land G_2)(x) \iff G_1(x) \land G_2(x) \iff (\exists s)[x \in N_s & (P_1(s) & P_2(s))],
\]
where \( f \) is the recursive function given by Problem 1A.10.

If \( G \subseteq X \times \omega \) is \( \Sigma^0_1 \), then
\[
G(x, n) \iff (\exists s, t)[x \in N(X, s) & n \in N(\omega, t) & P(s, t)]
\]
with a semirecursive \( P \subseteq \mathbb{N}^2 \), and then using the codings,
\[
(\exists^* G)(x) \iff (\exists n)G(x, n)
\]
\[
\iff (\exists n, s, t)[x \in N(X, s) & n \in N(\omega, t) & P(s, t)]
\]
\[
\iff (\exists s)[x \in N(X, s) & (\exists n, t)[n \in N(\omega, t) & P(s, n)]];
\]
now the relation
\[
P^*(s) \iff (\exists n, t)[n \in N(\omega, t) & P(s, n)]
\]
is \( \Sigma^0_1 \), and so \( \exists^* G \) is \( \Sigma^0_1 \) by Lemma 1B.2 again.

For the more general case of closure under \( \exists^Y \), we use the fact that if \( G \subseteq X \times Y \) is \( \Sigma^0_1 \), then it is open; and so for each \( x \in X \) the section \( G_x = \{ y \in Y : G(x, y) \} \) is open; and so
\[
G_x \neq \emptyset \iff (\exists i)[0 \neq N(Y, i) \subseteq G_x]
\]
which then allows us to complete the computation as above.

\[\square\]

**Problem 1B.7.** Prove that the pointclass \( \Sigma^0_1 \) is closed under trivial substitutions, \( \forall^\leq \) and \( \exists^Y \).

It is worth emphasizing the usefulness of closure under trivial substitutions, which allows us to identify, permute or introduce new arguments in relations. For example, suppose
\[
P(x, y, z) \iff (\exists n)[Q(x, n) \land R(n, y, z)]
\]
1. Recursion on Polish spaces

Figure 1. $P^*$ uniformizes $P$.

with $Q$, $R$ in $\Sigma^0_1$, then

$$P(x, y, z) \iff (\exists n)[Q^*(x, y, z, n) \& R^*(x, y, z, n)]$$

with

$$Q^*(x, y, z, n) \iff Q(x, n),$$

$$R^*(x, y, z, n) \iff R(n, y, z)$$

with both $Q^*$ and $R^*$ in $\Sigma^0_1$ by closure under trivial substitutions, and hence $P$ is also $\Sigma^0_1$ by closure under $\&$ and $\exists^\omega$. (This use of explicit substitutions to simplify the formulation of closure properties is due to Gödel.)

1B.8. Products of $\omega$ and $N$. A space $X$ is of type 0 if it is $\omega$ or a product $X = X_1 \times \cdots \times X_k$ with each $X_i = \omega$, and of type 1 if it is $N$ or a product with each $X_i$ either $\omega$ or $N$ and at least one $X_i$ is $N$. Subsets of spaces of type 0 or 1 are pointsets of type 0 or 1 accordingly.

1B.9. Uniformization. A set $P^* \subseteq X \times Y$ uniformizes $P \subseteq X \times Y$ if

(a) $P^* \subseteq P$, and

(b) for all $x \in X$, $(\exists y)P(x, y) \implies (\exists y)P^*(x, y)$,

i.e., if $P^*$ is the graph of a function which selects a point $y$ from each non-empty section

$$P_x = \{ y \in Y : P(x, y) \}$$

of $P$. The uniformization result in (4) of the next theorem is rather trivial, but results and problems about the existence of definable uniformizations permeate the subject, and we will see many of them in the sequel.

1B.10. Recursive pointsets. A pointset $P \subseteq X$ is recursive if both $P$ and $\neg P = X \setminus P$ are $\Sigma^0_1$, which agrees with the definition we have for $P \subseteq \omega^n$ by Theorem P.13. Recursive pointsets are clopen, and so they are trivial in connected spaces like the reals. They are very important in studying pointsets of type 0 or 1.
Theorem 1B.11. (1) The pointclass of recursive sets contains $\emptyset$, every space $X$, every recursive relation on $\omega$, the application relations

$$\text{ap}(\alpha, t, w) \iff \alpha(t) = w$$
$$\overline{\text{ap}}(\alpha, t, w) \iff \overline{\alpha}(t) = w \ (\iff (\alpha(0), \ldots, \alpha(t-1)) = w),$$

and for each space $X$ of type 0 or 1 every basic nbhd $N(X, s)$ as well as the basic nbhd relation $\{(x, s) : x \in N(X, s)\}$; moreover, it is closed under substitution of trivial functions, $\neg$, $\&$, $\lor$, $\exists$, $\leq$, and $\forall$. (2) A pointset $P \subseteq X$ of type 0 or 1 is $\Sigma^0_1$ if and only if there is a recursive $R \subseteq X \times \omega$ such that

$$P(x) \iff (\exists n)R(x, n).$$

(3) A pointset $P \subseteq \omega^n \times X^m$ is $\Sigma^0_1$ if and only if there is a recursive $R \subseteq \omega^{n+m}$ such that

$$P(\vec{x}, \alpha_1, \ldots, \alpha_m) \iff (\exists t)R(\vec{x}, \overline{\alpha}_1(t), \ldots, \overline{\alpha}_m(t)),$$

and $R$ is monotone on sequence codes, i.e.,

$$u_1 \subseteq v_1, \ldots, u_m \subseteq v_m \ & \ R(\vec{x}, u_1, \ldots, u_m) \implies R(\vec{x}, v_1, \ldots, v_m).$$

(4) If $X$ is of type 0 or 1, then every semirecursive $P \subseteq X \times \omega$ can be uniformized by a semirecursive $P^* \subseteq P$.

Proof. The closure properties are immediate from 1B.6 and so are the facts that $\emptyset$, each $X$ and each recursive relation on $\omega$ are recursive, and the claim about the application relation follows easily from Lemma 1A.8. We skip the rest. $\dashv$

Problem 1B.12. Prove (2), (3) and (4) of the theorem.

The simple characterization in (2) of the theorem cannot be extended to arbitrary spaces, since it implies that every $\Sigma^0_1$ set—and hence every rational ball—is a countable union of clopen sets,

$$\{x : (\exists n)R(x, n)\} = \bigcup_n \{x : R(x, n)\},$$

which fails for $\mathbb{R}$ which has no non-trivial clopen subsets. (It is the characteristic property of 0-dimensional metric spaces.)

Additional problems for Section 1B

Problem 1B.13. Prove that for each $X$, the inequality relation

$$\{(x, y) : x \neq y\} \subseteq X \times X$$

is $\Sigma^0_1$.  

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1. Recursion on Polish spaces

Problem 1B.14. Prove that the relation $x < y$ on $\mathbb{R}$ is $\Sigma^0_1$ but not recursive.

Problem 1B.15. True or false:

1. For every $x_0 \in \mathbb{R}$, the set $\{x : x < x_0\}$ is $\Sigma^0_1$.
2. The set of reals $\{x : x < \pi = 3.14\ldots\}$ is $\Sigma^0_1$.

Problem 1B.16. Prove that if $r : \omega \to \mathcal{X}$ is a recursive presentation of $\mathcal{X}$, then the relations

$P(x, i, k) \iff d(x, r(i)) < q_k,$

$Q(x, i, k) \iff q_k < d(x, r(i))$

are both $\Sigma^0_1$, and the relation

$R(i, s) \iff r(i) \in N(\mathcal{X}, s)$

on $\omega^2$ is recursive.

Problem 1B.17. Prove that for every $\mathcal{X}$, the relations

$P(x, y, k) \iff d(x, y) < q_k,$

$Q(x, y, k) \iff d(x, y) > q_k,$

are $\Sigma^0_1$.

Problem 1B.18. Prove that $\Sigma^0_1$ is the smallest pointclass which contains all recursive pointsets of type 0 and for each recursively presented metric space $\mathcal{X}$ the relation $P^X \subseteq \mathcal{X} \times \omega^2$,

$P^X(x, i, k) \iff d(r_i, x) < q_k,$

and which is closed under trivial substitutions, $\&$, $\lor$, $\exists \subseteq$, $\forall \subseteq$ and $\exists^\omega$.

One consequence of this problem is that $\Sigma^0_1$ does not depend on the coding of nbhds that we chose, even though this coding was used in its definition.

Problem 1B.19. Prove that a pointset $P \subseteq \mathcal{X}$ is open if and only if it is a section of some semirecursive $Q \subseteq N \times \mathcal{X}$, i.e., for some $\varepsilon \in N$,

$P(x) \iff Q(\varepsilon, x) \iff Q_\varepsilon(x).$

1C. Recursive Polish spaces

A recursive Polish space is a set $\mathcal{X}$ together with a family $\mathcal{R} = \mathcal{R}(\mathcal{X})$ of subsets of $\omega \times \mathcal{X}$ such that for some $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and $r : \omega \to \mathcal{X}$ the following conditions hold:

(RP1) $(\mathcal{X}, d, r)$ is a recursively presented metric space, and
1C. Recursive Polish spaces

(RP2) $\mathcal{R}$ is the family of semirecursive subsets of $\omega \times \mathcal{X}$.

In particular, every recursively presented metric space $(\mathcal{X}, d, r)$ determines a unique recursive Polish $(\mathcal{X}, \mathcal{R}(\mathcal{X}))$, taking

$$\mathcal{R}(\mathcal{X}) = \text{the family of semirecursive subsets of } \omega \times \mathcal{X}.$$ 

A pair $(d, r)$ is compatible with $\mathcal{X}$ if it satisfies (RP1) and (RP2) so that it determines the (recursive) frame $\mathcal{R}$ of $\mathcal{X}$. We will also say that a metric $d$ is compatible with $\mathcal{X}$ if $(d, r)$ is compatible for some $r$, and that a sequence $r$ is compatible with $\mathcal{X}$ if $(d, r)$ is compatible for some $d$.

A subset $G \subseteq \mathcal{X}$ of a recursive Polish space is recursively open if it is the union of the sections of some $S \in \mathcal{R}(\mathcal{X})$,

$$(1C-1) \quad G = \bigcup_t S_t = \bigcup_{t \in \omega} \{x : (t, x) \in S\};$$

and it is open if

$$G = \bigcup_t G_t$$

is a union of a sequence of recursively open sets. We set

$$\mathcal{T} = \mathcal{T}(\mathcal{X}) = \text{the family of open subsets of } \mathcal{X}.$$ 

Lemma 1C.1. Let $(\mathcal{X}, \mathcal{R})$ be a recursive Polish space and $(d, r)$ any compatible pair.

1. A set $G \subseteq \mathcal{X}$ is recursively open if and only if it is semirecursive as a subset of $\mathcal{X}$ presented by $(d, r)$.

2. A set $G \subseteq \mathcal{X}$ is open if and only if it is open in the metric space $(\mathcal{X}, d)$.

In particular, $\mathcal{T}(\mathcal{X})$ is a Polish topology on $\mathcal{X}$ generated by $d$.

Proof. (1) If $G = \bigcup_t S_t$ is recursively open, then

$$x \in G \iff (\exists t) S(t, x)$$

with some semirecursive $S$, and so $G$ is semirecursive in $(\mathcal{X}, d, r)$ by Theorem 1B.6. Conversely, if $G$ is semirecursive in $(\mathcal{X}, d, r)$, then

$$x \in G \iff (\exists s)[x \in N(\mathcal{X}, s) \& P^*(s)]$$

with a semirecursive $P^* \subseteq \omega$ by Lemma 1B.2, and the relation

$$S(s, x) \iff x \in N(\mathcal{X}, s) \& P^*(s)$$

is semirecursive by Theorem 1B.6 again, so $S \in \mathcal{R}(\mathcal{X})$ and $G$ is recursively open.

(2) If $G$ is open in $(\mathcal{X}, \mathcal{R})$, then $G$ is a union of $(\mathcal{X}, d)$-semirecursive and hence open sets, so $G$ is open in $(\mathcal{X}, d)$. For the converse, notice that each open ball $N(\mathcal{X}, s)$ in $(\mathcal{X}, d)$ is semirecursive, once more by Theorem 1B.6; and hence every open set in $(\mathcal{X}, d)$ is a union of recursively open sets and hence open in $(\mathcal{X}, \mathcal{R})$. 

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The last claim follows because every recursively presented metric space is separable and complete by definition.

1C.2. Terminological convention. From now on and for the remainder of these notes we will work in the category of recursive Polish spaces, i.e.,

- a pointset is a subset of some recursive Polish space, formally a triple $(X, R, P)$ where $P \subseteq X$, and
- a pointclass is any collection of pointsets.

Since, however, the frame $R(X)$ and the topology $T(X)$ are determined by any compatible pair $(d, r)$, we will often use notions of and results about recursively presented metric spaces, sometimes assuming a compatible pair $(d, r)$ without explicit mention.

In particular, we will use “recursively open”, “semirecursive” and $\Sigma^0_1$ as synonyms.

1C.3. Relativization. An $\varepsilon$-recursive Polish space is a pair $(X, R)$ such that (RP1) and (RP2) hold, except that $(X, d, r)$ is an $\varepsilon$-recursively presented metric space and $R$ is the family of $\varepsilon$-semirecursive subsets of $\omega \times X$.

Problem 1C.4. Prove that every (classical) Polish space is $\varepsilon$-recursive for some $\varepsilon$.

All the results we will prove about recursive Polish spaces have obvious formulations for $\varepsilon$-recursive spaces constructed by inserting $\varepsilon$ in all the appropriate places. These are proved by “relativizing” the proofs for the absolute case and they yield “refined” results about all Polish spaces.

We view the natural numbers $\omega$, the reals $\mathbb{R}$, Baire space $\mathcal{N}$ and the Cantor set $C$ as recursive Polish spaces, with their frames specified by the compatible pairs listed in (1) – (4) in Section 1A. For the product operations, however, we need to verify that the frames of the products are determined by the frames of the factors and do not depend on any accidental properties of the compatible pairs chosen in definitions (5) and (6) of Section 1A. For finite products, this is a simple consequence of Lemma 1B.2:

Lemma 1C.5. For any two recursive Polish spaces $X, Y$, if $X \times Y$ is defined using any compatible pairs for $X$ and $Y$, then $S \in R(X \times Y)$ if and only if $S$ satisfies an equivalence

\[(x, y) \in S \iff (\exists s, t)[x \in S^1 \land y \in S^2 \land P^*(i, s, t)]\]

with some $S^1 \in R(X), S^2 \in R(Y)$, and some semirecursive $P^* \subseteq \omega^3$.

It follows that $R(X \times Y)$ is determined by $R(X)$ and $R(Y)$ (and similarly for products of more factors).
Proof. We show first a simple fact about recursively presented metric spaces:

Sublemma. Suppose \( X, Y \) are recursively presented metric spaces. A set \( P \subseteq \omega \times X \times Y \) is semirecursive if and only if it satisfies an equivalence

\[
P(n, x, y) \iff (\exists s, t) [x \in N(X, s) \& y \in N(Y, t) \& P^*(n, s, t)]
\]

with some semirecursive \( P^* \subseteq \omega^3 \).

Proof. If \( P \) satisfies (1C-3), then it is \( \Sigma_0^1 \), by the closure properties of \( \Sigma_0^1 \). For the converse we appeal to the representation of \( \Sigma_0^1 \) relations in Lemma 1B.2:

\[
P(n, x, y) \iff (\exists i, s, t)[n \in N(\omega, i) \& x \in N(X, s) \& y \in N(Y, t) \& P^*(i, s, t)]
\]

which gives the required representation for \( P \).

\( \dashv \) (Sublemma)

The claim in the Lemma now follows because the sets

\[
S^1 = \{(s, x) : x \in N(X, s)\}, \quad S^2 = \{(s, x) : x \in N(Y, s)\}
\]

are \( \Sigma_0^1 \) and hence in \( \mathcal{R}(X) \) and \( \mathcal{R}(Y) \) respectively, and so (1C-3) implies (1C-2).

Problem 1C.6. Prove that every recursive Polish space has a compatible pair \( (d, r) \) for which \( d(x, y) \leq 1 \) for all \( x, y \); and that if the infinite power

\[
X^\omega = \prod_{n \in \omega} X
\]

is defined using such a compatible pair by (6) in Section 1A, then \( \mathcal{R}(X^\omega) \) is determined from \( \mathcal{R}(X) \).

1C.7. The pointclass \( \Sigma_1^0 \). For any recursive Polish space \( X \), set

\[
\Sigma_1^0 | X = \text{the family of recursively open subsets of } X,
\]

and let

\[
\Sigma_1^0 = \bigcup_{X} \Sigma_1^0 | X = \text{the pointclass of all recursively open pointsets.}
\]

By Lemma 1C.1,

\[
P \in \Sigma_1^0 \iff P \text{ is } \Sigma_1^0 \text{ as a set in } (X, d, r)
\]

for any compatible pair \( (d, r) \), and so \( \Sigma_1^0 \) has all the basic properties enumerated in Theorem 1B.6.
Additional problems for Section 1C

Problem 1C.8. Prove that for any $i \in \omega$, a set $A \subseteq X$ is recursively open in a recursive Polish space $X$ if and only if $A = S_i$ for some sequence $S \in \mathcal{R}(X)$.

Problem 1C.9 (Disjoint unions). Suppose $(X, \mathcal{R}(X))$ and $(Y, \mathcal{R}(Y))$ are disjoint recursive Polish spaces, i.e., $X \cap Y = \emptyset$. Define the disjoint union $X \uplus Y$ so that it is a recursive Polish space whose topology $\mathcal{T}(X \uplus Y)$ is the classical topology on the union, i.e., for all $G \subseteq X \uplus Y$

$$G \in \mathcal{T}(X \uplus Y) \iff \left( \exists G_X \in \mathcal{T}(X), G_Y \in \mathcal{T}(Y) \right) [G = G_X \cup G_Y].$$

Problem 1C.10. Suppose $X_1 = (X, d_1, r_1)$ and $X_2 = (X, d_2, r_2)$ are two recursively presented metric spaces on the same universe $X$ and set

$$A_i(x, t, s) \iff d_i(x, r_i(t)) < q, \quad (i = 1, 2).$$

Prove that $\mathcal{R}(X_1) = \mathcal{R}(X_2)$ if and only if

(*) $A_1$ is semirecursive in $X_2$ and $A_2$ is semirecursive in $X_1$,

i.e., $X_1$ and $X_2$ determine the same recursive Polish space exactly when (*) holds. Infer that the two metrics on the product $X \times Y$ defined in (5) of Section 1A determine the same recursive Polish space.

1D. Recursive and $\Gamma$-recursive functions

A function

$$f : X \to Y$$
on one recursive Polish space to another is recursive if for every $S \in \mathcal{R}(Y)$, the pointset

(1D-1) $S^f(t, x) \iff S(t, f(x))$

is in $\mathcal{R}(X)$.

The definition makes it clear that the recursiveness of $f$ depends only on the frames of $X$ and $Y$, but for some computations, the next characterization which uses compatible pairs is more useful.

Lemma 1D.1 (Dellacherie). A function $f : X \to Y$ is recursive if and only if the nbhd diagram

(1D-2) $G^f(x, s) \iff f(x) \in N(Y, s)$
of $f$ is semirecursive.
1D. Recursive and $\Gamma$-recursive functions

Proof. We compute relative to fixed compatible pairs of $\mathcal{X}$ and $\mathcal{Y}$.

The pointset $S = \{(s, y) : y \in N(\mathcal{Y}, s)\}$ is semirecursive by Theorem 1B.6, and so it is a member of $R(\mathcal{Y})$. If $f$ is recursive, then

$$S^f = \{(s, x) : (s, f(x)) \in S\} = \{(s, x) : f(x) \in N(\mathcal{Y}, s)\}$$

is semirecursive, and then so is its “converse” $G^f = \{(x, s) : (s, x) \in S^f\}$. Conversely, if $G^f$ is semirecursive and $S \in R(\mathcal{Y})$, then by Lemma 1B.2

$$(t, y) \in S \iff (\exists s)[y \in N(\mathcal{Y}, s) & P^*(t, s)]$$

with a semirecursive $P^*(t, s)$, and so

$$(t, x) \in S^f \iff (t, f(x)) \in S \iff (\exists s)[f(x) \in N(\mathcal{Y}, s) & P^*(t, s)] \iff (\exists s)[G^f(x, s) & P^*(t, s)];$$

hence $S^f$ is semirecursive and so in $R(\mathcal{X})$. $\dashv$

Corollary 1D.2. Every recursive $f : \mathcal{X} \rightarrow \mathcal{Y}$ is continuous.

Proof. The inverse image $f^{-1}(N(\mathcal{Y}, s))$ of every basic nbhd is semirecursive in $\mathcal{X}$, and hence open. $\dashv$

In fact, by the theorem, if $f$ is recursive, then we can effectively compute arbitrarily good approximations to $f(x)$ if we have access to sufficiently good approximations of $x$; because

$$f(x) \in N_s \iff (\exists t)[x \in N_t & P^*(t, s)]$$

with a semirecursive $P^*$, and so given any $k$ we can search for some $t$ and $s$ such that

$$x \in N_t & P^*(t, s) \& \text{radius}(N_s) < 2^{-k}$$

and be assured that $d(f(x), \text{center}(N_s)) < 2^{-k}$. In other words, recursiveness is a “refinement” of continuity. Note however that not all “simple” continuous functions are recursive and that some of the most elementary properties of continuous functions do not carry over to recursive functions.

For example,

\begin{itemize}
  \item not all constant functions are recursive—only those whose constant value can be effectively approximated to any desired degree of accuracy;
  \item it makes no sense to ask if “$f$ is recursive at a single point $x$”;
  \item and it also makes no sense to ask whether a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of two variables is “separately recursive”, at least not immediately.
\end{itemize}

It is convenient to establish the simplest properties of recursive functions in a wider context which also covers many classes of functions, including the continuous ones.
1. Recursion on Polish spaces

1D.3. Σ-pointclasses. A Σ-pointclass is any collection of pointsets which contains $\Sigma^0_1$ and has its basic closure properties, i.e., it is closed under trivial substitutions, $\&, \vee, \exists \leq, \forall \leq$ and $\exists^\omega$—but is not necessarily closed under $\exists^Y$ for all spaces $Y$. A simple but important example beyond $\Sigma^0_1$ is

$\Sigma^0_1 = \text{the pointclass of all open pointsets},$

but we will meet many additional, natural Σ-pointclasses in the sequel.

Fix now a Σ-pointclass $\Gamma$. A function $f : \mathcal{X} \rightarrow \mathcal{Y}$ on one recursive Polish space to another is $\Gamma$-recursive if for every $S \in R(\mathcal{Y})$, the pointset $S^f$ defined in (1D-1) is in $\Gamma$. Dellacherie’s Lemma takes the following form for $\Gamma$-recursion which is proved exactly as before:

**Problem 1D.4.** Suppose $\Gamma$ is a Σ-pointclass. Prove that a function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is $\Gamma$-recursive if and only if its nbhd diagram $G^f \subseteq \mathcal{X} \times \omega$ defined in (1D-2) is in $\Gamma$.

In this terminology,

$f : \mathcal{X} \rightarrow \mathcal{Y}$ is recursive $\iff$ $f$ is $\Sigma^0_1$-recursive,

and $\Sigma^0_1$-recursion captures continuity:

**Problem 1D.5.** A function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is $\Sigma^0_1$-recursive if and only if it is continuous.

We start with a laundry list of simple properties of recursive and $\Gamma$-recursive functions.

**Theorem 1D.6.** Let $\Gamma$ be a Σ-pointclass.

1. A function $f : \mathcal{X} \rightarrow \omega$ is $\Gamma$-recursive if and only if $\text{Graph}(f) \in \Gamma$, where $\text{Graph}(f) = \{(x, n) : f(x) = n\}$.
   
   It follows that a function $f : \omega^n \rightarrow \omega$ is recursive in the present sense exactly when it is recursive as a function on $\omega$.

2. A function $f : \mathcal{X} \rightarrow \mathcal{N}$ is $\Gamma$-recursive if and only if the associated “unfolding function”

   \[
   f^*(x, s) = f(x)(s) \quad (f^* : \mathcal{X} \times \omega \rightarrow \omega)
   \]

   is $\Gamma$-recursive.

3. Every trivial function $f : \mathcal{X} \rightarrow \mathcal{Y}$ is recursive, and every recursive $f : \mathcal{X} \rightarrow \mathcal{Y}$ is $\Gamma$-recursive.

4. A function $f : \mathcal{X} \rightarrow \mathcal{Y} = \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_l$ is $\Gamma$-recursive if and only if

   \[
   f(x) = (f_1(x), \ldots, f_l(x))
   \]

   with suitable $\Gamma$-recursive functions $f_1, \ldots, f_l$. 

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(5) If \( f : X \rightarrow Y \) is \( \Gamma \)-recursive and \( g : Y \rightarrow Z \) is recursive, then the composition \( h : X \rightarrow Z \),

\[ h(x) = g(f(x)) \]

is \( \Gamma \)-recursive.

In particular, the class of recursive functions is closed under composition.

**Proof.** We outline proofs for some of these, mostly trivial facts.

1. Assume first that \( f : X \rightarrow \omega \) is \( \Gamma \)-recursive and take \( S(m, n) \iff m = n \); then \( S \) is in \( \Sigma^0_1 \), so \( S \in \cal R(\omega) \), and

\[ \text{Graph}(x, m) \iff f(x) = m \iff S^f(m, x) \]

in the notation of (1D-1), so \( \text{Graph}(f) \) is in \( \Gamma \). Conversely, if \( \text{Graph}(f) \in \Gamma \) and \( P \subseteq \omega \times \omega \) is in \( \Sigma^0_1 \), then

\[ P^f(n, x) \iff P(n, f(x)) \iff (\exists m) \{ f(x) = m \& P(n, m) \}, \]

so \( P^f \) is in \( \Gamma \) by the closure properties of a \( \Sigma \)-pointclass and \( f \) is \( \Gamma \)-recursive by Problem 1D.4.

2. Assume first that \( f : X \rightarrow N \) is \( \Gamma \)-recursive and take \( R(u, \alpha) \iff \alpha((u)_0) = (u)_1 \),

so that

\[ R^f(u, x) \iff f(x)((u)_0) = (u)_1. \]

Clearly \( R \) is \( \Sigma^0_1 \) and

\[ f^*(x, n) = m \iff R^f((n, m), x), \]

so \( f^* \) is \( \Gamma \)-recursive by 1D.6.

To prove the converse we appeal to 1A.8 by which there are recursive functions \( g \) and \( h \) such that

\[ \alpha \in N(N, s) \iff q(s)_1 > 0 \& (\forall i < g(s))[f(x)(i) = h(s, i)]. \]

Hence for \( f : X \rightarrow N \),

\[ G^f(x, s) \iff f(x) \in N(N, s) \]

\[ \iff q(s)_1 > 0 \& (\forall i < g(s))[f(x)(i) = h(s, i)] \]

\[ \iff q(s)_1 > 0 \& (\forall i < g(s))[f^*(x, i) = h(s, i)] \]

so that if \( f^* \) is \( \Gamma \)-recursive, then \( G^f \) is in \( \Gamma \) by 1D.6 again and the closure properties of \( \Gamma \).
1. Recursion on Polish spaces

(3) The second claim follows from the general fact that if \( \Gamma_1 \subseteq \Gamma_2 \) are both \( \Sigma \)-pointclasses, then for every \( f \),

\[
F \text{ is } \Gamma_1\text{-recursive } \implies f \text{ is } \Gamma_2\text{-recursive}.
\]

(5) We compute the nbhd diagram of the composition \( h \):

\[
g(f(x)) \in N(Z, s) \iff f(x) \in \{ y : g(y) \in N(Z, s) \}

\iff G^g(f(x), s)

\iff (\exists t)[f(x) \in N(Y, t) & P^*(t, s)]
\]

with a semirecursive \( P^* \) by Problem 1B.4, since \( G^g \) is \( \Sigma^0 \). Now the closure properties of \( \Gamma \) imply that the nbhd diagram of \( h \) is in \( \Gamma \).

It is not true of every \( \Sigma \)-pointclass \( \Gamma \) that the \( \Gamma \)-recursive functions are closed under composition or that \( \Gamma \) is closed under substitution of \( \Gamma \)-recursive functions, cf. Problem 1F.41. \( \Sigma^0 \) has both of these useful properties, by (5) of the theorem and

**Theorem 1D.7.** The pointclass \( \Sigma^0_1 \) of semirecursive sets is closed under recursive substitutions.

**Proof.** Suppose \( P \subseteq Y \) is \( \Sigma^0 \), so

\[
P(y) \iff (\exists s)[y \in N(Y, s) & P^*(s)]
\]

with a semirecursive \( P^* \) by 1B.2, and suppose \( f : X \to Y \) is recursive. Then

\[
Q(x) \iff P(f(x)) \iff (\exists s)[f(x) \in N(Y, s) & P^*(s)],
\]

and this is obviously semirecursive.

The **characteristic function** \( \chi_P : X \to \omega \) of a pointset \( P \subseteq X \) is defined as usual:

\[
\chi_P(x) = \begin{cases} 1, & \text{if } x \in P, \\ 0, & \text{otherwise}. \end{cases}
\]

**Problem 1D.8.** Suppose \( \Gamma \) is a \( \Sigma \)-pointclass and \( P \subseteq X \). Prove that

\[
P \in \Gamma \land \neg P = X \setminus P \in \Gamma \iff \chi_P \text{ is } \Gamma\text{-recursive}.
\]

In particular, \( P \) is recursive if and only if \( \chi_P \) is recursive.

**1D.9. Pointclass relativization.** With each pointclass \( \Gamma \) and each point \( z \in Z \) we associate the **relativization** \( \Gamma(z) \) of \( \Gamma \) to \( z \): \( P \subseteq X \) is in \( \Gamma(z) \) if there exists some \( Q \subseteq Z \times X \) in \( \Gamma \) such that

\[
P(x) \iff Q(z, x).
\]

The sets in \( \Sigma^0_1(z) \) are called **semirecursive in** \( z \) and the functions which are \( \Sigma^0_{1}(z) \)-recursive are called **recursive in** \( z \).
1D. Recursive and \(\Gamma\)-recursive functions

1D.10. \(\Gamma\)-recursive points. A point \(x \in \mathcal{X}\) is \(\Gamma\)-recursive if the set of codes of nbhds of \(x\) is in \(\Gamma\), i.e., if (the codeset of nbhds of \(x\))
\[U(x) = \{s : x \in N(X, s)\} \in \Gamma\]
We often call these simply the points in \(\Gamma\) and write
\[x \in \Gamma \iff x\text{ is }\Gamma\text{-recursive.}\]
The points in \(\Sigma^0_1\) are called recursive, the points in \(\Sigma^0_1(z)\) are called recursive in \(z\).

**Theorem 1D.11.** Let \(\Gamma\) be a \(\Sigma\)-pointclass.

1. For each point \(z\), \(\Gamma(z)\) is a \(\Sigma\)-pointclass.
2. A point \(x\) is \(\Gamma\)-recursive
   - if and only if \(x\) is \(\Delta\)-recursive,
   - if and only if for each \(Y\), the constant function \((y \mapsto x)\) is \(\Gamma\)-recursive.
3. If \(x\) is recursive in \(y\) and \(y\) is \(\Gamma\)-recursive, then \(x\) is \(\Gamma\)-recursive.
4. If \(f : \mathcal{X} \to \mathcal{Y}\) is \(\Gamma\)-recursive, then for each \(x \in \mathcal{X}\), \(f(x)\) is \(\Gamma(x)\)-recursive.

In particular, if \(f : \mathcal{X} \to \mathcal{Y}\) is recursive and \(x\) is recursive, then \(f(x)\) is also recursive.

**Proof.** (1) is very easy and the first claim in (2) follows from
\[x \notin N_x \iff (\exists t)[x \in N_t \& N_x \cap N_t = \emptyset],\]
using the recursive, “intensional” version of \(\cap\) in Problem 1A.9. The second part of (2) holds because if \(f : \mathcal{X} \to \mathcal{Y}\) is the constant function \(y \mapsto x\), then
\[G^f(y, s) \iff x \in N(X, s) \iff s \in U(x).\]
(3) If \(x\) is recursive in \(y\), then \(U(x)\) is in \(\Sigma^0_1(y)\), i.e., there is a \(\Sigma^0_1\) set \(P \subseteq Y \times \omega\) such that
\[s \in U(x) \iff P(y, s) \iff (\exists t)[y \in N(Y, t) \& P^*(t, s)] \iff (\exists t)[t \in U(y) \& P^*(t, s)]\]
with some semirecursive \(P^* \subseteq \omega^2\) by Problem 1B.4; now the hypothesis on
\(y\) and the closure properties of a \(\Sigma\)-pointclass imply that \(U(x) \in \Gamma\).
(4) is immediate since
\[s \in U(f(x)) \iff G^f(x, s)\]
and the last claim comes from (4) and (3).

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Additional problems for Section 1D

**Problem 1D.12.** Prove that if $r : \omega \to \mathcal{X}$ is compatible with $\mathcal{X}$, then it is recursive (as a function from $\omega$ to $\mathcal{X}$).

**Problem 1D.13.** Prove that every natural number is a recursive point in $\omega$, $\varepsilon \in \mathcal{N}$ is recursive if and only if the function $n \mapsto \varepsilon(n)$ is recursive, and $(x_1, \ldots, x_k)$ is recursive if and only if $x_1, \ldots, x_k$ are all recursive.

**Problem 1D.14.** Suppose $(d, r)$ is a compatible pair for $\mathcal{X}$. Prove that a point $x \in \mathcal{X}$ is recursive if and only if there is a recursive $\varepsilon \in \mathcal{N}$ such that
\[
\lim_{i \to \infty} r_{\varepsilon(i)} = x
\]
and for each $i$,
\[
d(r_{\varepsilon(i)}, r_{\varepsilon(i+1)}) < 2^{-i}.
\]

**Problem 1D.15.** Prove that the following functions with arguments and/or values in $\mathcal{N}$ and $\omega$ are recursive.
\[
f(\alpha, n) = \alpha(n), \quad n^\alpha = (n, \alpha(0), \alpha(1), \ldots),
\]
\[
f(\alpha, n) = \overline{\alpha}(n) = (\alpha(0), \ldots, \alpha(n-1)),
\]
\[
f(\alpha, i) = (\alpha)_i = (t \mapsto \alpha(i, t)),
\]
\[
f(\alpha) = \alpha^* = (t \mapsto \alpha(t+1))
\]
\[
f(\alpha_0, \ldots, \alpha_{k-1}) = \langle \alpha_0, \ldots, \alpha_{k-1} \rangle,
\]
where
\[
\langle \alpha_0, \ldots, \alpha_{k-1} \rangle(i, t) = \alpha_i(t) \quad \text{for } i = 0, \ldots, k-1,
\]
\[
\langle \alpha_0, \ldots, \alpha_{k-1} \rangle(n) = 0 \quad \text{if } n \neq \langle i, t \rangle \text{ for all } t, i < k.
\]

**Problem 1D.16.** Prove that a function $f : \mathcal{N} \to \omega$ is recursive if and only if
\[
(1D-1) \quad f(\alpha) = g(\mu t R(\overline{\alpha}(t)))
\]
with a recursive function $g : \omega \to \omega$ and a recursive relation $R \subseteq \omega$. More generally, $f : \omega^n \times \mathcal{N}^m \to \omega$ is recursive if and only if
\[
f(\overline{x}, \alpha_1, \ldots, \alpha_m) = g(\mu t R(\overline{x}, \overline{\alpha}_1(t), \ldots, \overline{\alpha}(t)))
\]
with suitable, recursive $g, R$.

**Hint:** If (1D-1) holds, then $\text{Graph}(f)$ is $\Sigma^0_1$ and so $f$ is recursive. For the converse use Lemma 1B.2 on $\mathcal{N} \times \omega$ and the standard representation of the nbhds in $\mathcal{N}$.
Problem 1D.17 (Minimalization). Assume that $\Gamma$ is a $\Sigma$-pointclass, $g : \omega \times X \rightarrow \omega$ is $\Gamma$-recursive and for each $x$, there is some $n$ such that $g(n, x) = 0$. Prove that the function $f$ defined by minimalization

$$f(x) = \mu n [g(n, x) = 0]$$

is $\Gamma$-recursive.

Problem 1D.18 (Primitive recursion). Suppose $f : \omega \times X \rightarrow Y$ is defined by the primitive recursion

$$(1D-2) \quad f(0, x) = g(x), \quad f(n + 1, x) = h(f(n, x), n, x),$$

where $g : X \rightarrow Y$ and $h : Y \times \omega \times X \rightarrow Y$ are recursive. Prove that $f$ is recursive. HINT: Define a semirecursive relation $P(n, t, s)$ on $\omega$ such that $f(n, x) \in N(Y, s) \iff (\exists t)[x \in N(X, t) \& P(n, t, s)]$ using the Second Recursion Theorem for $\Sigma^0_1$ on $\omega$, (3) of Corollary P.12.

It is not true that the set of $\Gamma$-recursive functions is closed under primitive recursion for every $\Sigma$-pointclass $\Gamma$, because of the following simple result:

Problem 1D.19. Prove that if $\Gamma$ is a $\Sigma$-pointclass closed under primitive recursion (1D-2) with $Y = Z = X$ for some specific space $X$, then the composition of any two $\Gamma$-recursive functions $\varphi, \psi : X \rightarrow X$ is also $\Gamma$-recursive.

Since it is not true that the $\Gamma$-recursive functions on $\mathcal{N}$ to $\mathcal{N}$ are closed under composition for every $\Sigma$-pointclass $\Gamma$ by Problem 1F.41, one import of this problem is that 1D.18 cannot be extended to all $\Sigma$-pointclasses.

Problem 1D.20. Prove that a function $f : X \rightarrow \mathbb{R}$ into the reals is recursive if and only if the relation

$$P(x, i, j) \iff q_i < f(x) < q_j$$

is semirecursive. Infer that

- a real number $x$ is recursive by 1D.10 exactly when it is a recursive real number by 1A.12;
- for each $X$ and every compatible pair, the metric $d : X \times X \rightarrow \mathbb{R}$ is recursive; and
- the embeddings $(n \mapsto n)$ and $q \mapsto q$ of $\mathbb{N}$ and $\mathbb{Q}$ into $\mathbb{R}$ are recursive.

Problem 1D.21. Prove that the operations of addition $+ : \mathbb{R}^2 \rightarrow \mathbb{R}$, multiplication $\cdot : \mathbb{R}^2 \rightarrow \mathbb{R}$ and absolute value $| \cdot | : \mathbb{R} \rightarrow \mathbb{R}$ are recursive. Infer that the functions $\max, \min : \mathbb{R}^2 \rightarrow \mathbb{R}$ are recursive as well.

Problem 1D.22. Prove that the set of recursive real numbers is a subfield of $\mathbb{R}$.
1. Recursion on Polish spaces

Problem 1D.23. Prove that if \( f : \omega \times \mathcal{X} \to \mathbb{R} \) is recursive, then the finite sum and product functions
\[
g(n, x) = \sum_{i \leq n} f(i, x), \quad h(n, x) = \prod_{i \leq n} f(i, x)
\]
are also recursive.

Problem 1D.24. Prove that if \( g : \omega \times \mathcal{X} \to \mathbb{R} \) is recursive and for all \( k \in \omega, x \in \mathcal{X}, g(k, x) \leq 1 \), then the function
\[
f(x) = \sum_{k \in \omega} 2^{-k} \cdot g(k, x) \quad (x \in \mathcal{X})
\]
is also recursive.

By routine arguments about infinite series, this result extends to the wider case of a bounded \( g \), where it is only assumed that \(|g(n, x)| \leq M\) for some \( M \) and all \( n, x \). We will not go into such topics here.

Problem 1D.25 (\( \Sigma^0_1 \)-Selection for type 0). Suppose \( \mathcal{X} \) is of type 0 or 1, \( \mathcal{Y} \) is of type 0, and \((\forall x)(\exists y)P(x, y)\). Prove that there is a recursive function \( f : \mathcal{X} \to \mathcal{Y} \) such that \((\forall x)P(x, f(x))\).

A homeomorphism \( \pi : \mathcal{X} \to \mathcal{Y} \) is recursive if both \( \pi \) and its inverse \( \pi^{-1} \) are recursive functions.

Problem 1D.26. Prove that if \( \mathcal{X} \) and \( \mathcal{Y} \) are of the same type 0 or 1, then they are recursively homeomorphic.

HINT: For type 1, take \( \mathcal{X} = \mathcal{N} \) and use induction on the number of factors in \( \mathcal{Y} \) after producing trivial homeomorphisms of \( \omega \times \mathcal{N} \) and \( \mathcal{N} \times \mathcal{N} \) with \( \mathcal{N} \).

A pointset \( A \subseteq \mathcal{X} \) is recursively enumerable if \( A = \emptyset \) or
\[
A - f[\mathbb{N}] = \{ f(0), f(1), \ldots \}
\]
for some recursive \( f : \mathbb{N} \to \mathcal{X} \).

Problem 1D.27. Prove that the set of recursive real numbers is not a recursively enumerable subset of \( \mathbb{R} \).

Problem 1D.28. Suppose \( d : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) is a recursive function which is a metric on a recursive Polish space \( \mathcal{X} \). Prove that there is a recursive real number \( \bar{a} > 0 \) such that the metric
\[
d_\bar{a}(x, y) = \bar{a}d(x, y)
\]
has a recursive presentation; in fact, \( \bar{a} \) can be chosen to be arbitrarily close to and above or below 1.

Problem 1D.29. Prove that if a function \( f : \mathcal{X} \to \mathcal{Y} \) is recursive in some \( z \), then \( f \) is continuous, and every continuous function is recursive in some \( \varepsilon \in \mathcal{N} \).
Careful! This does not mean that every continuous \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is a section

\[ f(x) = g(\varepsilon, x) \]

of some recursive \( g : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y} \), which fails even for \( \mathcal{X} = \mathcal{N} \) and \( \mathcal{Y} = \omega \), cf. Problem 1F.38.

**Problem 1D.30.** Prove that every recursively enumerable, infinite set \( A \subseteq \mathcal{X} \) can be enumerated without repetitions, i.e.,

\[ A = f[\omega] \text{ with } f : \omega \rightarrow \mathcal{X} \text{ a recursive injection.} \]

**Problem 1D.31.** Prove that every two discrete recursive Polish spaces are recursively homeomorphic, and give an example of two countable spaces which are not (even classically) homeomorphic. (*Discrete* means that every point is isolated.)

### 1E. Surjecting \( \mathcal{N} \) and embedding \( \mathbb{C} \)

Our main aim in this section is to prove that every recursive Polish space is the recursive image of the Baire space \( \mathcal{N} \), and that the Cantor set \( \mathbb{C} \) can be embedded into every uncountable recursive Polish space by a map \( \rho : \mathbb{C} \rightarrow \mathcal{X} \) which is recursive in some \( p(\mathcal{X}) \in \mathcal{N} \), an important parameter associated with \( \mathcal{X} \). Both results have numerous applications, and the second one illustrates how in some cases we need to introduce parameters which are not (necessarily) recursive.

We will also prove here the classical *Cantor-Bendixson Theorem* which helps understand the meaning of the parameter \( p(\mathcal{X}) \)—and which, perhaps, we should have proved earlier.

**Theorem 1E.1.** For every recursive Polish space \( \mathcal{X} \), there is a recursive surjection

\[ \pi : \mathcal{N} \rightarrow \mathcal{X} \]

of the Baire space onto it.

**Proof.** Fix a compatible pair \((d, r)\) of \( \mathcal{X} \) with countable dense subset \( r[\omega] = \{r_0, r_1, \ldots\} \), and to each \( \alpha \in \mathcal{N} \) assign the sequence \((n \mapsto x_n = x_n^\alpha)\) by the primitive recursion

\[
x_0 = r_\alpha(0) \\
x_{n+1} = \begin{cases} 
  r_\alpha(n+1) & \text{if } d(x_n, r_\alpha(n+1)) < 2^{-n}, \\
  x_n & \text{if } d(x_n, r_\alpha(n+1)) \geq 2^{-n}.
\end{cases}
\]

Now for each \( n \),

\[ d(x_n, x_{n+1}) < 2^{-n}, \]
so \((n \mapsto x^\alpha_n)\) is Cauchy and we can set 
\[
\pi(\alpha) = \lim_n x^\alpha_n.
\]

Notice that 
\[
d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+m-1}, x_{n+m}) \\
< 2^{-n} + 2^{-n-1} + \cdots + 2^{-n-m+1} < 2^{-n} \sum_{i=0}^{\infty} 2^{-i} = 2^{-n+1},
\]
so that taking limits as \(m \to \infty\), we have that for every \(\alpha\) and \(n\), 
\[
d(\pi(\alpha), x_n) < 2^{-n+1};
\]
hence \(\pi\) is continuous, since 
\[
\alpha(0) = \beta(0), \ldots, \alpha(n) = \beta(n) \implies x^\alpha_0 = x^\beta_0, \ldots, x^\alpha_n = x^\beta_n
\]
from which it follows immediately that 
\[
d(\pi(\alpha), \pi(\beta)) \leq d(\pi(\alpha), x^\alpha_n) + d(x^\alpha_n, \pi(\beta)) \\
< 2^{-n+1} + 2^{-n+1} = 2^{-n+2}.
\]

On the other hand, for each \(x \in X\) let 
\[
\alpha(n) = \text{least } k \text{ such that } d(x, r_k) < 2^{-n-1}
\]
and check that 
\[
\pi(\alpha) = \lim_n r_{\alpha(n)} = x.
\]

To see that \(\pi\) is recursive, check that 
\[
(1E-1) \quad \pi(\alpha) \in \mathcal{N}(X, s) \iff (\exists n)[d(\text{center}(N_s), x^\alpha_n) + 2^{-n+1} < \text{radius}(N_s)]
\]
which witnesses that the nbhd diagram \(G^\pi\) is semirecursive. \(\dash\)

**Problem 1E.2.** Why does (1E-1) imply that \(G^\pi\) is semirecursive?

We will appeal to this result many times, to infer that properties of Baire space which are preserved by recursive surjections hold of all recursive Polish spaces. We will also strengthen it in more than one ways, cf. Theorem 1F.12.

**Problem 1E.3.** Prove that every Polish space is the image of a continuous surjection \(\pi : \mathcal{N} \to X\).

**1E.4. Perfect sets.** Recall that a set \(F\) in a topological space is **perfect** if it is closed and has no isolated points. In particular, a recursive Polish space \(X\) is perfect exactly when 
\[
(\forall s) (N(X, s) \neq \emptyset \implies (\exists i, j)[r(i), r(j) \in N(X, s) \& r(i) \neq r(j)])
\]
and the closure \(\overline{N}_s\) of every basic nbhd in a perfect space \(X\) is perfect.
Theorem 1E.5 (Cantor-Bendixson). If $F$ is a closed pointset, then

\[(1E-2)\quad F = P \cup S\]

where $P$ is perfect, $S$ is countable and $P \cap S = \emptyset$. Moreover, there is only one such decomposition of $F$ into two disjoint sets, one perfect the other countable.

Proof. A point $x$ is a \textit{condensation point} of $F$ if every nbhd of $x$ intersects $F$ in an uncountable set. Put

\[P = \{ x : x \text{ is a condensation point of } F \}, \quad S = F \setminus P.\]

Since condensation points are clearly limit points and $F$ is closed, we have $P \subseteq F$, and by definition $P \cap S = \emptyset$, $F = P \cup S$.

We will show that $S$ is countable, $P$ is perfect and if $F = P' \cup S'$ with $P'$ perfect, $S'$ countable and $P' \cap S' = \emptyset$, then $P' = P$, $S' = S$.

To each $y \in S$ we can assign some basic nbhd $N_y$ such that $N_y \cap F$ is countable. Since there are only countably many basic nbhds altogether, there is a countable sequence $N_0, N_1, \ldots$ such that $S \subseteq \bigcup_{i \in \omega} (N_i \cap F)$ with each $N_i \cap F$ countable.

To prove that $P$ is closed, let $x$ be a limit point of $P$, $N$ any nbhd of $x$. Then some $x' \in N \cap P$, so $N$ is also a nbhd of $x'$ and it contains uncountably many points of $F$, so $x$ is a condensation point; hence $x \in P$. To prove $P$ perfect: if $x \in P$, then every nbhd of $x$ contains uncountably many points of $F$ of which only countably many can be in $S$—hence at least two are in $P$.

Finally, assume that $F = P' \cup S'$ with $P'$, $S'$ as above. If $x \in P'$ and $N$ is any nbhd of $x$, choose some nbhd $N_1$ of $x$ with $N_1 \subseteq N$ and check that $N_1 \cap P'$ is a perfect subset of $N_1 \cap P' \subseteq N \cap P'$, so $N \cap P'$ is uncountable and hence $x \in P'$; this proves $P' \subseteq P$. On the other hand, if $y \in S'$, then there is some nbhd $N$ of $y$ such that $N \cap P' = \emptyset$, since $P'$ is closed; hence $N \cap F = N \cap S'$, i.e., $N \cap F$ is countable and $y \in S$. \hfill ⊥

In the canonical decomposition (1E-2) of a closed pointset, we call $P$ the \textit{(perfect) kernel} and $S$ the \textit{scattered part} of $F$. It is sometimes useful to recall that the kernel comprises the condensation points of $F$, as in the following

\textbf{Corollary 1E.6} (to the proof). If a nbhd $N_x$ in a space $X$ is uncountable, then for every $n$ there exists uncountable nbhds $N^1_x, N^2_x$ such that

\[N^1_x \cup N^2_x \subset N_x, \quad N^1_x \cap N^2_x = \emptyset, \quad \text{radius}(N^1_x), \text{radius}(N^2_x) < 2^n.\]
Recursion on Polish spaces

1. Recursion on Polish spaces

\[ N_\sigma(\emptyset) \]

\[ N_\sigma(0) \]

\[ N_\sigma(0,0) \]

\[ N_\sigma(0,1) \]

\[ N_\sigma(1) \]

**Figure 2.** Embedding \( \mathbb{C} \) into an uncountable nhbd.

**Proof.** Choose a (slightly) smaller uncountable \( N_t \) such that \( N_t \subseteq N_s \) and then apply the Cantor-Bendixson Theorem to the closed set \( N_t \) which is uncountable; it has a non-empty kernel, and so it contains at least two condensation points \( x_1, x_2 \), and then we can choose sufficiently small nbhds around them with disjoint closures which are necessarily uncountable. \( \dashv \)

There is an obvious way to embed the Cantor set \( \mathbb{C} \) into any uncountable Polish space \( X \) by applying repeatedly this Corollary: start with an uncountable \( N_\sigma(\emptyset) \) of radius \( \leq 1 \), then choose uncountable \( N_\sigma(0), N_\sigma(1) \) with radii \( \leq 2^{-1} \) and disjoint closures within \( N_\sigma(\emptyset) \), etc. Every \( \alpha \in \mathbb{C} \) determines in this way an infinite sequence of closed, nested nbhds which contains a single point \( x^\alpha \) in its intersection, and the map \( \alpha \mapsto x^\alpha \) is an injection.

The construction applies to any perfect \( A \subseteq X \) which can be viewed as a (classical) Polish space with the induced topology. It verifies the following important fact, the main tool for showing that specific pointsets are equinumerous with the continuum:

**Corollary 1E.7.** Every pointset \( A \) which contains a non-empty perfect set has cardinality \( 2^{\aleph_0} \).

The next, basic result gives a definable version of this construction.

**1E.8. The local size parameter.** For any recursive Polish space \( X \), let

\[ P_X(s) \iff N(X, s) \text{ is uncountable}, \]

and let \( p(X) \in \mathcal{N} \) be the characteristic function of \( P_X \). If \( X \) is perfect then

\[ P_X(s) \iff N(X, s) \neq \emptyset \iff \text{radius}(N(X, s)) > 0, \]

because if \( N_s \) is non-empty, then it contains the closure \( \overline{N_t} \) of a bit smaller non-empty nbhd; \( N_t \) cannot have any isolated points, since they would also be isolated in \( X \); and so the closure \( \overline{N_t} \subseteq N_s \) is perfect has cardinality \( 2^{\aleph_0} \).
So for a perfect $X$, $P_X$ is a recursive relation and $p(X)$ is a recursive point in $\mathcal{N}$.

**Theorem 1E.9.** For every uncountable, recursive Polish space $X$, there is an injection

$$\pi : \mathbb{C} \to X$$

which is recursive in $p(X)$.

In particular, the Cantor set can be injected recursively into every perfect, recursive Polish space.

**Proof.** We need a simple

Sublemma. There is a function $\sigma : \{0, 1\}^{<\omega} \to \omega$ on finite binary sequences such that the following conditions hold, with $N_s = N(X, s)$:

(i) If $u$ is a proper initial segment of $v$, then $N_{\sigma(v)} \not\subseteq N_{\sigma(u)}$.

(ii) If $u$ and $v$ are incompatible, then $N_{\sigma(u)} \cap N_{\sigma(v)} = \emptyset$.

(iii) If $u = (t_0, \ldots, t_{n-1})$ has length $n$, then $\text{radius}(N_{\sigma(u)}) \leq 2^{-n}$.

(iv) For some $\tilde{\sigma} : \omega \to \omega$ which is recursive in $p(X)$,

$$\sigma(t_0, \ldots, t_{n-1}) = \tilde{\sigma}((t_0, \ldots, t_{n-1}))$$

Proof. Put

$$P(n, i, j) \iff N_n \text{ is countable}$$

$$\vee \Big( N_i \text{ and } N_j \text{ are both uncountable}$$

$$& N_i \subseteq N_n \& N_j \subseteq N_n \& N_i \cap N_j = \emptyset$$

$$& \text{radius}(N_i), \text{radius}(N_j) < \frac{1}{2} \text{radius}(N_n) \Big).$$

Now $(\forall n)(\exists j) P(n, i, j)$ by Corollary 1E.6, and $P$ is semirecursive (in fact recursive) in $p(X)$, so by the $\Sigma^0_1$-Selection Lemma (4) of Theorem P.13, there is a $p(X)$-recursive $f(n) = (g(n), h(n))$ so that $(\forall n)P(n, g(n), h(n))$.

Fix some $z_0 \in \omega$ so that $N(X, z_0)$ is uncountable and $\text{radius}(N(X, z_0)) \leq 1$, and put

$$Q(u, m) \iff \text{Seq}(u) \& \text{lh}(u) > 0 \& (\forall i < \text{lh}(u))[i]((u)_i \leq 1]$$

$$\& (\exists z)[(z)_0 = z_0 \& (\forall i < \text{lh}(u) - 1) \{(i) = 0 \implies (z)_{i+1} = g((z)_i)]$$

$$\& [(i) = 1 \implies (z)_{i+1} = h((z)_i) \& (z)_{\text{lh}(u)} = m].$$

Clearly $Q$ is $p(X)$-semirecursive and with

$$A = \{ u : \text{Seq}(u) \& (\forall i < \text{lh}(u))[(u)_i \leq 1] \}$$

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we have $(\forall u \in A)(\exists m)Q(u,m)$, so by $\Sigma^0_1$-Selection again, there is a $p(X)$-recursive $\tilde{\sigma}$ such that $(\forall u \in A)Q(u,\tilde{\sigma}(u))$. It is not hard to chase the definitions and check that the function

$$
\sigma(t_0, \ldots, t_{n-1}) = \tilde{\sigma}(\langle t_0, \ldots, t_{n-1} \rangle)
$$

has the required properties. ⊣ (Sublemma)

We now work as in the proof of Theorem 1E.1: set for each $\alpha \in C$

$$
x^\alpha_n = \text{center}\left(N(X, \sigma(\alpha(0), \ldots, \alpha(n-1)))\right),
$$

$$
\pi(\alpha) = \lim_{n \to \infty} x^\alpha_n,
$$

and check easily by the properties of $\sigma$ that $\pi$ is continuous and injective. That it is $p(X)$-recursive follows from the equivalence

$$
\pi(\alpha) \in N_s \iff (\exists n)[N(X, \tilde{\sigma}(\langle \alpha(0), \ldots, \alpha(n-1) \rangle)) \subseteq^s N(X, s)].
$$

Problem 1E.10. Prove that for every uncountable Polish space $X$, there is a continuous injection $\pi : C \to X$.

Additional problems for Section 1E

It was once (early on and briefly) thought that the Continuum Hypothesis could be settled by proving that every uncountable set of real numbers has a non-empty perfect subset and then applying Lemma 1E.7. This does not work:

Problem 1E.11 (AC). Prove that there is an uncountable subset of the real interval $[0, 1]$ which has no non-empty perfect subset.

The proof of this requires the Axiom of Choice, and the set “produced” by it is necessarily not definable, by classical results of Solovay (that we will not discuss here). So it may be possible to show that there are no such “bad” pointsets in a specific pointclass $\Gamma$, which then implies that $\Gamma$ satisfies the Continuum Hypothesis. Put

(1E-1) $P(\Gamma) \iff \Gamma$ has property $P$

$\iff$ every uncountable $A \in \Gamma$ has a non-empty perfect subset,

so the pointclass of closed sets has property $P$ by the Cantor-Bendixson Theorem. We will show in the sequel some substantial generalizations and refinements of this result.
1F. The Kleene pointclasses

We introduce here some of the most significant pointclasses of effective and classical descriptive set theory, by starting with $\Sigma^0_0$ or $\Sigma^0_1$ and applying repeatedly complementation and quantification on $\omega$ and $\mathbb{N}$. The method goes back to Lebesgue [1905] and provides a natural method for establishing properties of these pointclasses by induction on their definitions.

For any pointclass $\Gamma$ and any space $\mathcal{X}$, let $\Gamma|\mathcal{X} = \{P \subseteq \mathcal{X} : P \in \Gamma\}$; this is the restriction of $\Gamma$ to $\mathcal{X}$. Most of the operations on pointclasses that we will study are defined by distribution, i.e., setting $\Phi\Gamma = \{\Phi(P_1, \ldots, P_n) : P_1, \ldots, P_n \in \Gamma & \Phi(P_1, \ldots, P_n) \text{ is defined}\}$ where $\Phi$ is an $n$-ary operation on pointsets whose domain of definition may be rather complex and is typically specified using the restriction operation. For example,

$$(\neg P)(x) \iff \neg P(x), \quad (P \subseteq \mathcal{X}),$$

$$(\exists^\mathcal{N} P)(x) \iff (\exists^\omega P(x, \alpha)) (P \subseteq \mathcal{X} \times \mathcal{N}),$$

and then by distribution,

$$(\neg \Gamma)|\mathcal{X} = \{\neg P : P \in \Gamma|\mathcal{X}\},$$

$$(\exists^\mathcal{N} \Gamma)|\mathcal{X} = \{\exists^\mathcal{N} P : P \in \Gamma| (\mathcal{X} \times \mathcal{N})\}.$$  

Kleene’s arithmetical pointclasses are defined by recursion on $n$, starting with the semirecursive sets and iterating complementation $\neg$ and $\exists^\omega$, quantification over $\omega$:

$$\Sigma^0_n = \text{all semirecursive pointsets},$$

$$\Pi^0_n = \neg \Sigma^0_n, \quad \Sigma^{0}_{n+1} = \exists^\omega \Pi^0_n, \quad \Delta^n_0 = \Sigma^0_n \cap \Pi^0_n.$$  

The analytical pointclasses are defined in the same way, starting (in effect) with $\Pi^0_n$ and iterating quantification over $\mathcal{N}$ and complementation:

$$\Sigma^1_n = \exists^\mathcal{N} \Pi^0_n, \quad \Pi^1_n = \neg \Sigma^1_n, \quad \Sigma^{1}_{n+1} = \exists^\mathcal{N} \Pi^1_n, \quad \Delta^{1}_n = \Sigma^1_n \cap \Pi^1_n.$$  

The pointsets in $\bigcup_n \Sigma^0_n$ are called arithmetical because $(\bigcup_n \Sigma^0_n)|\omega$ comprises exactly the sets of natural numbers which are definable in the first-order language of arithmetic, and those in $\bigcup_n \Sigma^1_n$ are called analytical because $(\bigcup_n \Sigma^1_n)|\mathcal{N}$ comprises those subsets of $\mathcal{N}$ which are definable in the formal language of analysis, i.e., second-order arithmetic. These characterizations of the Kleene pointclasses are trivial, once the definitions are given, and we will not go into them here.
The \textbf{relativized Kleene pointclasses} $\Sigma_0^n(z)$, $\Pi_0^n(z)$, $\Sigma_1^n(z)$, $\Pi_1^n(z)$, are defined by the general relativization process we described in Section 1D, and we also set

$$
\Delta_0^n(z) = \Sigma_0^n(z) \cap \Pi_0^n(z), \quad \Delta_1^n(z) = \Sigma_1^n(z) \cap \Pi_1^n(z).
$$

One should be careful with this notation, however, since it is not the case that $\Delta_0^n(z)$ is the relativization of $\Delta_0^n$ to $z$, see Problem 1F.38. The sets in $\bigcup_n \Sigma_0^n(z)$ and $\bigcup_n \Sigma_1^n(z)$ are naturally called (respectively) \textit{arithmetical} and \textit{analytical in $z$}.

The simple, elementary properties of the Kleene pointclasses can be established very easily, by induction following their definition, very much like the classical properties of the arithmetical hierarchy on $\omega$.

\textbf{1F.1. Normal forms.} For example, $P$ is $\Sigma_0^2$ if there is a $\Pi_0^1$ set $F$ so that

$$
P(x) \iff (\exists t)F(x, t),
$$

$P$ is $\Sigma_0^3$ if there is a $\Sigma_0^1$ (semirecursive) set $G$ so that

$$
P(x) \iff (\exists t_1)(\forall t_2)G(x, t_1, t_2),
$$

etc. Similarly, $P$ is $\Sigma_1^1$ if there is a $\Pi_0^1$ set $F$ such that

$$
P(x) \iff (\exists \alpha)F(x, \alpha),
$$

$P$ is $\Sigma_1^2$ if there is a semirecursive $G$ such that

$$
P(x) \iff (\exists \alpha_1)(\forall \alpha_2)G(x, \alpha_1, \alpha_2),
$$

etc. These forms become a bit simpler for spaces of type 0 or 1 because of the characterization in Theorem 1B.11. Thus, if $P$ is a pointset of type 0 or 1, then $P$ is $\Pi_0^1$ if there is a recursive $R$ such that

$$
P(x) \iff (\forall t)R(x, t),
$$

$P$ is $\Sigma_0^2$ if there is a recursive $R$ such that

$$
P(x) \iff (\exists t_1)(\forall t_2)R(x, t_1, t_2),
$$

$P$ is $\Sigma_1^2$ if there is a recursive $R$ such that its negation $\neg R$ is monotone in the sequence codes (as in (3) of Theorem 1B.11) and

$$
P(x) \iff (\exists \alpha)(\forall \beta)(\exists t)R(x, \alpha(t), \beta(t)),
$$

etc.

The key to the closure properties of the Kleene pointclasses are the closure properties of $\Sigma_0^1$ given in 1B.6 and 1D.7.

\textbf{Lemma 1F.2.} If a pointclass $\Lambda$ contains all recursive pointsets and is closed under recursive substitutions, $\&$, $\lor$, $\exists \subseteq$ and $\forall \subseteq$, then the pointclasses

$$
\neg \Lambda, \ \exists \neg \Lambda, \ \forall \neg \Lambda, \ \forall \forall \Lambda
$$

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are also closed under recursive substitutions, \& , \lor, \exists^\leq and \forall^\leq, and they include \Lambda. Moreover:

(a) \exists^\omega \Lambda is closed under \exists^\omega,
(b) \forall^\omega \Lambda is closed under \forall^\omega,
and for every \forall \forall

(c) \exists^N \Lambda is closed under \exists^\forall and
(d) \forall^N \Lambda is closed under \forall^\forall.

Proof is quite trivial, by the usual “pairing tricks” which allow us to contract quantifiers using recursive substitutions. For example, to show closure of \exists^N \Lambda under conjunction, suppose \exists P, Q \subseteq X \times N and compute:

(\exists \alpha) P(x, \alpha) \& (\exists \beta) Q(x, \beta) \iff (\exists \gamma)[P(x, (\gamma)_0) \& Q(x, (\gamma)_1)],

which shows that (\exists^N P \& \exists^N Q) \in \exists^N \Lambda because \Lambda is closed under recursive substitutions and conjunction. The proofs of (c) and (d) require an appeal to the basic Theorem 1E.1 and we leave them for problems. ⊣

Problem 1F.3. In the notation and with the hypotheses of the Lemma, prove that \forall^N \Lambda is closed under recursive substitutions; which means that if Q \in \forall^N \Lambda, Q \subseteq \forall, f : X \to \forall is recursive and

P(x) \iff Q(f(x)),

then P \in \forall^N \Lambda.

Problem 1F.4. Prove (c) and (d) in the Lemma.

Theorem 1F.5. All Kleene pointclasses contain all recursive pointsets and are closed under recursive substitutions, \& , \lor, \exists^\leq and \forall^\leq. Moreover:

• \Sigma^0_n is closed under \exists^\forall,
• \Pi^0_n is closed under \forall^\forall,
• \Sigma^1_n is closed under \forall^\forall and \exists^\forall for all \forall,
• \Pi^1_n is closed under \exists^\forall and \forall^\forall for every \forall; and
• \Delta^0_n and \Delta^1_n are closed under \neg.

In particular, for all n \geq 1, \Sigma^0_n, \Sigma^1_n, \Pi^1_n and \Delta^1_n are all \Sigma-pointclasses.

Proof is easy by two inductions on n, using Lemma 1F.2. ⊣

Problem 1F.6. Prove that \Delta^1_n is closed under \exists^\forall and \forall^\forall.

Theorem 1F.7. The diagram of inclusions in Figure 3 holds for the Kleene pointclasses, and every arithmetical pointset is \Delta^1.

Proof. There is only one, non-trivial argument that is needed:

Sublemma. \Sigma^1_1 \subseteq \Sigma^2_2.
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\[ \begin{array}{cccccc}
\Sigma^0_0 & \subseteq & \Sigma^0_1 & \subseteq & \Sigma^0_2 & \subseteq \\
\Delta^0_1 & \subseteq & \Delta^0_2 & \subseteq & \Delta^0_3 & \subseteq \\
\Pi^0_1 & \subseteq & \Pi^0_2 & \subseteq & \Pi^0_3 & \subseteq \\
\end{array} \]

\[ \begin{array}{cccccc}
\Sigma^1_0 & \subseteq & \Sigma^1_1 & \subseteq & \Sigma^1_2 & \subseteq \\
\Delta^1_1 & \subseteq & \Delta^1_2 & \subseteq & \Delta^1_3 & \subseteq \\
\Pi^1_1 & \subseteq & \Pi^1_2 & \subseteq & \Pi^1_3 & \subseteq \\
\end{array} \]

\textbf{Figure 3.} The Kleene pointclasses.

\textit{Proof.} All recursive pointsets of type 0 are clearly in \( \Sigma^0_2 \) (by vacuous quantification). Also, \( \Sigma^0_2 \) is closed under trivial substitutions, \& , \( \lor \), \( \exists \leq \), \( \forall \leq \) and \( \exists^\omega \) by 1F.5; hence to show that \( \Sigma^0_1 \subseteq \Sigma^0_2 \) by applying 1B.18 it is enough to verify that for each space \( X \) and compatible metric \( d \), the relation

\[ P^X(x, i, k) \iff d(r_i, x) < q_k \]

is in \( \Sigma^0_2 \), and this holds because of the equivalence

\[ P^X(x, i, k) \iff (\exists l)[q_l < q_k \& d(r_i, x) \leq q_l] \iff (\exists l)[q_l < q_k \& \neg(q_l < d(r_i, x))]. \]

and (again) the closure properties of \( \Sigma^0_2 \). \( \dashv \) (Sublemma)

Using this sublemma at the basis and a lot of vacuous quantification, we verify that

\[ \Sigma^0_n \cup \Pi^0_n \subseteq \Delta^0_{n+1} = \Sigma^0_{n+1} \cap \Pi^0_{n+1}. \]

The argument is similar for the analytical pointclasses, and the second claim follows by the closure properties of \( \Delta^1_1 \). \( \dashv \)

\textbf{1F.8.} Parametrizations and universal sets. A pointset \( G \subseteq Y \times X \) is \textbf{universal for} \( \Gamma \upharpoonright X \), if \( G \) is in \( \Gamma \) and the map

\[ y \mapsto G_y = \{ x \in X : G(y, x) \} \]

is a \textbf{parametrization} of \( \Gamma \upharpoonright X \) on \( Y \); i.e., for \( P \subseteq X \),

\[ P \in \Gamma \iff \text{for some } y \in Y, P = G_y. \]

A pointclass \( \Gamma \) is \textbf{\( Y \)-parametrized} if for every space \( X \), there is some \( G \subseteq Y \times X \) which is universal for \( \Gamma \upharpoonright X \).

Part (3) of Theorem 1B.6 says exactly that \( \Sigma^0_1 \) is \( \omega \)-parametrized—and a bit more, which we do not need right now. We show instead a more general result which then implies that all the inclusions in Figure 3 are generally proper.

\textbf{Theorem 1F.9} (Parametrization for the Kleene pointclasses). If \( \Gamma \) is any of the Kleene pointclasses \( \Sigma^0_n, \Pi^0_n, \Sigma^1_n, \Pi^1_n \), then:

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(1) $\Gamma$ is $\omega$-parametrized.

(2) $\Gamma$ is $\cal Y$-parametrized for every perfect, recursive Polish space $\cal Y$.

Proof. Both (1) and (2) follow by an easy induction once they are established for $\Sigma^0_1$. For example, if $H(y, n, x)$ is universal for $\Pi^0_3 \upharpoonright (\omega \times X)$, then

$$G(y, x) \iff (\exists n)H(y, n, x)$$

is clearly universal for $\Sigma^0_4 \upharpoonright X$. So it suffices to constructs universal sets for $\Sigma^0_1$, and for (1), this is part of (3) of Theorem 1B.6.

Proof of (2) for $\Sigma^0_1$. Recall from the proof of Theorem 1E.9 the function $\sigma$ which assigns to each finite binary sequence $u$ a nbhd $N(Y, \tilde{\sigma}(u))$ in $Y$ with certain properties (i), (ii), (iii) and (iv) listed there. Let

$$C(y, e) \iff (\exists u)\left[\text{Seq}(u) \land \text{lh}(u) = e + 1 \land (\forall i < e)[(u)_i = 0] \land (u)_e = 1 \land y \in N(Y, \tilde{\sigma}(u))\right].$$

This is $\Sigma^0_0$ (because we have assumed that $\cal Y$ is perfect and so the function $\tilde{\sigma}$ is recursive), and the properties of $\sigma$ imply that for any $y$,

$$C(y, e) \text{ can hold for at most one } e;$$

because if $C(y, e)$ and $C(y, e')$ are both true with $e \neq e'$, then there are incompatible, binary sequences coded by some $u, u'$ for which we have

$$y \in N(Y, \tilde{\sigma}(u)) \cap N(Y, \tilde{\sigma}(u')),$$

which cannot happen. Fix some $G(e, s)$ which parametrizes $\Sigma^0_1 \upharpoonright \omega$ and for any space $X$ set

$$H(y, x) \iff (\exists e)\left[C(y, e) \land (\exists s)[x \in N(X, s) \land G(e, s)]\right].$$

This is also $\Sigma^0_0$, and it is enough to show that its sections comprise exactly the $\Sigma^0_1$ subsets of $X$.

Fix $y \in \cal Y$. If $H_y \neq \emptyset$, then $H(y, x)$ holds for some $x$, then there is a unique $e$ such that $C(y, e)$, and for this $e = e(y)$ we have

$$H_y(x) \iff H(y, x) \iff (\exists s)[x \in N(X, s) \land G(e, s)],$$

so that $H_y$ is $\Sigma^0_1$. Conversely, if

$$P(x) \iff (\exists s)[x \in N(X, s) \land G(e, s)]$$

is any semirecursive subset of $X$, let

$$u = (0, \ldots, 0, 1)$$

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and let \( y \) be any point in \( N(\mathcal{Y}, \tilde{\sigma}(u)) \); chasing the definitions we get
\[
H_y(x) \iff P(x),
\]
which completes the argument. \( \dashv \)

**Lemma 1F.10** (Diagonalization). Let \( \Gamma \) be a pointclass such that for every space \( \mathcal{X} \) and every pointset \( P \subseteq \mathcal{X} \times \mathcal{X} \in \Gamma \), the diagonal
\[
P' = \{ x : P(x, x) \}
\]
is also in \( \Gamma \). If \( \Gamma \) is \( \mathcal{Y} \)-parametrized, then some \( P \in \Gamma \mid \mathcal{Y} \) is not in \( \neg \Gamma \).

**Proof.** Let \( G \subseteq \mathcal{Y} \times \mathcal{Y} \) be universal for \( \Gamma \mid \mathcal{Y} \) and take \( P = \{ y : G(y, y) \} \).

By hypothesis \( P \in \Gamma \). If \( \neg P \in \Gamma \), then for some fixed \( y^* \in \mathcal{Y} \) we would have
\[
G(y^*, y) \iff \neg P(y) \iff \neg G(y, y)
\]
which is absurd for \( y = y^* \).

Theorem 1F.9 and this Lemma imply immediately

**Corollary 1F.11** (The Kleene hierarchies). The diagram of proper inclusions in Figure 4 holds for every space of type 0 and for every perfect space.

A good exercise in using the closure properties of the arithmetical pointclasses is the proof of the next result which strengthens considerably Theorem 1E.1.

**Theorem 1F.12.** For every \( \mathcal{X} \), there is a recursive surjection
\[
\pi : \mathbb{N} \twoheadrightarrow \mathcal{X}
\]
and a \( \Pi_0^1 \) set \( A \subseteq \mathbb{N} \) such that \( \pi \) is injective on \( A \), \( \pi[A] = \mathcal{X} \), and the inverse \( \pi^{-1} : \mathcal{X} \twoheadrightarrow A \subseteq \mathbb{N} \) is \( \Sigma_2^0 \)-recursive.

**Proof.** To begin with, let
\[
\rho : \mathbb{N} \twoheadrightarrow \mathcal{X}
\]
be the surjection defined in the proof of 1E.1 and for \( x \in \mathcal{X} \), put
\[
g(x) = \alpha,
\]
where
\[ \alpha(n) = \text{least } k \text{ such that } d(x, r_k) \leq 2^{-n-2}. \]
It is very simple to check that for all \( x \in \mathcal{X} \), \( \rho(g(x)) = x \), so \( g \) is an injection. If we put \( B = g[\mathcal{X}] \), then \( g \) is precisely the inverse of \( \rho \) restricted to \( B \), since
\[ \alpha \in B \implies \alpha = g(x) \text{ for some } x, \]
and so \( \rho \) is injective on \( B \) since \( \rho \restriction B \) has an inverse. Moreover,
\[ \alpha \in B \iff (\forall n) \left[ d(\rho(\alpha), r_{\alpha(n)}) \leq 2^{-n-2} \right. \\
& \left. \& (\forall k < \alpha(n)) \left[ d(\rho(\alpha), r_k) > 2^{-n-2} \right] \right], \]
so \( B \) is a \( \Pi^0_2 \) subset of \( \mathcal{N} \). It also follows from the construction that if \( g(x) = \alpha \), then
\[ a(n) = k \iff d(x, r_k) \leq 2^{-n-2} \& (\forall s < k) [d(x, r_s) > 2^{-n-2}], \]
which is a \( \Sigma^0_2 \) relation; so the function
\[ g^*(x, n) = g(x)(n) \]
is \( \Sigma^0_2 \)-recursive, and then by (2) of Theorem 1D.6 so is \( g \), since \( \Sigma^0_2 \) is a \( \Sigma \)-pointclass.

We must now refine the construction a bit to get \( \pi \) and \( A \) with the same properties as \( \rho \) and \( B \), except that \( A \) is \( \Pi^0_1 \). We do this by \textit{unfolding}, a version of “Skolemization” which is very useful in descriptive set theory.

Put \( B \) in normal form
\[ \alpha \in B \iff (\forall n)(\exists s) R(\alpha, n, s), \]
where \( R \) is recursive, and define \( A \subseteq \mathcal{N} \times \mathcal{N} \) by
\[ (\alpha, \beta) \in A \iff (\forall n) \left[ R(\alpha, n, \beta(n)) \& (\forall k < \beta(n)) \neg R(\alpha, n, k) \right]. \]
Clearly \( A \) is \( \Pi^0_1 \). Moreover, the projection \( \sigma : \mathcal{N} \times \mathcal{N} \rightarrow \mathcal{N} \), \( \sigma(\alpha, \beta) = \alpha \) takes \( A \) onto \( B \) and is one-to-one on \( A \), since
\[ (\alpha, \beta) \in A \implies \beta(n) = \text{least } k \text{ such that } R(\alpha, n, k). \]
Hence the composition \( \pi = \rho \circ \sigma \) is recursive, it takes \( A \) onto \( \mathcal{X} \) and it is injective on \( A \). It is easy to check (as above for the second component) that the inverse of \( \pi \)
\[ f(x) = \left( g(x), n \mapsto \text{least } k \text{ such that } R(g(x), n, k) \right) \]
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is $\Sigma_0^2$-recursive. The proof is completed by carrying $A$ to $\mathcal{N}$ via some trivial, recursive homeomorphism of $\mathcal{N}$ with $\mathcal{N} \times \mathcal{N}$, e.g., the map

$$n_0, n_1, n_2, \ldots \mapsto ((n_0, n_2, n_4, \ldots), (n_1, n_3, n_5, \ldots)).$$

This result is best possible in more ways than one. For example:

**Problem 1F.13.** Prove that every continuous $\rho : \mathbb{R} \to \mathcal{N}$ is constant, and so there is no continuous injection of $\mathbb{R}$ into $\mathcal{N}$.

**1F.14. The finite Borel and Lusin pointclasses.** These are defined exactly like the Kleene pointclasses, only starting with $\Sigma_{\tilde{0}}^1$ (the open sets) instead of $\Sigma_0^0$:

$$\Sigma_1^0 = \text{all open pointsets},$$
$$\Pi_1^0 = \neg \Sigma_1^0, \quad \Sigma_{n+1}^0 = \exists^\mathcal{N} \Pi_n^0, \quad \Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0,$$
$$\Sigma_1^1 = \exists^\mathcal{N} \Pi_1^0, \quad \Pi_1^1 = \neg \Sigma_1^1, \quad \Sigma_{n+1}^1 = \exists^\mathcal{N} \Pi_n^1, \quad \Delta_1^1 = \Sigma_1^1 \cap \Pi_1^1.$$

The sets in $\bigcup_n \Sigma_n^0$ are the **Borel sets of finite order**, and $\bigcup_n \Sigma_n^1$ comprises the **projective sets**. In the classical terminology for the most important first two of the Lusin pointclasses,

$$\Sigma_1^1 = \text{the analytic sets and } \Pi_1^1 = \text{the co-analytic sets}.$$ Missing from the list are the **Borel sets**, the most important pointclass of classical descriptive set theory. We will get to it in the next chapter.

**Problem 1F.15.** Prove that all finite Borel and Lusin pointclasses are closed under continuous substitutions.

This is verified by two easy inductions on $n$, and it is the main difference between these and the Kleene pointclasses.

**1F.16. Boldification.** For each pointclass $\Gamma$ and any space $\mathcal{X}$, let

$$(1F-1) \quad b\Gamma | \mathcal{X} = \{ P \subseteq \mathcal{X} : P \in \Gamma | \mathcal{N} \times \mathcal{X}, \varepsilon \in \mathcal{N} \},$$

so that $b\Gamma$ comprises all sections of pointsets in $\Gamma$ above some $\varepsilon \in \mathcal{N}$. This is the **boldification** or **boldface version** of $\Gamma$, its awful name stemming by the notation convention

$$\Gamma^\mathcal{N} = b\Gamma$$

which is widely used (and very useful). First, however, we must justify it in the cases we have already defined $\Gamma^\mathcal{N}$.

**Theorem 1F.17.** For every $n \in \omega$,

$$b\Sigma_n^0 = \Sigma_n^0, \quad b\Sigma_n^1 = \Sigma_n^1,$$

and similarly with $\Pi$ in place of $\Sigma$.  

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Proof is by two inductions on $n$, with the first basis

$$b\Sigma_0^0 = \Sigma_1^0 = \text{all open sets}$$

supplied by Problem 1B.19. The idea is that boldification “commutes” with negation and projection; we put down two of the four, basically trivial computations that are needed.

(a) If $\Sigma_0^0_n = b\Sigma_0^n$, then $\Pi_0^n = b\Pi_0^n$:

$$P \in b\Pi_0^n \iff P(x) \iff P_1(\varepsilon, x) \text{ with } P_1 \in \Pi_0^n, \varepsilon \in \mathcal{N}$$

$$\iff P(x) \iff \neg Q_1(\varepsilon, x) \text{ with } Q_1 \in \Sigma_0^n, \varepsilon \in \mathcal{N}$$

$$\iff P(x) \iff \neg Q(x) \text{ with } Q \in \Sigma_0^n$$

$$\iff P \in \Pi_0^n.$$

(b) If $\Pi_1^n = b\Pi_1^n$, then $\Sigma_1^{n+1} = b\Sigma_n^{n+1}$:

$$P \in b\Sigma_1^{n+1} \iff P(x) \iff Q(\varepsilon, x) \text{ with } Q \in \Sigma_1^{n+1}, \varepsilon \in \mathcal{N}$$

$$\iff P(x) \iff (\exists \alpha)Q_1(\varepsilon, x, \alpha) \text{ with } Q_1 \in \Pi_1^n, \varepsilon \in \mathcal{N}$$

$$\iff P(x) \iff (\exists \alpha)P_1(x, \alpha) \text{ with } P_1 \in \Pi_0^n$$

$$\iff P \in \Sigma_1^{n+1}.$$

\[\Box\]

Corollary 1F.18 ($\mathcal{N}$-parametrizations). For every $n \geq 1$ and every space $X$, there are sets

$$S_0^n, P_0^n, S_1^n, P_1^n \subseteq \mathcal{N} \times X$$

with the following properties:

1. $S_0^n \in \Sigma_0^n, P_0^n \in \Pi_0^n, S_1^n \in \Sigma_1^n, P_1^n \in \Pi_1^n$.
2. For every $A \subseteq X$

$$A \in \Sigma_0^n \iff (\exists \varepsilon)[\varepsilon \text{ is recursive and } A = S_{0,n,\varepsilon}^0],$$

and similarly for $\Pi_0^n, \Sigma_1^n, \Pi_1^n$.

3. For every $A \subseteq X$

$$A \in \Sigma_1^n \iff (\exists \varepsilon)[A = S_{1,n,\varepsilon}^0],$$

and similarly for $\Pi_0^n, \Sigma_1^n, \Pi_1^n$.

Proof. For $\Sigma_0^n$, for example, choose $G \subseteq \omega \times \mathcal{N} \times X$ by Theorem 1F.9 which parametrizes $\Sigma_0^n \upharpoonright (\mathcal{N} \times X)$ and set

$$S_0^n(\varepsilon, x) \iff G(\varepsilon(0), \varepsilon^*, x).$$

(1) is immediate and for (2), if $P = G_\varepsilon$, then also $P = S_{0,n,\varepsilon}^0$ with $\varepsilon(0) = e$ and $\varepsilon^* = \lambda 0$ (the constant function 0). (3) follows from the theorem,

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because for $A \subseteq X$,

$$
A \in \Sigma^0_n \implies x \in A \iff B(\alpha, x) \text{ for some } B \in \Sigma^0_n, \alpha \in \mathcal{N},
$$

$$
\implies x \in A \iff G(\varepsilon, \alpha, x) \text{ for some } \varepsilon, \alpha \in \mathcal{N},
$$

$$
\implies x \in A \iff S^0_n(\varepsilon, x) \text{ for some } \varepsilon.
$$

Corollary 1F.19. If $\Gamma$ is any of the pointclasses $\Sigma^0_n, \Sigma^1_n, \Pi^1_n$, then for each $X$, $\Gamma \mid X$ is closed under countable unions.

It follows that $\Pi^0_n$ is also closed under countable intersections, and the Lusin pointclasses $\Sigma^1_n, \Pi^1_n, \Delta^1_n$ are all closed under both countable unions and countable intersections.

Proof. If $A_i \subseteq X$, $A_i \in \Sigma^0_n$, then

$$
x \in \bigcup_i A_i \iff (\exists i)[x \in A_i]
$$

$$
\iff (\exists i)S^0_n(\varepsilon_i, x) \text{ for some map } i \mapsto \varepsilon_i
$$

$$
\iff (\exists i)S^0_n((\varepsilon_i), x) \text{ for some } \varepsilon.
$$

For $\Pi^1_n$ with $n > 1$, it is simpler to consider the complement:

$$
x \notin \bigcup_i A_i \iff (\forall i)(\exists \alpha)B_i(x, \alpha) \text{ with each } B_i \in \Pi^1_{n-1}
$$

$$
\iff (\forall i)(\exists \alpha)P^1_{n-1}(\varepsilon_i, x, \alpha) \text{ with each map } i \mapsto \varepsilon_i
$$

$$
\iff (\forall i)(\exists \alpha)P^1_{n-1}(\varepsilon_i, x, \alpha) \text{ for some } \varepsilon
$$

$$
\iff (\forall i)(\exists \alpha)P^1_{n-1}(\varepsilon_i, x, (\beta_i)) \text{ for some } \varepsilon
$$

and the relation on the right of the last equivalence is $\Sigma^1_n$.

**Problem 1F.20.** Prove the theorem for $\Pi^1_n$ and $\Sigma^1_n$.

**Problem 1F.21.** For each space $X$ and $A \subseteq X$,

$$
A \in \Sigma^0_1 \iff A \text{ is open }, A \in \Pi^0_1 \iff (X \setminus A) \in \Sigma^0_1,
$$

$$
A \in \Sigma^0_{n+1} \iff A = \bigcup_i A_i \text{ with each } A_i \in \Pi^0_n.
$$

This is, in fact, the classical definition of these pointclasses, given separately for each $\mathbb{R}^d$ in Lebesgue [1905] using countable unions and intersections rather than quantification over $\omega$. Lebesgue introduced directly its natural extension into the transfinite by the recursion

$$
(1F-2) \quad A \in \Sigma_\xi = \bigcup_i A_i \text{ with each } A_i \text{ in } \bigcup_{\eta < \xi} \Pi^0_\eta,
$$

which yields new sets (on uncountable spaces) for each countable ordinal. This is a very interesting hierarchy, but we will not study it in any serious way in these notes.

The classical notation for the finite Borel classes uses word subscripts:

$$
G = \Sigma^0_1, \quad F = \Pi^0_1, \quad G_\delta = \Sigma^0_2, \quad F_\sigma = \Sigma^0_2, \quad G_{\delta\sigma} = \Sigma^0_3, \ldots
$$

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1F. The Kleene pointclasses

It is pretty useless for $n > 2$ and we will not use it, except for an occasional reference to $G_δ$ or $F_σ$ sets in stating classical results.

Corollary 1F.18 implies that the Hierarchy Theorem illustrated in Figure 4 holds also for the boldface pointclasses for perfect $X$.

1F.22. The Suslin quantifier $\mathcal{S}$ and the operation $\mathcal{A}$. For any pointset $P \subseteq X \times \omega$, put

\[(\mathcal{S}u)P(x, u) \iff (\forall \alpha)(\exists t) P(x, \pi(t))\]  

This is the “effective dual” of the classical operation $\mathcal{A}$, which assigns to each sequence of pointsets $\{P_i\}_{i \in \omega}$ in the same space $X$ the set

\[\mathcal{A}_i P_i = \{x \in X : (\exists \alpha)(\forall t)(x \in P_{\alpha}(t))\}.\]

Notice that $(\mathcal{S}u)P(x, u)$ depends only on the truth values of $P(x, u)$ when $u = \langle t_0, \ldots, t_{n-1}\rangle$ is the code of a finite sequence and, similarly, $x \in \mathcal{A}_i P_i$ depends only on the sets $P_u$ for sequence codes $u$. Because of this, $\mathcal{A}$ is usually defined as operating on systems of sets indexed by finite sequence from $\omega$, but it simplifies notation to view $\mathcal{S}$ as a quantifier like $\exists^ω$ or $\forall^ω$ and $\mathcal{A}$ as an operation on sequences of sets, like countable union and intersection.

The operation $\mathcal{A}$ was an important tool in the development of classical descriptive set theory, the original definition of analytic sets in Suslin [1917] was (very nearly) in the form

\[\Sigma^1_1 = \mathcal{A} \Pi^0_1.\]

This means that for each $P \subseteq X$,

\[P \in \Sigma^1_1 \iff \text{for some sequence } \{P_i\}_i \text{ with each } P_i \in \Pi^0_1, x \in P \iff (\exists \alpha)(\forall t)(x \in P_{\alpha}(t)).\]

**Problem 1F.23.** Prove

\[(1F-6) \quad \Pi^1_1 = \mathcal{S} \Sigma^0_1\]

and infer (1F-5).

And the precise meaning of the “effective dual” remark is the following

**Problem 1F.24.** Prove that for all $P \subseteq X \times \omega$,

\[(\mathcal{S}u)P(x, u) \iff x \notin \mathcal{A}_i \{x : \neg P(x, i)\},\]

and for any sequence $\{P_i\}_i$ of subsets of $X$,

\[x \in \mathcal{A}_i P_i \iff \neg (\mathcal{S}u)\{x, i : x \notin P_i\}.\]

The most interesting result about the Suslin quantifier and the operation $\mathcal{A}$ is the following:
1. Recursion on Polish spaces

Problem 1F.25. (1) (Addison and Kleene [1957]). Prove that for each \( n \geq 2 \), \( \Sigma^1_n \) and \( \Pi^1_n \) are both closed under \( S \), i.e., (for the non-trivial part), if \( Q \subseteq X \times \omega \) is in \( \Sigma^1_n \) with \( n \geq 2 \), then so is the relation

\[
P(x) \iff (S u)Q(x,u) \iff (\forall \alpha)(\exists !)Q(x, \alpha(t)).
\]

(2) (Kantorovitch and Livenson [1932]). Infer that for \( n \geq 2 \), both \( \Sigma^n \tilde{1} \) and \( \Pi^n \tilde{1} \) are closed under the operation \( A \).

Hint: For the (typical) case \( n = 2 \), we need to show that if \( Q(x, \beta, u) \) is \( \Pi^1_1 \) and

\[
P(x) \iff (\forall \alpha)(\exists !)Q(x, \beta, \alpha(t)),
\]

then \( P \in \Sigma^1_2 \). The key observation is that to verify the relation on the right for any particular \( x \), we only need countably many \( \beta \)'s, because the matrix depends only on \( \alpha(t) \). Make this precise by verifying that

\[
(1F-7) \quad (\forall \alpha)(\exists !)(\exists !)Q(x, \beta, \alpha(t))
\]

which then implies the claim.

This result supplies us with a wealth of pointclasses, e.g.,

\[
\mathcal{S}\Sigma^1_1, \mathcal{S}-\mathcal{S}\Sigma^1_1, \ldots,
\]

which behave very much like the analytical pointclasses, are larger than \( \Pi^1_1 \cup \Sigma^1_1 \) and are all contained in \( \Delta^1_2 \).

The original proof of this result (for the boldface classes) was very laborious, and the short, elegant proof in Addison and Kleene [1957] outlined in the hint was one of the early indications that logical notation and quantifier manipulation can be powerful tools in descriptive set theory.

Additional problems for Section 1F

Problem 1F.26. Let \( f : \mathbb{R} \to \mathbb{R} \) be an arbitrary function on the line. Prove that the set

\[
A = \{ x \in \mathbb{R} : f \text{ is continuous at } x \}
\]

is \( \Pi^0_2 \). Hint: Define the variation of \( f \) on an interval \( (a,b) \) by

\[
V(a,b) = \sup \{ f(x) : a < x < b \} - \inf \{ f(x) : a < x < b \},
\]

where the value may be \( \infty \) or \( -\infty \). The local variation of \( f \) is given by

\[
v(x) = \lim_{n \to \infty} V \left( x - \frac{1}{n}, x + \frac{1}{n} \right)
\]

and it is clear that \( f \) is continuous at \( x \) just in case \( v(x) = 0 \).
Problem 1F.27. Let $C[0,1]$ be the space of continuous real functions on the unit interval and define $Q \subseteq C[0,1] \times \mathbb{R}$ by
\[ Q(f,x) \iff 0 < x < 1 \& f'(x) \text{ exists.} \]
Prove that $Q$ is $\Pi^0_3$.

Problem 1F.28. On the space $C[0,1]$ of continuous real functions on the unit interval, put
\[ Q(f) \iff f \text{ is differentiable on } [0,1], \]
\[ R(f) \iff f \text{ is continuously differentiable on } [0,1], \]
where at the endpoints we naturally take the one-sided derivatives. Prove that $Q$ is $\Pi^1_1$ and $R$ is $\Sigma^1_1$.

Problem 1F.29. Prove that every non-empty $\Pi^0_1$ subset of the Cantor space $C$ has a $\Delta^0_2$ member, but there is a non-empty $\Pi^0_1$ set $A \subset C$ which has no recursive member. HINT: For the second claim use the existence of recursively inseparable r.e. subsets of $\omega$, Corollary P.14.

Problem 1F.30. Prove that for every $n \geq 2$, if $P \subseteq X \times \omega$ is in $\Sigma^0_n$, then there is some $P^*$ also in $\Sigma^0_n$ which uniformizes $P$.

Give an example of a $\Sigma^0_1$ pointset $P \subseteq \mathbb{R} \times \omega$ which cannot be uniformized by any $\Sigma^0_1$ set $P^* \subseteq P$.

1F.31. Reduction. Suppose $P$ and $Q$ are subsets of the same space $X$. We say that the pair $P^*, Q^*$ reduces the pair $P, Q$ if the following hold:
\[ P^* \subseteq P, \quad Q^* \subseteq Q, \quad P \cup Q = P^* \cup Q^*, \quad P^* \cap Q^* = \emptyset; \]
and a pointclass $\Gamma$ has the reduction property if every pair $P, Q \in \Gamma \upharpoonright X$ can be reduced by a pair $P^*, Q^*$ in $\Gamma$. 

\textbf{Figure 5. Reduction.}
Problem 1F.32. Prove that for each \( n > 1 \), \( \Sigma^0_n \) and \( \Sigma^0_{\tilde{n}} \) have the reduction property.

1F.33. Separation. Suppose that \( P \) and \( Q \) are disjoint subsets of the same space \( X \). We say that the set \( S \) separates \( P \) from \( Q \) if
\[
P \subseteq S, \quad Q \cap S = \emptyset;
\]
and \( \Gamma \) has the separation property if any two disjoint pointsets \( P, Q \in \Gamma \upharpoonright X \) can be separated by some set \( S \in \Delta = \Gamma \cap \neg \Gamma \).

Problem 1F.34. Prove that \( \Pi^0_1 \) does not have the separation property.

Problem 1F.35. Prove that if a pointclass \( \Gamma \) has the reduction property, then the pointclass \( \neg \Gamma \) of its complements has the separation property; so that by Problem 1F.32, for all \( n \geq 2 \), \( \Pi^0_n \) and \( \Pi^0_{\tilde{n}} \) have the separation property.

Problem 1F.36. Prove that if a \( \Sigma \)-pointclass \( \Gamma \) is \( \omega \)-parametrized and closed under recursive substitutions and has the reduction property, then there are sets \( A, B \in \Gamma \upharpoonright X \) which are disjoint but cannot be separated by a set in \( \Delta \), for
\[
(1) \ X = \omega, \quad (2) \ X = \mathcal{N} \text{ and (3) } X = \mathbb{R}.
\]
Infer that for all \( n > 1 \), \( \Sigma^0_n, \Sigma^0_{\tilde{n}} \) do not have the separation property. HINT: Use the method of proof of Corollary P.14.

It is an open question whether the conclusion holds for all perfect \( X \).

We will show in the next chapter that \( \Pi^1_1 \) and \( \Sigma^1_2 \) have the reduction property, so that \( \Sigma^1_1 \) and \( \Pi^2_1 \) do not. For \( n \geq 3 \), the question of whether
at least one (and which one) of $\Sigma^1_n$ or $\Pi^1_n$ has the reduction property is independent of ZFC. The problem received considerable attention in the 1930s, when the independence was not known, and guided the application of strong, set theoretic axioms to descriptive set theory in its initial stages in the 1960s.

**Problem 1F.37.** Prove that there is a $\Delta^1_1$ set of integers which is not arithmetical.

**Problem 1F.38.** Prove that for each $n \geq 1$, there is a set $A \subseteq \mathbb{N}$ in $\Delta^0_n$ such that for every space $Z$, every $\Delta^0_n$ set $Q \subseteq Z \times \mathbb{N}$ and every $z \in Z$,

$$A \neq Q_z = \{\alpha : Q(z, \alpha)\}.$$

Infer that the following plausible sounding conjecture is false: $P$ is $\Delta^0_n(z)$ if and only if there is some $\Delta^0_n$ set $Q$ such that $P(x) \iff Q(z, x)$.

Similarly with $\Delta^0_1$, $\Delta^1_1$ in place of $\Delta^0_0$, $\Delta^0_1$ throughout.

**Hint:** Let $Q^0, Q^1, \ldots$ be an enumeration of all the $\Delta^0_n$ subsets of $\mathbb{N} \times \mathbb{N}$, take

$$A = \{\alpha : \neg Q^{\alpha(0)}(\alpha^*, \alpha)\},$$

where $\alpha^* = t \mapsto \alpha(t + 1)$ and check that it is $\Delta^0_n$ and not a section $Q_\varepsilon$ of any $\Delta^0_n$ set $Q \subseteq \mathbb{N} \times \mathbb{N}$ for some $\varepsilon \in \mathbb{N}$.

In the case of $\Delta^0_1$, this construction gives a clopen $A \subseteq \mathbb{N}$ which is not $Q_z$ for any recursive $Q \subseteq \mathbb{Z} \times \mathbb{N}$, any $z$. Its characteristic function $\chi_A$ is continuous and cannot be obtained from any recursive function by fixing one of the arguments.

**Problem 1F.39.** Let $\Gamma$ be any of the Kleene pointclasses $\Sigma^0_n, \Pi^0_n, \Sigma^1_n, \Pi^1_n$. Prove that for every perfect $Y$ and every $X$, there is a pointset $G \subseteq Y \times X$ such that:

(a) $G \in \Gamma \subseteq \Gamma$, and

(b) $G$ is universal for the boldface class $\mathbb{F}^\Gamma$.

Infer that for every perfect $X$, there is a pointset $P \subseteq X$ in $\Gamma \setminus \Delta$.

**Problem 1F.40.** Prove that $\Sigma^0_2$ is closed under substitution of $\Sigma^0_2$-recursive functions with values in $\omega$, i.e., if $Q \subseteq \omega$ is in $\Sigma^0_2$, $f : \mathbb{X} \to \omega$ is $\Sigma^0_2$-recursive, and

$$P(x) \iff Q(f(x)),$$

then $P$ is $\Sigma^0_2$.

Similarly, if $g : \mathbb{X} \to \omega$ and $h : \omega \to \mathbb{Y}$ are both $\Sigma^0_2$-recursive, then their composition

$$f(x) = h(g(x))$$

is $\Sigma^0_2$-recursive.

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A $\Sigma^0_1$ set $K \subset \omega$ is $\Sigma^0_1$-complete, if for any $\Sigma^0_1$ set $P \subseteq \omega^n$,

$$P(\vec{x}) \iff f(\vec{x}) \in K$$

with some recursive $f : \omega^n \to \omega$. Such sets exist, e.g.,

$$K = \{ e : (\exists t) T_1((e)_0, (e)_1, t) \}.$$

**Problem 1F.41.** Prove that $\Sigma^0_2$ is not closed under substitution of $\Sigma^0_2$-recursive functions with values in $\mathcal{N}$.

Show also that there are $\Sigma^0_2$-recursive functions $g : \mathcal{X} \to \mathcal{N}, h : \mathcal{N} \to \mathbb{N}$ whose composition $f(x) = h(g(x))$ is not $\Sigma^0_3$-recursive.

**Hint:** Show that if $K \subset \omega$ is $\Sigma^0_1$-complete, then

$$\Sigma^0_3 \upharpoonright \omega = \Sigma^0_3(\chi_K) \upharpoonright \omega,$$

i.e., a set $P \subseteq \omega$ is $\Sigma^0_3$ if and only if it satisfies an equivalence

$$P(x) \iff Q(\chi_K, x)$$

with some $\Sigma^0_2$ set $Q \subseteq \mathcal{N} \times \omega$.

**Problem 1F.42.** Prove that $\Sigma^0_{\tilde{2}}$ is not closed under substitution of $\Sigma^0_{\tilde{2}}$-recursive functions, and that the class of $\Sigma^0_{\tilde{2}}$-recursive functions on $\mathcal{N}$ to $\mathcal{N}$ is not closed under composition.

**1G. $\Delta^1_1$ recursion**

One consequence of Theorem 1F.5 is that all the results about $\Gamma$-recursion in Section 1D hold when $\Gamma$ is $\Sigma^0_n$ or any of the analytical pointclasses $\Sigma^1_n, \Pi^1_n, \Delta^1_n$. The most important case is $\Delta^1_1$-recursion, which is an effective refinement of Borel measurability as we will show further down. We prove here its basic properties and use them to derive refinements of the “transfer” Theorems 1E.1 and 1E.9 which have many consequences.

We will often say “$\Delta^1_1$ function” instead of “$\Delta^1_1$-recursive function”.

**Lemma 1G.1.** For every function $f : \mathcal{X} \to \mathcal{Y}$ on one recursive Polish space to another, the following four conditions are equivalent:

(a) $f$ is $\Delta^1_1$-recursive.
(b) $f$ is $\Sigma^1_1$-recursive.
(c) Graph$(f) = \{(x, y) : f(x) = y\}$ is $\Sigma^1_1$.
(d) Graph$(f)$ is $\Delta^1_1$.

**Proof.** (a) $\Rightarrow$ (b) is immediate and (b) $\Rightarrow$ (c) follows from the equivalence

$$f(x) = y \iff (\forall s)\{ y \in N(\mathcal{Y}, s) \Rightarrow f(x) \in N(\mathcal{Y}, s) \}.$$
To prove (c) ⇒ (d) notice that
\[ f(x) \neq y \iff (\exists z)\{f(x) = z \& z \neq y\}, \]
and for (d)⇒(a) use
\[ f(x) \in N(\mathcal{Y}, s) \iff (\exists y)\{f(x) = y \& y \in N(\mathcal{Y}, s)\} \iff (\forall y)\{f(x) \neq y \vee y \in N(\mathcal{Y}, s)\}. \]

Part (c) of the Lemma implies, in particular, that every \(\Delta^1_1\) bijection \(f : \mathcal{X} \to \mathcal{Y}\) is a \(\Delta^1_1\)-isomorphism, i.e., its inverse is also \(\Delta^1_1\).

**1G.2. Good \(\Delta^1_1\) injections.** The function \(f\) in this proof is an example of an interesting class of functions: \(f : \mathcal{X} \to \mathcal{Y}\) is a **good \(\Delta^1_1\) injection** if

1. \(f\) is a \(\Delta^1_1\) injection, and
2. there is a \(\Delta^1_1\) surjection \(g : \mathcal{Y} \to \mathcal{X}\) such that \(g \circ f\) is the identity on \(\mathcal{X}\), i.e.,
\[ g(f(x)) = x \quad (x \in \mathcal{X}). \]

We refer to any such \(g\) as a \(\Delta^1_1\)-inverse of \(f\).

**Lemma 1G.3.** (1) A \(\Delta^1_1\) injection \(f : \mathcal{X} \to \mathcal{Y}\) is good if and only if its image \(f[\mathcal{X}]\) is a \(\Delta^1_1\) subset of \(\mathcal{Y}\).

(2) If \(f : \mathcal{X} \to \mathcal{Y}\) is a good \(\Delta^1_1\) injection, then the image \(f[P]\) of every \(\Delta^1_1\) set \(P \subseteq \mathcal{X}\) is \(\Delta^1_1\).

**Proof.** (1) If \(f[\mathcal{X}]\) is \(\Delta^1_1\), define \(g : \mathcal{Y} \to \mathcal{X}\) by
\[ g(y) = x \iff [y \notin f[\mathcal{X}] \& x = r_0] \vee y = f(x), \]
where \(r_0\) is the first element in some compatible, dense subset of \(\mathcal{X}\). The converse implication holds because if \(f\) is good with inverse \(g\), then
\[ y \in f[\mathcal{X}] \iff f(g(y)) = y; \]

because
\[ y \in f[\mathcal{X}] \implies y = f(x) \text{ for some } x \]
\[ \implies y = f(g(f(x))) \text{ for some } x \text{ such that } y = f(x) \]
\[ \implies y = f(g(y)), \]

and the converse \(f(g(y)) = y \implies y \in f[\mathcal{X}]\) is trivial.

(2) holds because
\[ y \in f[P] \iff y \in f[\mathcal{X}] \& g(y) \in P. \]

Notice also that the class of good \(\Delta^1_1\) injections is closed under composition—this is immediate.

It will turn out that every \(\Delta^1_1\) injection is good, but this is a consequence of a rather deep result which we will prove in the next Chapter. Here we only need show that enough good \(\Delta^1_1\) injections exist.
1. Recursion on Polish spaces

Lemma 1G.4. For every perfect space \( X \), there are good \( \Delta^1_1 \) injections 
\( f : X \rightarrow N, \ h : N \rightarrow X \).

Proof. We have already constructed \( f \) in Theorem 1F.12, which does not need the hypothesis that \( X \) is perfect: it was called \( \pi^{-1} \) there and we showed the stronger result, that it is \( \Sigma^0_2 \)-recursive.

To construct \( h \), define first \( h_1 : N \rightarrow C \) by 
\[
\beta(n) = \begin{cases} 
0 & \text{if } \alpha((n)_0) = (n)_1, \\
1 & \text{if } \alpha((n)_0) \neq (n)_1.
\end{cases}
\]

This is obviously a \( \Delta^1_1 \) injection, and 
\[ \beta \in h_1[N] \iff (\forall n) \left[ \beta(n) = \beta((n)_0, (n)_1) \right] \]
\& (\forall n)(\forall k) \left[ [\beta(n) = 0 \& \beta(k) = 0 \& (n)_0 = (k)_0] \implies (n)_1 = (k)_1 \right] \]
\& (\forall n)(\exists k)[\beta((n, k)) = 0],
so that the image \( h_1[N] \) is \( \Delta^1_1 \) and hence \( h_1 \) is good. Next let 
\[ \pi : C \rightarrow X \]
be the recursive injection constructed in Theorem 1E.9 for perfect \( X \) and compute using the definition of \( \pi \):
\[ x \in \pi[C] \iff (\forall n)(\exists u)[\text{Seq}(u) \& x \in N(X, \tilde{\sigma}(u))]; \]
so \( \pi[C] \) is \( \Delta^1_1 \), and so \( \pi \) is good. The composition \( h = \pi \circ h_1 \) is then a good \( \Delta^1_1 \) injection of \( N \) into \( X \).

1G.5. The \( \Delta^1_1 \)-isomorphism Theorem. For every perfect recursive Polish space \( X \), there is a \( \Delta^1_1 \) bijection 
\[ g : N \rightarrow X . \]

Proof. Recall the classical Schröeder-Bernstein Theorem, whose proof constructs from given injections \( h : N \rightarrow X \) and \( f : X \rightarrow N \) a bijection \( g : N \rightarrow X \). We will give a brief outline of this classical proof and check that if \( h, f \) are good \( \Delta^1_1 \) injections, then the resulting bijection is a \( \Delta^1_1 \) isomorphism.

Define the sequences of sets \( N_0, N_1, \ldots, X_0, X_1, \ldots \) recursively by the equations 
\[
N_0 = N, \quad X_0 = X, \\
N_{n+1} = h[N_n], \quad X_{n+1} = f[X_n],
\]
as in Figure 7. An easy induction shows that
\[
\begin{align*}
\mathcal{N}_0 \supseteq f[\mathcal{X}_0] & \supseteq \mathcal{N}_{n+1}, \\
\mathcal{X}_0 \supseteq h[\mathcal{N}_0] & \supseteq \mathcal{X}_{n+1},
\end{align*}
\]
so that
\[
\begin{align*}
\mathcal{N} = \mathcal{N}_0 \supseteq & f[\mathcal{X}_0] \supseteq \mathcal{N}_1 \supseteq & f[\mathcal{X}_1] \supseteq \mathcal{N}_2 \supseteq & f[\mathcal{X}_2] \supseteq \cdots, \\
\mathcal{X} = \mathcal{X}_0 \supseteq & f[\mathcal{N}_0] \supseteq \mathcal{X}_1 \supseteq & f[\mathcal{N}_1] \supseteq \mathcal{X}_2 \supseteq & f[\mathcal{N}_2] \supseteq \cdots.
\end{align*}
\]
Put also
\[
\mathcal{N}^* = \bigcap_n \mathcal{N}_n, \quad \mathcal{X}^* = \bigcap_n \mathcal{X}_n
\]
and notice that
\[
\begin{align*}
\mathcal{X}^* = \bigcap_n \mathcal{X}_n & \supseteq \bigcap_n h[\mathcal{N}_n] \supseteq \bigcap_n \mathcal{X}_{n+1} = \mathcal{X}^*,
\end{align*}
\]
and since \(h\) is an injection,
\[
h[\mathcal{N}^*] = h[\bigcap_n \mathcal{N}_n] = \bigcap_n h[\mathcal{N}_n] = \mathcal{X}^*.
\]
Thus \(h\) gives a bijection on \(\mathcal{N}^*\) with \(\mathcal{X}^*\). On the other hand,
\[
\begin{align*}
\mathcal{N} = (\mathcal{N}_0 - f[\mathcal{X}_0]) & \cup (f[\mathcal{X}_0] - \mathcal{N}_1) \cup (\mathcal{N}_1 - f[\mathcal{X}_1]) \cup (f[\mathcal{X}_1] - \mathcal{N}_2) \cup \cdots \cup \mathcal{N}^*, \\
\mathcal{X} = (\mathcal{X}_0 - h[\mathcal{N}_0]) & \cup (h[\mathcal{N}_0] - \mathcal{X}_1) \cup (\mathcal{X}_1 - h[\mathcal{N}_1]) \cup (h[\mathcal{N}_1] - \mathcal{X}_2) \cup \cdots \cup \mathcal{X}^*.
\end{align*}
\]
where the sets in these unions are disjoint. Moreover, \( h \) is a bijection of \( \mathcal{N}_n \setminus f[X_n] \) with \( h[\mathcal{N}_n] \setminus X_{n+1} \), since \( h \) is an injection and \( f[X_n] \subseteq \mathcal{N}_n \), so that

\[
h[\mathcal{N}_n \setminus f[X_n]] = h[\mathcal{N}_n] \setminus h f[X_n] = h[\mathcal{N}_n] \setminus X_{n+1},
\]

and similarly, \( f \) is a bijection of \( \mathcal{N}_n \setminus h[X_n] \) with \( f[X_n] \setminus \mathcal{N}_{n+1} \). So we have a bijection of \( \mathcal{N} \) with \( \mathcal{X} \),

\[
g(\alpha) = \begin{cases} h(\alpha) & \text{if } \alpha \in \mathcal{N}^* \text{ or } \alpha \in \mathcal{N}_n \setminus f[X_n] \text{ for some } n, \\ f^{-1}(\alpha) & \text{if } \alpha \notin \mathcal{N}^* \text{ and } \alpha \in f[X_n] \setminus \mathcal{N}_{n+1} \text{ for some } n. \end{cases}
\]

It remains to verify that \( g \) is \( \Delta^1_1 \), and for this it is enough to prove that the five relations

\[
\alpha \in \mathcal{N}^*, \quad \alpha \in \mathcal{N}_n, \quad x \in \mathcal{X}_n, \quad \alpha \in f[X_n], \quad x \in h[\mathcal{N}_n]
\]

are \( \Delta^1_1 \), since the graph of \( g \) is explicitly defined from them.

Let us concentrate on the relation \( \alpha \in \mathcal{N}_n \). To begin with, it is almost trivial that

\[
(1G-1) \quad \alpha \in \mathcal{N}_n \iff (\exists \beta) \{(\forall i < n)[(\beta)_{i+1} = fh((\beta)_i)] \& \alpha = (\beta)_n\}. \tag{1G-1}
\]

We prove direction \( (\Rightarrow) \) of (1G-1) by choosing \( \beta_0, \beta_1, \ldots, \beta_n \) so that \( \beta_1 = fh(\beta_0), \beta_2 = fh(\beta_1), \ldots, \beta_n = fh(\beta_{n-1}) = \alpha \) and then picking \( \beta \) so that for \( i < n, (\beta)_i = \beta_i \). For the direction \( (\Leftarrow) \), choose any \( \beta \) which satisfies the matrix on the right of (1G-1) and verify by induction on \( i < n \) that \( (\beta)_{i+1} \in \mathcal{N}_{i+1} \), so that \( \alpha = (\beta)_n \in \mathcal{N}_n \).

Equivalence (1G-1) establishes that the relation \( \alpha \in \mathcal{N}_n \) is \( \Sigma^1_1 \). To show that this relation is also \( \Pi^1_1 \), we need the slightly less perspicuous equivalence

\[
(1G-2) \quad \alpha \in \mathcal{N}_n \iff (\forall \beta) \{(\forall i < n)[(\beta)_i = h^* f^*((\beta)_{i+1}) \& (\beta)_n = \alpha] \implies (\forall i < n)[(\beta)_{i+1} = fh((\beta)_i)] \}, \tag{1G-2}
\]

where \( f^* \) and \( h^* \) are \( \Delta^1_1 \) “inverses” of the good \( \Delta^1_1 \) injections \( f \) and \( h \).

**Proof of direction \( (\Rightarrow) \) of (1G-2)** is by induction on \( n \). If \( n = 0 \), then \( \alpha \in \mathcal{N}_0 = \mathcal{N} \) and the right hand side is vacuously true. Assume \( \alpha \in \mathcal{N}_{n+1} \), so that for some \( \gamma \in \mathcal{N}_n \) we have

\[
\alpha = fh(\gamma)
\]

and therefore

\[
h^* f^*(\alpha) = \gamma.
\]

Any \( \beta \) which satisfies

\[
(\forall i < n + 1)[(\beta)_i = h^* f^*((\beta)_{i+1}) \& (\beta)_{n+1} = \alpha]
\]

obviously satisfies

\[
(\forall i < n)[(\beta)_i = h^* f^*((\beta)_{i+1}) \& (\beta)_n = \gamma],
\]

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so by the induction hypothesis applied to $\gamma \in \mathcal{N}_n$,

\[(\forall i < n)[(\beta)_{i+1} = fh((\beta)_i)].\]

Since also

\[(\beta)_{n+1} = \alpha = fh(\gamma) = fh((\beta)_n),\]

we have

\[(\forall i < n + 1)[(\beta)_{i+1} = fh((\beta)_i)]\]

and we have shown the right hand side of (1G-2) for $n + 1$.

Proof of direction (⇐) of (1G-2). Given $\alpha$ so that the right hand side of (1G-2) holds, choose $\beta$ so that

\[(\beta)_n = \alpha,\ (\beta)_{n-1} = h^*f^*((\beta)_n),\ (\beta)_{n-2} = h^*f^*((\beta)_{n-1}),\ 
\ldots, (\beta)_0 = h^*f^*((\beta)_1).\]

We then have that $(\forall i < n)[(\beta)_{i+1} = fh((\beta)_i)]$ from which it follows immediately that for each $i < n$, $(\beta)_{i+1} \in \mathcal{N}_{i+1}$, so that $\alpha = (\beta)_n \in \mathcal{N}_n$.

A symmetric argument establishes that the relation $x \in X_n$ is $\Delta_1^1$, and then

\[\alpha \in f[X_n] \iff \alpha \in f[X] \& f^*(\alpha) \in X_n\]

\[\iff ff^*(\alpha) = \alpha \& f^*(\alpha) \in X_n\]

so that $\alpha \in f[X_n]$ is $\Delta_1^1$, and similarly for the relation $x \in h[\mathcal{N}_n]$. Finally, $X^* \Delta 1 \Delta_1^1$ because $\alpha \in \mathcal{N}^* \iff (\forall n)\exists \alpha \in N^\alpha].$

The proof of this basic fact depended (via Lemma 1G.4) on the existence of a $\Sigma_0^2$-recursive injection $f : \mathcal{N} \rightarrow X$, whose proof, in turn, depended on Theorem 1E.9 and the hypothesis that $X$ is perfect. To get a version of it for all uncountable spaces $X$, we need to relativize the argument to the local size parameter $p(X)$ defined in 1E.8. We leave this for a

Problem 1G.6. Prove that for every uncountable recursive Polish space $X$, there is a $\Delta_1^1(p(X))$-recursive bijection

\[g : \mathcal{N} \rightarrow X,\]

and infer that every uncountable (classical) Polish space $X$ is $\Delta_1^1$-isomorphic with $\mathcal{N}$.

Additional problems for Section 1G

Problem 1G.7. Prove that all analytical pointclasses $\Sigma_n^1$, $\Pi_n^1$, $\Delta_n^1$ are closed under substitutions of $\Delta_1^1$-recursive functions.
1H. Partial functions and the substitution property

We have postponed discussing partial functions on recursive Polish spaces because their treatment involves some technicalities. Now, however, we need them.

1H.1. Potential $\Gamma$-recursion. Fix a $\Sigma$-pointclass $\Gamma$. A partial function $f : X \rightarrow Y$ is potentially $\Gamma$-recursive or $\Gamma$-recursive on its domain if there is a set $P \subseteq X \times \omega$ in $\Gamma$ such that

$$f(x) \downarrow \implies \mathcal{U}(f(x)) = P_x = \{s : P(x,s)\},$$

in other words, if

$$(1H-1) \quad f(x) \downarrow \implies (\forall s)(f(x) \in N(Y,s) \iff P(x,s));$$

and $f$ is $\Gamma$-recursive if it is potentially $\Gamma$-recursive and in addition

$$(1H-2) \quad \text{Domain}(f) = \{x \in X : f(x) \downarrow\} \in \Gamma.$$  

As with total functions, we say (potentially) recursive for (potentially) $\Sigma^0_1$-recursive.

A pointset $P$ computes $f$ if $(1H-1)$ holds, and it computes $f$ in $\Gamma$ if in addition $P \in \Gamma$. It is important for some of the applications that we do not (in general) put any restrictions on $P(x,s)$ for $x$ such that $f(x) \uparrow$.

We collect in one easy problem the obvious, trivial properties of potential $\Gamma$-recursion:

Problem 1H.2. Fix a $\Sigma$-pointclass $\Gamma$ and prove the following:

(1) A total $f : X \rightarrow Y$ is potentially $\Gamma$-recursive if and only if it is $\Gamma$-recursive.

(2) The restriction $f \upharpoonright A$ of a potentially $\Gamma$-recursive partial function to any pointset is also potentially $\Gamma$-recursive.

(3) A partial function $f : X \rightarrow Y$ is $\Gamma$-recursive if and only if its nbhd diagram is in $\Gamma$, where

$$(1H-3) \quad G^f(x,s) \iff f(x) \downarrow \& f(x) \in N_s.$$  

(4) (Trivial substitutions). The collection of potentially $\Gamma$-recursive partial functions is closed under trivial substitutions,

$$f(x_1, \ldots, x_n) = g(x_{\pi(1)}, \ldots, x_{\pi(m)}) \quad (\pi : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\});$$

and $f : X \rightarrow Y_1 \times \cdots \times Y_m$ is potentially $\Gamma$-recursive if and only if $f(x) = (f_1(x), \ldots, f_m(x))$ with potentially $\Gamma$-recursive $f_i : X \rightarrow Y_i$, $i = 1, \ldots, m$.  

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(5) (Definition by cases). If \( f : \omega \times X \to Y \) is defined by cases on the first argument,
\[
f(0, x) = g(x), \quad f(n + 1, x) = h(x)
\]
where \( g, h : X \to Y \) are potentially \( \Gamma \)-recursive, then \( f \) is also potentially \( \Gamma \)-recursive.

Notice that
\[
f(m, x) \downarrow \iff [m = 0 \& g(x) \downarrow] \lor [m > 0 \& h(x) \downarrow]
\]
in (5), and there are similar, easy definitions of the domains of \( f \) and the \( f_i \)'s in (4).

It is also easy to identify the recursive and potentially recursive partial functions on \( \omega \) and \( N \):

**Problem 1H.3.** (1) A partial function \( f : \omega^n \to \omega \) is recursive if and only if it recursive by the classical definition (in the preliminary chapter).

(2) If \( X \) is of type 0 or 1 and \( f : X \to \omega \) is potentially \( \Gamma \)-recursive, then there is a recursive \( \bar{f} : X \to \omega \) which extends \( f \), i.e., \( f \subseteq \bar{f} \).

(3) A partial function \( f : X \to N \) is potentially \( \Gamma \)-recursive if and only if its unfolding \( f^* \) is potentially \( \Gamma \)-recursive, where for a partial \( f : X \to N \),
\[
f^*(x, s) \downarrow \iff f(x) \downarrow, \quad f^*(x, s) = f(x)(s).
\]

The situation is not so pleasant for arbitrary \( X, Y \), when the domain of convergence of a potentially \( \Gamma \)-recursive \( f : X \to Y \) may be very complex, and it is not always true that \( f \) has a \( \Gamma \)-recursive extension.

For any two spaces \( X, Y \) and any \( P \subseteq X \times \omega \), define a partial function
\[
\Phi_P : X \to Y
\]
as follows:
\[
\Phi_P(x) \downarrow \iff \text{there exists a unique } y \in Y \text{ such that } \\
(\forall s) \left( y \in N_s \iff P(x, s) \right),
\]
\[
\Phi_P(x) = \text{the unique } y \text{ such that } (\forall s) \left( y \in N_s \iff P(x, s) \right).
\]

**Theorem 1H.4.** Fix a \( \Sigma \)-pointclass \( \Gamma \) and spaces \( X, Y \).

(1) A set \( P \subseteq X \times \omega \) computes a partial function \( f : X \to Y \) if and only if \( f \subseteq \Phi_P \) and in particular, \( P \) computes \( \Phi_P \).

(2) For every \( P \subseteq X \times \omega \),
\[
\Phi_P(x) \downarrow \iff (\forall s, t) \left( [P(x, s) \& P(x, t)] \implies N_s \cap N_t \neq \emptyset \right) \land (\forall n)(\exists s) \left( P(x, s) \& \text{radius}(N_s) < 2^{-n} \right).
\]
(3) Every potentially recursive \( f : X \to Y \) has a potentially recursive extension \( \tilde{f} : X \to Y \) whose domain of convergence is \( \Pi^0_2 \).

Proof. (1) is immediate because directly from the definitions, \( P \) computes \( \Phi^P \) and if \( P \) computes any \( f : X \to Y \), then
\[
f(x) \downarrow \implies (f(x) = \text{the unique } y \text{ such that } \mathcal{U}(y) = P_x) \implies f(x) = \Phi^P(x).
\]

(2) The direction \( (\implies) \) of (1H-8) is trivial, because
\[
\Phi^P(x) = y \implies \{ s : P(x, s) \} = \{ N_s : y \in N_s \}.
\]

Proof of \( (\Longleftarrow) \) in (1H-8). Assume the right-hand-side.

The fact that at most one \( y \) can belong to every \( N_s \) such that \( P(x, s) \) is obvious from the second conjunct, and so we only need show that some \( y \) has this property.

Choose a sequence of nbhds \( N^0, N^1, \ldots \) such that
\[
\text{radius}(N^i) < 2^{-i-2} \text{ and } N^1 = N_s \text{ for some } s_i \text{ such that } P(x, s_i).
\]

Let \( y_i = \text{center}(N^i) \). If \( z \) is any point in the intersection \( N^i \cap N^{i+1} \) (which is non-empty), then
\[
d(y_i, y_{i+1}) \leq d(y_i, z) + d(z, y_{i+1}) < 2^{-i-2} + 2^{-i-3} < 2^{-i-1},
\]
\[
d(y_i, y_{i+m}) \leq d(y_i, y_{i+1}) + \cdots + d(y_{i+m-1}, y_{i+m}) < 2^{-i-1} \sum_{j=0}^{\infty} 2^{-j} = 2^{-i},
\]
and so \( \lim_{i \to \infty} y_i = y \) for some \( y \) such that
\[
d(y, y) = \lim_{m \to \infty} d(y_i, y_{i+m}) \leq 2^{-i}.
\]

To verify that \( y \in N_s \) for every \( s \) such that \( P(x, s) \), note that for every \( i \) there is some \( z \in N_s \cap N^i \), and so
\[
d(\text{center}(N_s), y_i) \leq d(\text{center}(N_s), z) + d(z, y_i) < \text{radius}(N_s) + \text{radius}(N^i),
\]
so that
\[
d(\text{center}(N_s), y) \leq d(\text{center}(N_s), y_i) + d(y_i, y) < \text{radius}(N_s) + \text{radius}(N^i) + 2^{-i},
\]
and taking the limit as \( i \to \infty \) we get \( d(\text{center}(N_s), y) < \text{radius}(N_s) \), so \( y \in N_s \).

\[\text{Problem 1H.5. \ Prove (3) of the theorem.}\]

The next Corollary explains, in some sense, why not every continuous \( f : X \to Y \) is the section of a recursive \( \tilde{f} : N \times X \to Y \), cf. Problem 1F.38: we need to allow \( \tilde{f} \) to be partial:
Corollary 1H.6. Suppose $A \subseteq \mathcal{X}$ and $f : A \to \mathcal{Y}$ is continuous (in the topology on $A$ induced by $\mathcal{X}$).

(1) There is a potentially recursive $\tilde{f} : \mathcal{N} \times \mathcal{X} \to \mathcal{Y}$ and some $\varepsilon \in \mathcal{N}$ such that
\[ x \in A \implies f(x) = \tilde{f}(\varepsilon, x). \]
(1H-9)

(2) There is a $G_5$ set $\overline{A} \supseteq A$ and a continuous extension of $f$ to some continuous $\overline{f} : \overline{B} \to \mathcal{Y}$.

In particular, every (total) continuous $f : \mathcal{X} \to \mathcal{Y}$ is a section $\tilde{f}_\varepsilon$ of some potentially recursive $\tilde{f} : \mathcal{N} \times \mathcal{X} \to \mathcal{Y}$.

Proof. (1) Since $f$ is continuous on $A$, for every $s$
\[ f^{-1}[N_s] = Q_s \cap A \]
for some open set $Q_s \subseteq \mathcal{X}$. It follows that the set
\[ Q = \bigcup_s Q_s \times \{s\} \]
is open in $\mathcal{X} \times \omega$, and so there is some semirecursive $P \subseteq \mathcal{N} \times \mathcal{X} \times \omega$ and some $\varepsilon \in \mathcal{N}$ such that $Q = P_\varepsilon$, so that
\[ x \in A \implies (\forall s) \left( f(x) \in N_s \iff P(\varepsilon, x, s) \right). \]
\[ (\ast) \]
Set $\tilde{f} = \Phi_P$. This is potentially recursive by (1) of the theorem, and by $(\ast)$,
\[ x \in A \implies \mathcal{U}(f(x)) = \{s : P(\varepsilon, x, s)\} \]
\[ \implies f(x) \text{ is the unique } y \text{ such that } \mathcal{U}(y) = \{s : P(\varepsilon, x, s)\} \]
\[ \implies \Phi_P(\varepsilon, x) = f(x) \]
by the definition of $\Phi_P$ and the fact that $\mathcal{U}(y)$ determines $y$.

(2) In the notation of (1), set $\overline{f} = \tilde{f}_\varepsilon$, i.e.,
\[ \overline{f}(x) = \Phi_P(\varepsilon, x). \]

By (2) of the theorem, the domain $\{\langle \alpha, x \rangle : \Phi_P(\alpha, x) \downarrow\}$ of $\Phi_P$ is $\Pi^0_2$, and so the domain $\{x : \Phi_P(\varepsilon, x) \downarrow\}$ of the section $\Phi_{P,\varepsilon} = \overline{f}$ is $\Pi^0_2(\varepsilon)$ and so a $G_5$ set.

Problem 1H.7. Prove that $\Sigma^0_1$ is not closed under recursive partial substitutions: specifically, define a recursive partial $f : \omega \to \mathcal{N}$ and a $\Sigma^0_1$ set $P \subseteq \mathcal{N}$ such that the relation
\[ Q(n) \iff P(f(n)) \iff f(n) \downarrow \land P(f(n)) \]
is not $\Sigma^0_1$.
To support a reasonable theory of \( \Gamma \)-recursion, a pointclass \( \Gamma \) should (at the least) be closed under total \( \Gamma \)-recursive substitutions. This is true for \( \Sigma_0^1 \) but fails for \( \Sigma_0^2 \) (by Problem 1F.41) and is not, in fact, easy to establish for complex pointclasses. The key tool for both proving it and using it is the following stronger—and somewhat more subtle—property.

1H.8. The substitution property. A pointclass \( \Gamma \) has the substitution property if for each \( Q \subseteq Y \) in \( \Gamma \) and for each partial function \( f : X \rightarrow Y \) which is potentially \( \Gamma \)-recursive, there is some \( Q^* \subseteq X \) in \( \Gamma \) such that for all \( x \in X \),

\[
f(x) \downarrow \implies [Q^*(x) \Leftrightarrow Q(f(x))].
\]

Lemma 1H.9. If \( \Gamma \) is a \( \Sigma \)-pointclass with the substitution property, then

(1) the collection of potentially \( \Gamma \)-recursive partial functions is closed under composition, and
(2) \( \Gamma \) is closed under the substitution of total \( \Gamma \)-recursive functions.

In particular, \( \Gamma \) is closed under (total) recursively defined functions.

Proof. (2) is immediate from the substitution property, and the last claim follows because \( \Gamma \) is a \( \Sigma \)-pointclass, and so every total recursive function is \( \Gamma \)-recursive.

(1) Suppose \( g : X \rightarrow Y, f : Y \rightarrow Z \) are both \( \Gamma \)-recursive on their domains and \( P_f \subseteq Y \times \omega \) computes \( f \) in \( \Gamma \);

\[
f(y) \downarrow \implies (\forall s)(f(y) \in N_s \Leftrightarrow P_f(y, s))
\]

Let \( h(x) = f(g(x)) \). If \( h(x) \downarrow \), then \( g(x) \downarrow \), so setting \( y = g(x) \) we have

\[
h(x) = f(g(x)) \downarrow \implies (\forall s)(f(g(x)) \in N_s \Leftrightarrow P_f(g(x), s)).
\]

Since \( g \) is \( \Gamma \)-recursive on its domain, so is (easily) the map \( (x, s) \mapsto (g(x), s) \);

thus by the substitution property, there is some \( Q^* \) in \( \Gamma \) so that

\[
g(x) \downarrow \implies \big( P_f(g(x), s) \Leftrightarrow Q^*(x, s) \big).
\]

It follows that

\[
h(x) \downarrow \implies (\forall s)(h(x) \in N_s \Leftrightarrow Q^*(x, s))
\]

so that \( h \) is \( \Gamma \)-recursive on its domain.

Theorem 1H.10. (1) \( \Sigma_0^0 \) has the substitution property.
(2) If a \( \Sigma \)-pointclass \( \Gamma \) has the substitution property, then so does every relativization \( \Gamma(w) \) as well as the boldface version \( \mathbf{\Gamma} \) of \( \Gamma \).
(3) If a $\Sigma$-pointclass $\Gamma$ is closed under $\forall^\omega$ and either $\exists^\forall^\omega$ or $\forall^\forall^\omega$, then $\Gamma$ has the substitution property.

In particular, $\Sigma^0_1$, $\Sigma^1_n$, $\Pi^0_n$ and all their relativizations and boldface versions have the substitution property.

PROOF. (1) Suppose $Q \subseteq \mathcal{Y}$ is semirecursive, so that
\[ Q(y) \iff (\exists s)[y \in N_s & Q'(s)] \]
with a semirecursive $Q'$. If $f : \mathcal{X} \to \mathcal{Y}$ is partial and computed on its domain by some semirecursive $P \subseteq \mathcal{X} \times \omega$, put
\[ Q^*(x) \iff (\exists s)[P(x, s) & Q'(s)]; \]
now if $f(x)$, then
\[ f(x) \in N_s \iff P(x, s), \]
so that
\[ Q^*(x) \iff (\exists s)[f(x) \in N_s & Q'(s)] \]
\[ \iff Q(f(x)). \]

(2) Suppose $Q \subseteq \mathcal{Y}$ is in $\Gamma(w)$, so that
\[ Q(y) \iff Q'(w, y) \]
for some $Q'$ in $\Gamma$ and suppose that $f : \mathcal{X} \to \mathcal{Y}$ is computed on its domain by some $P \subseteq \mathcal{X} \times \omega$ in $\Gamma(w)$; again
\[ P(x, s) \iff P'(w, x, s) \]
for some $P'$ in $\Gamma$. Now $P'$ computes on its domain the partial function $f' : \mathcal{W} \times \mathcal{X} \to \mathcal{Y}$ defined by
\[ f'(w', x) \downarrow \iff \text{for some } y, \mathcal{U}(y) = \{s : P'(w', x, s)\}, \]
\[ f'(w', x) \uparrow \iff (\forall s)[f'(w', x) \in N_s \iff P'(w', x, s)]. \]
Notice that for the specific fixed $w$ we have
\[ f(x) \downarrow \implies f'(w, x) \downarrow \quad f(x) = f'(w, x). \]
The partial function
\[ g(w', x) = (w', f'(w', x)) \]
is $\Gamma$-recursive on its domain, so by the substitution property for $\Gamma$, there is some $Q'' \subseteq \mathcal{W} \times \mathcal{X}$ in $\Gamma$ so that
\[ g(w', x) \downarrow \implies [Q''(w', x) \iff Q'(w', f'(w', x))]; \]
setting $w' = w$ then, we have
\[ f(x) \downarrow \implies [Q''(w, x) \iff Q'(w, f'(w, x))]; \]
\[ \implies [Q''(w, x) \iff Q(f(x))]. \]
and the relation
\[ Q^*(x) \iff Q'(w, x) \]
witnesses that \( \Gamma \) is closed under substitutions by \( f \).

(3) Suppose the partial function \( f : X \rightarrow Y \) is computed on its domain by \( P \subseteq X \times \omega \) in \( \Gamma \), \( Q \subseteq Y \) is in \( \Gamma \) and \( \Gamma \) is closed under \( \forall \omega \) and \( \exists \mathbb{N} \). Put
\[ Q^*(x) \iff (\exists \alpha \in \mathbb{N})[Q(\pi(\alpha)) \land (\forall s)[\pi(\alpha) \in N_s \implies P(x, s)]], \]
where \( \pi : \mathbb{N} \rightarrow Y \) is a recursive surjection. Notice that \( x \notin N_s \iff (\exists t)[x \in N_t \land N_t \cap N_s = \emptyset] \), so that \( \{ (x, s) : x \notin N_s \} \in \Gamma \). Using this and the fact that \( \Gamma \) is closed under recursive substitutions (by Lemma 1H.9), we check easily that \( Q^*(x) \) is in \( \Gamma \). If \( f(x) \downarrow \), then for any \( \alpha \),
\[ (\forall s)[\pi(\alpha) \in N_s \implies P(x, s)] \]
\[ \implies (\forall s)[\pi(\alpha) \in N_s \implies f(x) \in N_s] \implies \pi(\alpha) = f(x), \]
so that
\[ Q^*(x) \iff Q(f(x)). \]
The argument is similar if \( \Gamma \) is closed under \( \forall \mathbb{N} \), taking
\[ Q^*(x) \iff (\forall \alpha \in \mathbb{N})[Q(\pi(\alpha)) \lor (\exists s)[P(x, s) \land \pi(\alpha) \notin N_s]]. \]

One way to appreciate the usefulness of potential \( \Gamma \)-recursion is to spend a few minutes trying to prove directly that if a \( \Sigma \)-pointclass \( \Gamma \) is closed under \( (\text{total}) \) \( \Gamma \)-recursive substitutions, then each relativization \( \Gamma(w) \) is closed under \( \Gamma(w) \)-recursive substitutions. But we will also see many situations where potential \( \Gamma \)-recursive functions with bad domains—not in \( \Gamma \)—occur naturally.

**1H.11. \( \Sigma^* \)-pointclasses.** At this point we bundle in one definition three properties of pointclasses which we have already introduced and which insure that they carry a reasonable theory of recursion.

A \( \Sigma^* \)-pointclass is any collection \( \Gamma \) of pointsets such that:

1. \( \Gamma \) is a \( \Sigma \)-pointclass, i.e., it contains \( \Sigma^0_1 \) and it is closed under trivial substitutions, \&, \lor, \exists \mathbb{N}, \forall \mathbb{N} \); 
2. \( \Gamma \) has the substitution property; and 
3. \( \Gamma \) is \( \omega \)-parametrized, i.e., for every \( \mathcal{X} \) there is some \( G \subseteq \omega \times \mathcal{X} \) in \( \Gamma \) such that
\[ P \in \Gamma \iff (\exists e \in \omega)[P = G_e = \{ x : G(e, x) \}] \ (P \subseteq \mathcal{X}). \]

We have already shown that \( \Sigma^0_1, \Sigma^1_1, \Pi^1_1 \) and all their relativizations are \( \Sigma^* \)-pointclasses, but there are many more, cf. Problem 1H.23.
1H. Partial functions and the substitution property

1H.12. Good universal sets. The property of \( \omega \)-parametrization is much easier to use if we move it to \( \mathcal{N} \) and “massage” a bit the universal sets it guarantees.

A pointset \( G \subseteq \mathcal{N} \times \mathcal{X} \) is a good universal set (or parametrization) for a pointclass \( \Gamma \) at \( \mathcal{X} \), if:

1. \( G \in \Gamma \).
2. There is a recursive function \( S_\mathcal{X} = S : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \) such that for every \( P \subseteq \mathcal{N} \times \mathcal{X} \) which is in \( \Gamma \) and some recursive \( \varepsilon \in \mathcal{N} \),

\[
P(\alpha, x) \iff G(S(\varepsilon, \alpha), x).
\]

(1H-10)

Theorem 1H.13 (Good parametrizations). Every \( \Sigma^* \)-pointclass \( \Gamma \) has a good universal set \( G \subseteq \mathcal{N} \times \mathcal{X} \) at every \( \mathcal{X} \).

Moreover, if \( G(\varepsilon, x) \) is a good universal set for \( \Gamma \), then:

1. \( G \) is a universal set for \( \Gamma \) and \( \Gamma \), i.e., for \( P \subseteq \mathcal{X} \),

\[
P \in \Gamma \iff (\exists \text{ recursive } \varepsilon)[P = G_\varepsilon = \{ x : G(\varepsilon, x) \}],
\]

\[
P \in \Gamma \iff (\exists \varepsilon)[P = G_\varepsilon = \{ x : G(\varepsilon, x) \}].
\]

2. (The Second Recursion Theorem). For every \( P \subseteq \mathcal{N} \times \mathcal{X} \) in \( \Gamma \), there is a recursive \( \varepsilon \in \mathcal{N} \) such that

\[
P(\varepsilon, x) \iff G(\varepsilon, x) \quad (x \in \mathcal{X}).
\]

Proof. For the first claim, let \( H \subseteq \omega \times \mathcal{N} \times \mathcal{X} \) be universal for \( \Gamma \) as in 1F.8, and set

\[
G(\varepsilon, x) \iff H(\varepsilon(0), \varepsilon^*, x).
\]

This is in \( \Gamma \) by closure under recursive substitutions. If \( P \subseteq \mathcal{N} \times \mathcal{X} \) is in \( \Gamma \), then there is some \( \varepsilon \) such that \( P = H_\varepsilon \) and then

\[
P(\alpha, x) \iff H(\varepsilon, \alpha, x) \iff G(S(\varepsilon, \alpha), x)
\]

if we set

\[
S(\varepsilon, \alpha) = \beta \iff \beta(0) = \varepsilon(0) & (\forall s)[\beta(s + 1) = \alpha(s)].
\]

(1) Suppose \( P \subseteq \mathcal{X} \) is in \( \Gamma \), let

\[
P'(\alpha, x) \iff P(x),
\]

choose a recursive \( \varepsilon \) which satisfies (1H-10) for \( P' \), let \( \alpha_0 = \lambda 0 \) be the constant 0 function, and compute:

\[
P(x) \iff P'(\alpha_0, x) \iff G(S(\varepsilon, \alpha_0), x).
\]

This completes the argument because \( S(\varepsilon, \alpha_0) \) is recursive.

The proof for the boldface case is a bit easier.
1. Recursion on Polish spaces

(2) By the classical diagonal argument for the Second Recursion Theorem, given \( P(\alpha, x) \) in \( \Gamma \), set

\[
Q(\alpha, x) \iff P(S(\alpha, \alpha), x)
\]

which is also in \( \Gamma \) because \( \Gamma \) is closed under recursive substitutions. Choose a recursive \( \tilde{\varepsilon} \) such that \( Q = G_{\tilde{\varepsilon}} \) and compute:

\[
P(S(\alpha, \alpha), x) \iff Q(\alpha, x) \iff G(S(\tilde{\varepsilon}, \alpha), x);
\]

and if we set \( \alpha = \tilde{\varepsilon} \) in this and take

\[
\varepsilon = S(\tilde{\varepsilon}, \tilde{\varepsilon}),
\]

we get the required equivalence

\[
P(\varepsilon, x) \iff Q(\tilde{\varepsilon}, x) \iff G(\varepsilon, x).
\]

Problem 1H.14. Prove (1) in the theorem for \( \Gamma' \).

1H.15. The Kleene calculus for \( \Sigma^* \)-recursion. Suppose \( \Gamma \) is a \( \Sigma^* \)-pointclass, let

\[
G^{\mathcal{X} \times \omega}_{\Gamma} = G \subseteq \mathcal{N} \times \mathcal{X} \times \omega
\]

be a good universal set for \( \Gamma \) at \( \mathcal{X} \times \omega \), and for each \( \mathcal{Y} \) set

\[
\{ \varepsilon \}^{\mathcal{X} \times \mathcal{Y}}_{\Gamma}(x) = \{ \varepsilon \}_\Gamma(x) = \Phi_G(\varepsilon, x)
\]

in the notation of (1H-6), (1H-7).

This is an adaptation to \( \Sigma^* \)-recursion of the classical Kleene notation for recursive partial functions. Following the usual practice, we will typically omit the cumbersome superscripts \( \mathcal{X}, \mathcal{Y} \), unless they are necessary for clarity, and we will skip the subscript in the most common use of this notation, when \( \Gamma = \Sigma^0_1 \); i.e., , by convention,

\[
\{ \varepsilon \}_\mathcal{X}(x) = \{ \varepsilon \}_{\Sigma^0_1}(x).
\]

We collect in one result, for easy reference, some of the basic properties of “the Kleene calculus” which we have already established:

**Theorem 1H.16** (after Kleene). Let \( \Gamma \) be a fixed \( \Sigma^* \)-pointclass.

1. For all \( \mathcal{X} \) and \( \mathcal{Y} \), the partial function \( (\varepsilon, x) \mapsto \{ \varepsilon \}_\mathcal{X}(x) \) defined from any good universal set is potentially \( \Gamma \)-recursive.

2. A partial function \( f: \mathcal{X} \rightarrow \mathcal{Y} \) is potentially \( \Gamma \)-recursive if and only if there is some recursive \( \varepsilon \in \mathcal{N} \) such that \( f \subseteq \{ \varepsilon \}_\mathcal{Y} \), i.e.,

\[
f(x) \downarrow \implies f(x) = \{ \varepsilon \}_\mathcal{Y}(x).
\]

3. A partial function \( f: \mathcal{X} \rightarrow \mathcal{Y} \) is potentially \( \Gamma \)-recursive if and only if there is some \( \varepsilon \in \mathcal{N} \) such that \( f \subseteq \{ \varepsilon \}_\Gamma \), i.e.,

\[
f(x) \downarrow \implies f(x) = \{ \varepsilon \}_\Gamma(x).
\]
(4) (The Second Recursion Theorem). For every potentially $\Gamma$-recursive $f : \mathcal{N} \times \mathcal{X} \rightarrow \mathcal{Y}$, there is a recursive $\tau$ such that

\[
(1H-12) \quad f(\tau, x) \downarrow \iff f(\tau, x) = \{\tau\}_\Gamma(x).
\]

**Proof.** (1) – (3) follow immediately from the Theorems 1H.4 and 1H.13, and (4) is proved by the usual argument or by quoting appropriately (2) of Theorem 1H.13. \(\square\)

**Problem 1H.17.** Prove (4) of the theorem.

The classical Second Recursion Theorem for recursion on $\omega$ is often described as a *fixed point theorem* on the class of recursive partial functions. This is not a very useful way to think about it in the present context, where the potentially $\Gamma$-recursive partial functions may have very complex domains and the key equation in (1H-12) holds only for $x$ such that $f(\tau, x) \downarrow$. Its most important applications involve showing that the class of potentially $\Gamma$-recursive partial functions is closed under various kinds of recursive definitions, like the following:

**Theorem 1H.18.** If $\Gamma$ is a $\Sigma^*$-pointclass, then the collection of potentially $\Gamma$-recursive partial functions is closed under primitive recursion.

**Proof.** We are given potentially $\Gamma$-recursive partial functions

\[
g : \mathcal{X} \rightarrow \mathcal{Y}, \quad f : \mathcal{Y} \times \omega \times \mathcal{X},
\]

and we define $f : \omega \times \mathcal{X} \rightarrow \mathcal{Y}$ by the recursion

\[
\begin{cases}
  f(0, x) = g(x) \\
  f(m + 1, x) = h(f(m, x), m, x).
\end{cases}
\]

It is important to register the domain of convergence of $f$:

\[
f(0, x) \downarrow \iff g(x) \downarrow,
\]

\[
f(n + 1, x) \downarrow \iff f(n, x) \downarrow \& h(f(n, x), n, x) \downarrow,
\]

so that in particular,

\[
f(n, x) \downarrow \iff (\forall i < n)[f(i, x) \downarrow].
\]

Let

\[
\varphi(\varepsilon, m, x) = \begin{cases}
  g(x), & \text{if } m = 0, \\
  h(\{\varepsilon\}(m - 1, x), m - 1, x), & \text{if } m > 0,
\end{cases}
\]

This is potentially $\Gamma$-recursive by Problem 1H.2 and Lemma 1H.9, and so there is a recursive $\tau$ such that

\[
\varphi(\tau, m, x) \downarrow \iff \left(\varphi(\tau, m, x) = \{\tau\}(m, x)\right).
\]
1. Recursion on Polish spaces

For this \( \varepsilon \) then we have

\[
g(x) \downarrow \implies \{ \varepsilon \}(0, x) = g(x),
\]

\[
h(\{ \varepsilon \}(n, x), n, x) \downarrow \implies \{ \varepsilon \}(n + 1, x) = h(\{ \varepsilon \}(n, x), n, x),
\]

which by an easy induction on \( n \) proves

\[
f(n, x) \downarrow \implies \big( f(n, x) = \{ \varepsilon \}(n, x) \big),
\]

so that \( f \) is potentially \( \Gamma \)-recursive.

We include in the problems a similar result about “nested recursion”, and in the next section we will take up the more interesting applications of the Second Recursion Theorem to effective transfinite recursion.

Additional problems for Section 1H

One of the most useful properties of \( \Gamma \)-recursion is embodied in the following trivial extension of Theorem 1D.11 (2), which we put down for the record.

**Problem 1H.19.** If \( \Gamma \) is a \( \Sigma \)-pointclass and \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is potentially \( \Gamma \)-recursive, then for each \( x \),

\[
f(x) \downarrow \implies f(x) \text{ is } \Delta(x)\text{-recursive}.
\]

**Problem 1H.20 (Minimalization).** Prove that if \( \Gamma \) is a \( \Sigma^* \)-pointclass, then the class of potentially \( \Gamma \)-recursive partial functions is closed under minimalization,

\[
f(x) = \mu n [g(n, x) = 0].
\]

**Problem 1H.21 (Nested recursion).** Prove that if \( \Gamma \) is a \( \Sigma^* \)-pointclass, then the class of potentially \( \Gamma \)-recursive partial functions is closed under definitions by *nested recursion*,

\[
f(0, x) = g_1(x), \quad f(n + 1, x) = h(f(n, g_2(n, x)), n, x).
\]

**1H.22. The dual Suslin quantifier.** For any \( P \subseteq \mathcal{X} \times \omega \), put

\[
(S) \forall u P(x, u) \iff \neg(S) \forall P(x, u) \iff (\exists \alpha)(\forall t) P(x, \pi(t)).
\]

**Problem 1H.23.** Suppose that \( \Gamma \) is a \( \Sigma \) pointclass. Prove:

1. \( S \Gamma \) and \( S \bar{\Gamma} \) are \( \Sigma \)-pointclasses which contain \( \Gamma \) and are closed under both \( \exists \Sigma^0_1 \) and \( \forall \Sigma^0_1 \).

2. \( S \Gamma \) is closed under \( S \), and so \( S \Gamma \) is closed under \( S \).

3. If \( \Gamma \) is a \( \Sigma^* \) pointclass, then so are \( S \Gamma \) and \( S \bar{\Gamma} \).

Infer that

\[
\Sigma^0_1 \subseteq S \Sigma^0_1 \subseteq \bar{S} \Sigma^0_1 \subseteq \cdots
\]
II. CLASSICAL AND EFFECTIVE BOREL SETS AND FUNCTIONS

are \( \Sigma^* \)-pointclasses which include \( \Pi^1_1 \cup \Sigma^1_1 \) and are all contained in \( \Delta^2_1 \).

HINT: Show first that \( S \Gamma \) is closed under \( S \), without using the fact that \( \Gamma \) is closed under \( \exists^\omega \). For the last claim, prove that for any quantifier \( Q \) and any pointclass \( \Gamma \),

\[
(1H-2) \quad \neg \Gamma \subseteq Q \Gamma \Longrightarrow \neg Q \Gamma \subseteq \mathcal{Q} \mathcal{Q} \Gamma.
\]

The classical (boldface versions of) these pointclasses were extensively studied through the 1950s, especially by Russian mathematicians.

**Problem 1H.24** (Dellacherie’s Lemma for potential recursion). Suppose \( \Gamma \) is a \( \Sigma \)-pointclass. Prove that whether a partial function \( f : X \to Y \) is potentially \( \Gamma \)-recursive depends only on the frames of \( X \) and \( Y \).

II. Classical and effective Borel sets and functions

Recall the definitions of Lebesgue’s *additive Borel pointclasses* in (1F-2): starting with the open sets in \( \Sigma^0_0 \), we define \( \Sigma^0_{\xi} \) for every ordinal \( \xi > 0 \) by the ordinal recursion

\[
(1I-1) \quad A \in \Sigma^0_{\xi} \iff A = \bigcup_{i \in \omega} (X \setminus A_i) \text{ with each } A_i \in \bigcup_{\eta < \xi} \Sigma^0_{\eta},
\]

and we set, as usual,

\[
\Pi^0_{\xi} = \neg \Sigma^0_{\xi}, \quad \Delta^0_{\xi} = \Sigma^0_{\xi} \cap \Pi^0_{\xi}.
\]

The most important pointclass of Borel (or Borel measurable) sets is the union of all these,

\[
(1I-2) \quad \mathcal{B} = \bigcup_{\xi} \Sigma^0_{\xi}.
\]

A function \( f : X \to Y \) is Borel measurable if the inverse image \( f^{-1}[G] \) of every open set \( G \subseteq Y \) is Borel, i.e.,

\[
(1I-3) \quad \text{for every } s, \ f^{-1}[N(Y, s)] \in \mathcal{B}.
\]

Borel sets and Borel measurable functions are indispensable tools for the development of classical analysis, especially measure theory, and many concrete sets and functions one meets in analysis are Borel measurable. Moreover, Borel sets have most of the regularity properties one could hope for: they are \( \mu \)-measurable for every measure \( \mu \) on a space \( X \) which is determined by its values on the open sets, they have the property of Baire, every uncountable Borel set has a perfect subset, etc. We will prove some of these results in the next Chapter. The pointclasses \( \Sigma^0_{\xi}, \Pi^0_{\xi} \) also carry an interesting structure theory—the basic content of Lebesgue [1905]—which we will not cover in these notes.
1. Recursion on Polish spaces

Our main concern in this Section is to introduce the effective Borel or hyperarithmetic pointsets which we will then study in some detail in Chapter 2.

We start with a laundry list of elementary (and easy) properties of \( \mathcal{B} \).

**Theorem 1I.1.**

1. \( \mathcal{B} = \bigcup_{\xi < \aleph_1} \Sigma^0_\xi \subseteq \Delta^1_1 \).
2. \( \mathcal{B} \) is closed under complementation, countable unions, countable intersections and Borel measurable substitutions. In particular,
   \[
   \mathcal{B} = \bigcup_{\xi < \aleph_1} \Pi^0_\xi = \bigcup_{\xi < \aleph_1} \Delta^0_\xi.
   \]
3. For each \( X \), \( \mathcal{B} \upharpoonright X \) is the smallest family of subsets of \( X \) which contains all the open (or all the closed) subsets of \( X \) and is closed under complementation and countable unions.
4. For each \( X \), \( \mathcal{B} \upharpoonright X \) is the smallest family of subsets of \( X \) which contains all the open and all the closed subsets of \( X \) and is closed under countable unions and intersections.
5. Every continuous function is Borel measurable, and the class of Borel measurable functions is closed under composition and pointwise limits, i.e., if every \( f_n : X \to Y \) is Borel measurable and for every \( x \), \( \lim_n f_n(x) \) exists, then
   \[
   f(x) = \lim_n f_n(x),
   \]
   is also Borel measurable.
6. \( \mathcal{B} \) is closed under Borel substitutions.

The basic result of the classical theory is

**Theorem 1I.2** (Suslin [1917]). \( \mathcal{B} = \Delta^1_1 \).

In the terminology we have been using, Suslin’s Theorem implies that

\( f : X \to Y \) is Borel measurable \( \iff \) \( f \) is \( \Delta^1_1 \)-recursive,

and so by the (easy) boldface version of Lemma 1G.1:

**Corollary 1I.3.** The inverse \( f^{-1} : Y \to X \) of every Borel measurable bijection \( f : X \to Y \) is also Borel measurable.

This is one of many important consequences of Suslin’s Theorem, notoriously claimed (with a wrong proof) by Lebesgue [1905] and then proved correctly by Suslin [1917]. We will prove substantial refinements and extensions of Theorem 1I.2 (and some of what follows from it) in the next Chapter.

Recall the convention that recursive always means \( \Sigma^0_1 \)-recursive, so that for all \( X, Y \), by (1H-11),

\[
\{ \varepsilon \}(x) = \{ \varepsilon \}^{X, Y}(x) = \Phi_G(\varepsilon, x)
\]
II. Classical and effective Borel sets and functions

where $G \subseteq \mathcal{N} \times \mathcal{X} \times \omega$ is a good universal set for $\Sigma^0_1$ at $\mathcal{N} \times \mathcal{X}$. Let $S^\mathcal{X} = S : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ be the recursive function that comes with the good universal set $G$ by the definition 1H.12. The relevant (“local”) version of the classical $S^m_n$-Theorem P.8 for this notion of recursion is the following:

**Theorem 1I.4** (The $S^\mathcal{X}$-Theorem). If $f : \mathcal{N} \times \mathcal{X} \to \mathcal{Y}$ is potentially recursive, then there is a recursive $\hat{f} \in \mathcal{N}$ such that

$$f(\alpha, x) \downarrow \implies \left( f(\alpha, x) = \{S(\hat{f}, \alpha)\}(x) \right).$$

**Proof.** Fix a $\Sigma^0_1$ relation $P_f$ which computes $f$, so that

$$f(\alpha, x) \downarrow \implies (\forall s) \left( f(\alpha, x) \in N_s \iff P_f(\alpha, x, s) \right),$$

and let $G \subseteq \mathcal{N} \times \mathcal{X} \times \omega$ be the good universal set for $\Sigma^0_1$ at $\mathcal{X} \times \omega$ used to define the Kleene map $(\varepsilon, x) \mapsto \{\varepsilon\}(x)$. By the basic property of good universal sets, there is a recursive $\hat{f} \in \mathcal{N}$ such that

$$P_f(\alpha, x, s) \iff G(S(\hat{f}, \alpha), x, s),$$

so that

$$f(\alpha, x) \downarrow \implies (\forall s) \left( f(\alpha, x) \in N_s \iff G(S(\hat{f}, \alpha), x, s) \right) \implies (\forall s) \left( f(\alpha, x) \in N_s \iff \{S(\hat{f}, \alpha)\}(x) \in N_s \right),$$

the last by the definition of $\{\varepsilon\}(x)$, which then gives the required implication.\qed

1I.5. $\mathcal{N}$-codes for Borel sets. With each countable ordinal $\xi$ we associate the set $BC_\xi \subseteq \mathcal{N}$ by the following ordinal recursion:

$$BC_0 = \{\alpha : \alpha(0) = 0\},$$

and for $\xi > 0$,

$$BC_\xi = BC_0 \cup \left\{ \alpha : \alpha(0) \neq 0 \& (\forall n) \left[ \{\alpha^\ast\}(n) \downarrow \& \{\alpha^\ast\}(n) \in \bigcup_{\eta<\xi} BC_\eta \right] \right\},$$

where $\alpha^\ast(t) = \alpha(t + 1)$ as usual and $\{\alpha^\ast\} : \omega \to \mathcal{N}$ is the partial function of 1H.15 which is (\Sigma^0_1-) recursive on its domain. For each fixed space $\mathcal{X}$ and each $\xi$, we define the mapping

$$\pi_{c^\mathcal{X}_\xi} : BC_\xi \to \mathcal{P}(\mathcal{X})$$

by the recursion

$$\pi_{c^\mathcal{X}_\xi}(\alpha) = \begin{cases} \mathcal{N}(\mathcal{X}, \alpha(1)) & \text{if } \alpha(0) = 0, \\ \bigcup_n \left( \mathcal{X} \setminus \pi_{c^\mathcal{X}_{\eta(n)}}(\{\alpha^\ast\}(n)) \right) & \text{otherwise,} \end{cases}$$

where $\eta(n) = \text{least } \eta \text{ so that } \{\alpha^\ast\}(n) \in BC_\eta$. 

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**Lemma 1I.6.** For all \( \eta, \xi \)
\[
\eta \leq \xi \implies BC_\eta \subseteq BC_\xi \& \pi_\eta = \pi_\xi \upharpoonright BC_\eta.
\]

**Proof** is easy by induction on \( \xi \). \( \dashv \)

The lemma implies that the union of the \( \pi c^{X}_\xi \)'s is a function,
\[
\pi c^{X} = \bigcup_\xi \pi c^{X}_\xi : BC \to \mathcal{P}(X)
\]
where
\[
BC = \bigcup_\xi BC_\xi = \text{the set of Borel codes},
\]
and we also set
\[(1I-6)\quad B^{X}_\alpha = \pi c^{X}(\alpha) \quad (\alpha \in BC).\]

The terminology is justified by the following

**Lemma 1I.7.** Each \( B^{X}_\alpha \) is a Borel subset of \( X \), and every Borel subset of \( X \) is \( B^{X}_\alpha \) for some \( \alpha \in BC \).

**Proof** is easy, by two inductions on \( \xi \) following the recursive definitions of \( BC_\xi \) and \( \Sigma^0_\xi \). \( \dashv \)

The next lemma is rather obvious, in principle, but we include a full proof as an example of how the \( S^X \)-Theorem is used—and why it is needed.

**Lemma 1I.8.** The class of Borel subsets of \( X \) is uniformly closed under complementation, countable union and countable intersection in the following precise sense.

1. There is a recursive function \( u_1(\alpha) \) such that if \( \alpha \) is a Borel code of some \( A \subseteq X \), then \( u_1(\alpha) \) is a Borel code of \( X \setminus A \),
\[
\alpha \in BC \implies B^{X}_{u_1(\alpha)} = X \setminus B^{X}_\alpha.
\]

2. There is a recursive function \( u_2(\varepsilon) \) such that if for each \( i \), \( \{\varepsilon\}_i \downarrow \) and \( \{\varepsilon\}_i \) is a Borel code of some set \( A_i \subseteq X \), then \( u_2(\varepsilon) \) is a Borel code of \( \bigcup_i A_i \),
\[
(\forall i)[\{\varepsilon\}_i \in BC] \implies B^{X}_{u_2(\varepsilon)} = \bigcup_i B^{X}_{\{\varepsilon\}_i}.
\]

3. There is a recursive function \( u_3(\varepsilon) \) such that if for each \( i \), \( \{\varepsilon\}_i \downarrow \) and \( \{\varepsilon\}_i \) is a Borel code of some set \( A_i \subseteq X \), then \( u_3(\varepsilon) \) is a Borel code of \( \bigcap_i A_i \),
\[
(\forall i)[\{\varepsilon\}_i \in BC] \implies B^{X}_{u_3(\varepsilon)} = \bigcap_i B^{X}_{\{\varepsilon\}_i}.
\]
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Proof. For (1), choose a recursive \( \hat{f}_1 \) so that
\[
\{ \hat{f}_1 \}(\alpha, t) = \alpha \quad \text{(for all } t \in \omega, \alpha \in \mathcal{N})
\]
and define
\[
u_1(\alpha) = (1)^\sim S(\hat{f}_1, \alpha);
\]
now if \( \nu_1(\alpha) = \beta \), then \( \beta(0) = 1 \) and \( \beta^* = S(\hat{f}_1, \alpha) \) so that
\[
\{ \beta^* \}(i) = \{ S(\hat{f}_1, \alpha) \}(i) = \alpha
\]
and hence \( \beta \) codes \( \bigcup_i (X \setminus A_i) = X \setminus A \).

Similarly, for (2), choose a recursive \( \hat{f}_2 \) so that for all \( \varepsilon \), \( t \)
\[
u_1(\{ \varepsilon \}(t)) \downarrow \Rightarrow \{ \hat{f}_2 \}(\varepsilon, t) = \nu_1(\{ \varepsilon \}(t))
\]
and let \( \nu_2(\varepsilon) = (1)^\sim S(\hat{f}_2, \varepsilon) \);
now if \( \varepsilon \) satisfies the hypothesis so that each \( \{ \varepsilon \}(i) \) is a Borel code of some \( A_i \), then \( \{ S(\hat{f}_2, \varepsilon) \}(i) = \nu_1(\{ \varepsilon \}(i)) \) is a Borel code of \( X \setminus A_i \) for each \( i \) and hence \( \nu_2(\varepsilon) \) is a Borel code of \( \bigcup_i (X \setminus (X \setminus A_i)) = \bigcup_i A_i \).

The construction for (3) is similar. ⊣

1I.9. \( \mathcal{N} \)-codes for \( \Sigma^1_1, \Delta^1_1 \). Fix a good universal set \( G \subseteq \mathcal{N} \times X \) for \( \Pi^1_1 \) at \( X \), put
\[
CD^X = CD = \{ \alpha : (\forall x)[G((\alpha)_0, x) \Leftrightarrow \neg G((\alpha)_1, x)]\},
\]
and for each \( \alpha \in CD \), let
\[
D^X_\alpha = D_\alpha = \{ x \in X : G((\alpha)_0, x) \}.
\]
This is a coding of \( \Delta^1_1 \upharpoonright X \) which is effective in the following sense:

Problem 1I.10. Fix a space \( X \). Prove that there are recursive functions \( \nu_0 : \omega \to \mathcal{N} \) and \( \nu_1 : \mathcal{N} \to \mathcal{N} \) such that:

1. For every \( s \), \( \nu_0(s) \in CD \) and \( D_{\nu_0(s)} = N(X, s) \).

2. If for all \( i \), \( \{ \varepsilon \}(i) \downarrow \) \& \( \{ \varepsilon \}(i) \in CD \), then
\[
\nu_1(\varepsilon) \in CD \& D_{\nu_1}(\varepsilon) = \bigcup_i (X \setminus D_{\{ \varepsilon \}(i)}).
\]

The next result is an effective version of the easy half of Suslin’s Theorem, the inclusion \( B \subseteq \Delta^1_1 \), and its proof is our first example of definition by effective (transfinite) recursion.

Theorem 1I.11. Every Borel set is uniformly \( \Delta^1_1 \), in the following, precise sense: for each \( X \), there is a potentially recursive \( u = u^X : \mathcal{N} \to \mathcal{N} \) such that
\[
\alpha \in BC \Rightarrow u(\alpha) \downarrow \& u(\alpha) \in CD^X \& B^X_\alpha = D^X_{u(\alpha)}.
\]
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Proof. The inclusion $B \subseteq \Delta^1_1$ is proved by induction on $\xi$, and so the definition of $u$ must be given by some sort of effective recursion. The idea is to seek a potentially recursive function $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for a suitable $\varepsilon$, the required uniformity $u$ will be defined from it by the Second Recursion Theorem, i.e., by finding an $\varepsilon$ such that

$$f(\varepsilon, \alpha) \downarrow \Rightarrow f(\varepsilon, \alpha) = \{\varepsilon\}(\alpha)$$

and then setting $u(\alpha) = f(\varepsilon, \alpha)$. So we seek to formulate conditions on $f$ which will insure this.

1. If $\alpha(0) = 0$, then $B_\alpha = N_{\alpha(1)} = D_{\nu_0(\alpha(1))}$ (with the notation of Problem 1.10); so we need to have

$$f(\varepsilon, \alpha) = \nu_0(\alpha(1)) \quad (\alpha(0) = 0).$$

2. Assume $\alpha(0) \neq 0$, so that we may assume (pending a proof later) that

$$B_\alpha = \bigcup_i (X \setminus B_{i(\alpha)}(i)) = \bigcup_i (X \setminus D_{\{\varepsilon\}(\alpha)(i)}).$$

Define the potentially recursive function $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ by

$$g(\varepsilon, \alpha)(i) = \{\varepsilon\}(\{\alpha^*\}(i))$$

and choose a recursive $\hat{g}$ such that

$$g(\varepsilon, \alpha) \downarrow \Rightarrow \left( g(\varepsilon, \alpha) = \{S(\hat{g}, \varepsilon)\}(\alpha) \right),$$

so that at least if $g(\varepsilon, \alpha) \downarrow$ we have

$$B_\alpha = \bigcup_i \left( X \setminus D_{\{S(\hat{g}, \varepsilon)\}(\alpha)(i)} \right) = D_{\nu_1(\{S(\hat{g}, \varepsilon)\}(\alpha))}.'$$

We now choose by the Second Recursion Theorem a recursive $\varepsilon$ such that

$$\{\varepsilon\}(\alpha) = \begin{cases} \nu_0(\alpha(1)) & \text{if } (\alpha(0) = 0), \\ \nu_1(\{S(\hat{g}, \varepsilon)\}(\alpha)) & \text{otherwise}, \end{cases}$$

at least when the function on the right converges, The proof is completed by showing by induction on $\xi$, that

$$\alpha \in BC_\xi \Rightarrow \left( \{\varepsilon\}(\alpha) \downarrow \& B_\alpha = D_{\{\varepsilon\}(\alpha)} \right),$$

easily, following the analysis above, and finally setting $u(\alpha) = \{\varepsilon\}(\alpha)$. ⊢
11. Classical and effective Borel sets and functions

11.12. The hyperarithmetical (effective Borel) sets. A pointset $A$ is hyperarithmetical (hyp, effective Borel) if it is Borel with a recursive code,

$$A \in \text{HYP} \iff A = B^\omega_\alpha$$

and a function $f : \mathcal{X} \to \mathcal{Y}$ is effectively Borel measurable if it is HYP-recursive, which is the way we will usually refer to these functions.

Notice that there are only countably many sets in each $\text{HYP} \upharpoonright \mathcal{X}$, including $\text{HYP} \upharpoonright \omega$.

**Corollary 11.13.** $\text{HYP} \subseteq \Delta^1_1$.

**Proof.** If $A = B_\alpha \subseteq \mathcal{X}$, then by the theorem, $u(\alpha) \downarrow$ and

$$x \in A \iff G((u(\alpha))_0, x) \iff \neg G((u(\alpha))_1, x)$$

with a $\Pi^1_1$ set $G$, by the coding of $\Delta^1_1$; and if $\alpha$ is recursive, then $u(\alpha)$ is recursive, so $A \in \Delta^1_1$.

The converse of this is also true, of course, part of the effective version of Suslin’s Theorem that we will prove in the next chapter.

One can argue that the definition of HYP is natural, because a Borel code $\alpha$ for some $A \subseteq \mathcal{X}$ gives us a “construction” of $A$ from the basic nbhds by an iteration of complementation and countable union—and if $\alpha$ is recursive, then this construction is effective. On the other hand, much is built into the specific coding that we used, and it is not obvious that the definition of $\text{HYP} \upharpoonright \mathcal{X}$ depends only on the frame of $\mathcal{X}$ and not on any accidental properties of the compatible pair or the specific coding of the basic nbhds that we have used.

The key to getting a coding-invariant characterization of $\text{HYP} \upharpoonright \mathcal{X}$ is that this class of subsets of $\mathcal{X}$ is countable, and so it can be coded in $\omega$.

In the most general sense, a **coding in $\omega$** for a set $S$ is any surjection

$$c : I \to S$$

of some $I \subseteq \omega$ onto $S$. We call $I$ the index set of the coding $c$.

11.14. Effective $\sigma$-algebras on a recursive Polish space. A (countable) family $\mathcal{S}$ of subsets of $\mathcal{X}$ is an effective $\sigma$-algebra if there is a coding

$$i \mapsto A_i = c(i) \quad (i \in I)$$

of $\mathcal{S}$ in $\omega$ with the following properties:

1. For every semirecursive $P \subseteq \mathcal{X} \times \omega$, there is a recursive function
   $$u = u_P : \omega \to \omega$$
   such that
   $$(\forall s) \left( \{ x : P(x, s) \} = A_{u(s)} \right).$$
1. Recursion on Polish spaces

(2) There is a recursive partial function $v : \omega \to \omega$ such that for every $e$,

$$\forall i \left( \{e\}(i) \downarrow \& \{e\}(i) \in I \right) \implies \left( v(e) \downarrow \& A_v(e) = \bigcup \{X \setminus A(e(i)) : i \in \omega \} \right).$$

Notice that whether some $S \subseteq \mathcal{P}(X)$ is an effective $\sigma$-algebra depends only on the frame of $X$.

**Theorem 11.15.** For every $\mathcal{X}$, $\text{HYP} \upharpoonright \mathcal{X}$ is the smallest effective $\sigma$-algebra on $\mathcal{X}$.

**Outline of Proof.** To prove that $\text{HYP} \upharpoonright \mathcal{X}$ is an effective $\sigma$-algebra, set

$$I = \left\{ e : \{e\} : \omega \to \omega \text{ is a total recursive function and } \{e\} \in \mathcal{BC} \right\}$$

and set $c(e) = B^X_{\{e\}}$.

For the converse, suppose $S$ is an effective $\sigma$-algebra on $\mathcal{X}$ coded by some $c : J \to S$, and define by effective recursion a recursive partial $g : \omega \to \omega$ such that $e \in I \implies \left( g(e) \downarrow \& g(e) \in J \& B_{\{e\}} = c(g(e)) \right)$. ⊣

**Problem 11.16.** Prove Theorem 11.15.

### Additional problems for Section 11

Definitions by effective recursion come in all shapes (and sizes) and it is not feasible to prove once and for all an elegant result which can then be quoted and applied easily without getting one’s hands dirty. We describe one such effort here, which works in many cases and which, in any case, helps understand the method.

**Strict well founded relations.** A binary relation $\prec \subseteq S \times S$ is (strictly) well founded if every non-empty $A \subseteq \mathcal{X}$ has a $\prec$-minimal member,

$$\emptyset \neq A \subseteq S \implies (\exists m \in A)(\forall x \in S)[x \neq m].$$

A function $f : S \to \mathcal{Y}$ is defined by recursion on $\prec$ if it satisfies an equation

\begin{equation}
(11-1) \quad f(x) = F(f(\upharpoonright \{u \in \mathcal{X} : u \prec x\}, x) \quad (x \in S)
\end{equation}

where $F : \text{Domain}(F) \to \mathcal{Y}$ with

$$\text{Domain}(F) = \left\{ (h, x) : x \in S, h : S \to \mathcal{Y} \& (\forall u \prec x) h(u) \downarrow \right\}.$$  

If $\Gamma$ is a $\Sigma^*$-pointclass and $S \subseteq \mathcal{X}$, then $F$ is $\Gamma$-effective if there is a potentially $\Gamma$-recursive $g : \mathcal{N} \times \mathcal{X} \to \mathcal{Y}$ such that for all $\varepsilon \in \mathcal{N}, x \in \mathcal{X}$,

\begin{equation}
(11-2) \quad (\forall u)[u \prec x \in S \implies \{\varepsilon\}(u) \downarrow] \\
\quad \implies g(\varepsilon, x) \downarrow \& g(\varepsilon, x) = F(\{\varepsilon\} \upharpoonright \{u : u \prec x\}, x)
\end{equation}
II. Classical and effective Borel sets and functions

When (II-1) and (II-2) hold, we say that $f$ is defined by **Γ-effective recursion** on $\prec$.

**Problem 11.17** (after Kleene). Prove that if $\Gamma$ is a $\Sigma^*\text{-pointclass}$, $\prec$ is a wellfounded relation on some $S \subseteq \mathcal{X}$ and $f : S \to \mathcal{Y}$ is defined by $\Gamma$-effective recursion, then $f$ is potentially $\Gamma$-recursive.

Notice that there are no definability hypotheses on $S$ or the relation $\prec$ in this result and in fact we can define the relevant $\bar{\tau}$ in the proof directly from $g$, with no knowledge of the relation $\prec$; this is important in more subtle applications of this method where we define $\bar{\tau}$ before we even know that $\prec$ is well founded and then show that it has whatever properties we need if the relevant relation $\prec$ happens to be well founded.

**Problem 11.18.** Prove that every HYP-recursive $f : \mathcal{X} \to \mathcal{Y}$ is Borel measurable, and every Borel measurable $f : \mathcal{X} \to \mathcal{Y}$ is HYP($\varepsilon$)-recursive for some $\varepsilon \in \mathcal{N}$, where

$$A \in \text{HYP}(\varepsilon) \Leftrightarrow A = B^n_\alpha$$

with some $\alpha$ which is recursive in $\varepsilon$. 

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CHAPTER 2

STRUCTURE THEORY FOR POINTCLASSES

In this chapter we will study (primarily) the pointclasses $\Delta^1_1$, $\Pi^1_1$, and $\Sigma^1_2$ and their relativized and boldface versions. Some of the main results will be formulated in more general terms not immediately relevant to the purpose at hand, because they have applications to the study of the higher analytical pointclasses—and the proofs are no harder.

Much of the material in this chapter is extracted from Chapters 2, 4 and 7 of Moschovakis [2009], briefly DST. In a few places we will discuss some results and point to DST for the proofs.

2A. Trees and the Perfect Set Theorem

The main result of this section is Suslin’s Perfect Set Theorem, that every uncountable $\Sigma^1_1$ set has a non-empty perfect subset. We will later prove a very strong, effective refinement of this important, classical result.

Trees. In descriptive set theory, a tree on a (non-empty) set $X$ is a set $T$ of finite sequences of members of $X$ closed under initial segments,

$T \subseteq X^{<\omega} \& (\forall u, v)[u \sqsupseteq v \implies u \in T]$.

The members of $T$ are its nodes or finite paths, and the empty sequence $\emptyset$ is a node of every non-empty tree $T$—its root. In the case $X = \omega$ with which we will be most concerned, we associate with $T$ its code set

$T^c = \{u : \text{Seq}(u) \& ((u)_0, \ldots, (u)_{\text{lh}(u)}-1) \in T\} \subseteq \omega$

and we sometimes identify $T$ with $T^c$; for example, $T$ is recursive if $T^c$ is recursive. Similarly for trees on $\omega^m$, for any $m$.

A function $f : \omega \to X$ is an infinite branch (or path) of a tree $T$, if for every $n$,

$f \upharpoonright n = (f(0), \ldots, f(n-1)) \in T$,

and the body of $T$ is the set of all its infinite branches,

$[T] = \{f : \omega \to X : (\forall n)[f \upharpoonright n \in T]\}.$
Lemma 2A.1.  (1) If we put the product topology on $X^\omega$ taking $X$ discrete, then

$C \subseteq X^\omega$ is closed $\iff C = [T]$ for some tree $T$ on $X$.

(2) A pointset $C \subseteq N$ is $\Pi^0_1$ if and only if $C = [T]$ for some recursive tree on $\omega$.

Proof.  (1) Suppose $T$ is a tree on $X$ and $f \notin [T]$; then for some $n$,

$$(f(0), \ldots, f(n-1)) \notin T,$$

so that the basic nbhd $\{g : g(0) = f(0), \ldots, g(n-1) = f(n-1)\}$ of $X^\omega$ is disjoint from $[T]$ and hence the complement of $[T]$ is open. Conversely, if $C \subseteq X^\omega$ is closed, put

$$T = \{(f(0), \ldots, f(n-1)) : f \in C\};$$

clearly $C \subseteq [T]$ and $C$ is dense in $[T]$, so $C = \overline{C} = [T] = [T]$.

(2) If $T$ is recursive, then

$$\alpha \in [T] \iff (\forall n) [\overline{\alpha(n)}] \in T^c],$$

and so $T$ is $\Pi^0_1$. For the converse we appeal to (3) of Theorem 1B.11, by which every $\Pi^0_1$ set $C \subseteq N$ is of the form

$$\alpha \in C \iff (\forall n) R(\overline{\alpha(n)})$$

where $R(u)$ is recursive and “downwards monotone”, i.e.,

$$u \sqsubseteq v \& R(v) \implies R(u);$$

so $T^c = \{u : R(u)\}$ is (the code set of) a recursive tree $T$ and $C = [T]$.  $\dashv$
Trees of pairs. We are especially interested in trees on products
\[ X = \omega \times \kappa \]
where \( \kappa \) is some infinite cardinal \( \kappa \). There is an obvious bijection of \( (\omega \times \kappa)^\omega \) with \( \omega^\omega \times \kappa^\omega = N \times \kappa^\omega \) which sends \( g : \omega \to (\omega \times \kappa) \) to \( (\alpha, f) \), where
\[
(\ast) \quad g(n) = (\alpha(n), f(n)).
\]
Let us agree that when \( T \) is a tree on \( \omega \times \kappa \) for some \( \kappa \), then we will take the body of \( T \) to be the corresponding subset of \( N \times \kappa^\omega \),
\[
[T] = \{ (\alpha, f) : \text{for all } n, (\alpha(0), f(0)), \ldots, (\alpha(n-1), f(n-1)) \in T \}.\]
This “double meaning” of \([T]\) will not cause any confusion, because it will always be clear from the context when we view \( T \) as a tree of pairs.

We will also simplify notation further by denoting an arbitrary sequence \( ((t_0, \xi_0), \ldots, (t_{n-1}, \xi_{n-1})) \) in \( \omega \times \kappa \) by \( (t_0, \xi_0, \ldots, t_{n-1}, \xi_{n-1}) \).

Trees and the Perfect Set Theorem

2A.2. \( \kappa \)-Suslin pointsets. A set \( P \subseteq X \) is \( \kappa \)-Suslin if it is the projection to \( X \) of a closed set \( C \subseteq X \times \kappa^\omega \),
\[
(2A-1) \quad P = \text{proj}[C] = \{ x \in X : (\exists f)C(x, f) \} \quad (C \subseteq X \times \kappa^\omega, C \text{ closed}).
\]
The pointclass of \( \kappa \)-Suslin sets has a rich and interesting structure with many applications to the study of projective sets. It is studied in considerable detail in DST, starting with Chapter 2 and in many later chapters. Here we will use it only to derive results about \( \Sigma^1_1 \) which stem from the following simple

Lemma 2A.3. 1. A pointset \( P \subseteq X \) is \( \Sigma^1_1 \) if and only if it is \( \aleph_0 \)-Suslin.

2. A pointset \( P \subseteq N \) is \( \Sigma^1_1 \) if and only if \( P = \text{proj}[T] \) with a recursive tree \( T \) on \( \omega \times \omega \).

**Proof.** 1. This holds because \( N = \omega^\omega = \aleph_0^\omega \): if \( P \subseteq X \), then
\[
P \in \Sigma^1_1 \iff P = \exists C \exists N C \quad \text{for some closed } C \subseteq X \times N
\]
\[
\iff P = \exists C N C \quad \text{for some closed } C \subseteq X \times N^\omega
\]
\[
\iff P \text{ is } \aleph_0 \text{-Suslin.}
\]

2. If \( P \subseteq N \) is \( \Sigma^1_1 \), then by (3) of Theorem 1B.11
\[
P(\alpha) \iff (\exists \beta)(\forall t)R(\pi(t), \bar{\beta}(t))
\]
where $R(u,v)$ is recursive and downward monotone, as in the proof of Lemma 2A.1; so
\[
T = \{(u_0,v_0,\ldots,u_{t-1},v_{t-1}) : R((u_0,\ldots,u_{t-1}),(v_0,\ldots,v_{t-1}))\}
\]
is a recursive tree on $\omega \times \omega$ and clearly $P = \text{proj } T$.

**Subtrees.** Two finite sequences $u$ and $v$ from $X$ are compatible if they have a common extension, i.e., if there is some $w$ such that both $u$ and $v$ are initial segments of $w$. This simply means that either $u = v$ or one of $u$ and $v$ is an initial segment of the other.

For each tree $T$ on $X$ and each finite sequence $u$ from $X$, let
\[
T_u = \{ v \in T : v \text{ is compatible with } u \}.
\]
Evidently $T_u$ is always a tree, the result of pruning all the “side” branches of $T$ from the point of view of $u$. In particular,
\[
T_\emptyset = T.
\]

Notice that if $u = (x_0,\ldots,x_{n-1})$ is a sequence of length $n$, then
\[
[T_u] = [T] \cap \{ f \in \mathcal{P}X : f \upharpoonright n = u \} = \bigcup_{x \in X} [T_u^\sim(x)],
\]
where for each $x \in X$,
\[
u^\sim(x) = (x_0,\ldots,x_{n-1})^\sim(x) = (x_0,\ldots,x_{n-1},x).
\]
Similarly, for a tree of pairs, if
\[
u = (t_0,\xi_0,\ldots,t_{n-1},\xi_{n-1})
\]
is a finite sequence from $\omega \times \kappa$, then
\[
[T_\nu] = \bigcup_{t<\omega,\xi<\kappa} [T_\nu^\sim(t,\xi)],
\]
so that
\[(2A-2) \quad \text{proj}[T_u] = \bigcup_{t<\omega, \xi<\kappa} \text{proj}[T_u^{-}(t,\xi)] \quad (T \text{ a tree on } \omega \times \kappa).\]

We will need an elementary fact from topology:

**Problem 2A.4.** Prove that if \( f : \mathcal{X} \rightarrow \mathcal{Y} \) is a continuous injection and \( C \subseteq \mathcal{X} \) is perfect and compact, then its image \( f[C] \subseteq \mathcal{Y} \) is also perfect and compact.

It is easy to see that both hypotheses on \( f \) and \( C \) are needed for the image \( f[C] \) to be perfect, which is why we need this fact.

**2A.5. The Perfect Set Theorem.**

1. If \( \kappa \) is an infinite cardinal, then every \( \kappa \)-Suslin \( P \subseteq \mathcal{N} \) with more than \( \kappa \) members has a non-empty, compact perfect subset.

2. (Suslin) Every uncountable \( \Sigma^1_1 \) pointset has a non-empty, compact perfect subset.

**Proof.** (1) Assume that \( |P| > \kappa \), fix a tree \( T \) on \( \omega \times \kappa \) such that
\[ P = \text{proj}[T] = \{\alpha : (\exists f \in \omega^\kappa)(\alpha, f) \in [T]\}, \]
and define the sets \( T^\xi \subseteq T \) by the following ordinal recursion:
\[
T^0 = T, \\
T^\xi+1 = \{u \in T^\xi : \text{proj}[T^\xi_u] \text{ has more than one element (in } \mathcal{N})\}, \\
T^\lambda = \bigcap_{\xi<\lambda} T^\xi, \quad \text{if } \lambda \text{ is a limit ordinal.}
\]
It is immediate that each \( T^\xi \) is a tree and
\[ \eta < \xi \implies T^\eta \supseteq T^\xi. \]

There are at most \( \kappa \) nodes in \( T \), so there must be some ordinal \( \lambda < \kappa^+ \) of cardinality \( \kappa \) such that
\[ T^{\lambda+1} = T^\lambda. \]
Choose the least such \( \lambda \) and put
\[ S = T^\lambda. \]

The heart of the proof is the following simple lemma about \( S \).

**Lemma.** \( S \neq \emptyset \).

**Proof.** Assume towards a contradiction that \( S = \emptyset \).

For each \( \alpha \in P = \text{proj}[T] \) choose \( f \in \kappa^\omega \) so that \( (\alpha, f) \in [T] \) and notice that there must exist some \( \xi < \lambda \) such that
\[ (\alpha, f) \in [T^\xi] \setminus [T^\xi+1]; \]
this is because \( (\alpha, f) \notin [T^\lambda] \) and for limit \( \zeta \),
\[ (\forall \eta < \zeta)(\alpha, f) \in [T^\eta], \implies (\alpha, f) \in [T^\xi]. \]
It follows that for some \( n \),
\[
u = (\alpha(0), f(0), \alpha(1), f(1), \ldots, \alpha(n-1), f(n-1)) \in T^\xi \setminus T^{\xi+1},
\]
i.e., by definition
\[
\alpha \in \text{proj}[T^\xi_u] \text{ and } \text{proj}[T^\xi_u] \text{ has exactly one element.}
\]
Thus we have shown that
\[
P \subseteq \bigcup \{\text{proj}[T^\xi_u] : \xi \leq \lambda, u \in T^\xi \setminus T^{\xi+1}\}
\]
which is absurd since the set on the right is the union of at most \( \kappa \) singletons and \( P \) has cardinality greater than \( \kappa \).

If \( u = (t_0, \xi_0, \ldots, t_{n-1}, \xi_{n-1}) \) and \( v = (s_0, \zeta_0, \ldots, s_{m-1}, \zeta_{m-1}) \), call \( u \) and \( v \) incompatible in the first coordinate just in case \( t_i \neq s_i \) for some \( i < n \), \( i < m \). It is immediate that every \( u \) in \( S \) has extensions \( u', u'' \in S \) which are incompatible in the first coordinate—otherwise \( \text{proj}[S_u] \) would have at most one irrational in it and \( u \notin T^\lambda_{\xi+1} = T^\lambda = S \).

We now follow the method of proof of Theorem 1E.9. For each \( u \in S \), let \( l(u), r(u) \) be extensions of \( u \) in \( S \) which are incompatible in the first coordinate and for each \( f : \omega \to \{0, 1\} \) define the sequence \( u^f_0, u^f_1, \ldots \) of nodes in \( S \) by the recursion
\[
u^f_0 = \emptyset,
\]
\[
u^f_{n+1} = \begin{cases} l(u^f_n) & \text{if } f(n) = 0, \\ r(u^f_n) & \text{if } f(n) = 1. \end{cases}
\]
Let \( J \) be the set of initial segments of sequences \( u^f_n \) with \( f : \omega \to \{0, 1\} \). It is clear that \( J \) is a tree, \( J \subseteq S \) and every two distinct infinite paths in \( J \) are incompatible in the first coordinate. The set \( [J] \) is perfect and compact in \( N \times \kappa^\omega \) (by König’s Lemma), and since the projection mapping \( \text{proj} \) is continuous and one-to-one on \( [J] \), \( \text{proj}[J] \) is compact, perfect and a subset of \( P = \text{proj}[T] \).

(2) Suppose \( P \subseteq X \) is \( \Sigma^1_1 \) and uncountable and let \( \pi : N \to X \) be the recursive surjection supplied by Theorem 1F.12 which is injective on some \( \Pi^0_1 \) set \( A \subseteq N \). Let
\[
P^* = \pi^{-1}[P] \cap A \subseteq N;
\]
this is also \( \Sigma^1_1 \) and uncountable, since \( \pi \) is continuous and injective on \( A \). So \( P^* \) is \( \aleph_0 \)-Suslin by Problem 2A.3, and so by (1), \( P^* \) has a non-empty, compact, perfect subset \( C \); whose image \( \pi[C] \) is a compact, perfect subset of \( P \).

It cannot be proved in \( \text{ZFC} \) that \( \Pi^1_1 \) has property \( \mathcal{P} \), and extending it to the higher projective pointclasses has been one of the key motivations for introducing strong, set theoretic hypotheses.
Additional problems for Section 2A

**Problem 2A.6** (The leftmost branch). Suppose \(T\) is a tree on a cardinal \(\kappa\) with nonempty body \([T] \neq \emptyset\). Prove that there is a unique \(f \in [T]\) such that

\[
(\forall g \in [T]) \neg f = f \implies (\exists n) \left( (\forall t < n) f(t) = g(t) \land f(n) < g(n) \right).
\]

Use this to argue that the apparent appeal to the Axiom of Choice in the proof of Theorem 2A.5 can be bypassed.

**Problem 2A.7.** Prove that if \(P \subseteq X\) is \(\Sigma^1_1\), not empty and has no \(\Delta^1_1\) members, then \(P\) has a non-empty perfect subset.

**Hint:** For \(P \subseteq \mathcal{N}\) first, use the representation in (2) of Lemma 2A.3 and in the proof of the Perfect Set Theorem 2A.5 argue that if \(P\) has no \(\Delta^1_1\) members, then \(P = \text{proj}[T]\) with a recursive \(T\) for which \(\lambda = 0\) and so \(S = T\).

**Problem 2A.8.** Prove that if \(P \subseteq X\) and \(Q \subseteq Y\) are Borel sets such that

\[
\text{card}(P) = \text{card}(Q), \quad \text{card}(X \setminus P) = \text{card}(Y \setminus Q),
\]

then there exists a Borel isomorphism \(f : X \leftrightarrow Y\) such that \(f[P] = Q\).

**Problem 2A.9.** Prove that for every cardinal \(\kappa\), the collection of \(\kappa\)-Suslin pointsets is closed under \((\exists \mathcal{N})\).

**Hint:** Check that \(\mathcal{N} \times \kappa^\omega\) is homeomorphic with \(\kappa^\omega\).

**Problem 2A.10** (Martin). Suppose \(\kappa\) is a cardinal of cofinality \(> \omega\), i.e., if \(\xi_0, \xi_1, \ldots\) are all \(< \kappa\), then supremum\(\{\xi_n : n = 0, 1, 2, \ldots\} < \kappa\).

Prove that a pointset \(P \subseteq X\) is \(\kappa\)-Suslin if and only if

\[
P = \bigcup_{\xi < \kappa} P_\xi,
\]

where each \(P_\xi\) is \(\lambda\)-Suslin for some cardinal \(\lambda < \kappa\).

**Hint:** Suppose

\[
P(x) \iff (\exists f \in \omega^\kappa) C(x, f)
\]

with \(C\) closed and for each \(\xi < \kappa\), put

\[
P_\xi(x) \iff (\exists f \in \omega^\xi) C(x, f).
\]

2B. Bar induction and bar recursion

We establish here some additional facts about \(\kappa\)-Suslin sets and their complements which, in particular, imply Shoenfield’s Theorem 2B.4, that every \(\Sigma^1_2\) pointset is \(\aleph_1\)-Suslin. These results are mostly classical, but the techniques in the section title that we will introduce are indispensable tools of the effective theory, which we will put to good use in the next section.
2B.1. Well founded trees. A tree $T$ on $X$ is well founded if $[T] = \emptyset$ so that $T$ has no infinite branches and the relation of proper extension of finite sequences from $X$,

$$v < u \iff u \succ v \iff u, v \in T \& u \not\subset v$$

is well founded.

Let $T$ be a well founded tree on $X$ and suppose $P$ is a relation on the finite sequences from $X$ such that for all $u \in X^{<\omega}$:

$$(*) \quad \text{if } P(u^{-}(x)) \text{ holds for every } u^{-}(x) \in T, \text{ then } P(u) \text{ holds.}$$

It follows that $P(u)$ must hold for every sequence $u \in X^{<\omega}$: otherwise there is some $u_0$ such that $\neg P(u_0)$, hence there is some $x_0$ such that $u_0^{-}(x_0) \in T$ and $\neg P(u_0^{-}(x_0))$, hence there is some $x_1$ such that $u_0^{-}(x_0, x_1) \in T$ and $\neg P(u_0^{-}(x_0, x_1))$, etc., so we get an infinite branch $u_0^{-}(x_0, x_1, x_2, \ldots)$ in $T$ contradicting $[T] = \emptyset$.

This method of proof is called backwards or bar induction on $T$.

In the same way we can justify definition by backwards or bar recursion on a well founded tree $T$: in order to define $F(u)$ for every finite sequence $u$ from $X$, it is enough to show how to compute $F(u)$ if we know $F(u^{-}(x))$ for every $u^{-}(x) \in T$. Formally, a function $F(u)$ is defined on $T$ by bar recursion if it satisfies an equation of the form

$$(**) \quad F(u) = G\left( u, \left\{ \left( x, F(u^{-}(x)) \right) : u^{-}(x) \in T \right\} \right) \quad (u \in X^{<\omega})$$

with $G$ a given function. We then put

$$R(u, z) \iff \text{there is some function } f \text{ such that}$$

$$u \in \text{Domain}(f) \& f(u) = z$$

$$\& (\forall u', x)[[u' \in \text{Domain}(f) \& u'^{-}(x) \in T] \implies u'^{-}(x) \in \text{Domain}(f)]$$

$$\& (\forall u' \in \text{Domain}(f))[G\left( u', \left\{ \left( x, f(u'^{-}(x)) \right) : u'^{-}(x) \in T \right\} \right)]$$

and show by bar induction on $T$ that for every $u$ there is exactly one $z$ such that $R(u, z)$, so we can set

$$F(u) = \text{the unique } z \text{ such that } R(u, z).$$

This clearly satisfies (**). Another simple bar induction shows that no other $F'$ can satisfy the determining equation.

Ranks. A rank function for a tree $T$ on $X$ is any mapping $\rho$ defined on all the finite sequences from $X$, with ordinal values, such that

$$\text{if } u^{-}(x) \in T, \text{ then } \rho(u) > \rho(u^{-}(x)).$$

It is instructive at this point to prove by bar recursion the following, standard result from set theory:

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Problem 2B.2. Prove that a tree $T$ on $X$ is well founded if and only if it admits a rank function.

Moreover, if $\text{card}(X) = \kappa \geq \aleph_0$ and $T$ is well founded, then $T$ admits a (unique, canonical) rank function $\rho$ which is defined by the bar recursion

$$\rho(u) = \sup \{ \rho(u^-(x)) + 1 : u^-(x) \in T \}$$

(2B-1)

(with the standard understanding that $\sup(\emptyset) = 0$), and for which,

$$\rho(u) < \kappa^+ = \text{the least cardinal} > \kappa \quad (u \in \kappa^{<\omega}).$$

We will sometimes write $\rho = \rho^T$ for the canonical rank function of a well founded tree $T$. The length of $T$ is then defined by

$$|T| = \sup \{ \rho^T(u) : u \in T \},$$

and if $T \neq \emptyset$, clearly $|T| = \rho^T(\emptyset)$.

For any tree $T$ on $\omega \times \kappa$ and $\alpha \in \mathcal{N}$, put

$$T(\alpha) = \{ (\xi_0, \ldots, \xi_{n-1}) : (\alpha(0), \xi_0, \ldots, \alpha(n-1), \xi_{n-1}) \in T \}.$$  

(2B-2)

The notation gives us an easy but useful characterization of $\kappa$-Suslin subsets of $\mathbb{N}$:

Problem 2B.3. Suppose $\kappa$ is an infinite cardinal.

(1) For every tree $T$ on $\omega \times \kappa$ and all $\alpha, \beta, n$,

$$\alpha \upharpoonright n = \beta \upharpoonright n \implies [(\xi_0, \ldots, \xi_{n-1}) \in T(\alpha) \iff (\xi_0, \ldots, \xi_{n-1}) \in T(\beta)].$$

(2) A pointset $P \subseteq \mathcal{N}$ is $\kappa$-Suslin if and only if there is a tree $T$ on $\omega \times \kappa$ such that

$$P(\alpha) \iff T(\alpha) \text{ is not well founded}.$$  

We now put these two problems to good use:

2B.4. Shoenfield’s Theorem. Every $\Sigma^1_2$ pointset is $\aleph_1$-Suslin.

Proof. It is enough to prove the theorem for sets $P \subseteq \mathcal{N}$: because if $Q \subseteq X$ is $\Sigma^1_2$, then so is its pre-image $P = \pi^{-1}[Q] \subseteq \mathcal{N}$ by the recursive $\pi : \mathcal{N} \to X$ of Theorem 1E.1; and if we know that $P$ is $\aleph_1$-Suslin, then

$$Q(x) \iff (\exists \alpha)[\pi(\alpha) = x \land P(\alpha)]$$

$$\iff (\exists \alpha)(\exists f) \left( \pi(\alpha) = x \land C(\alpha, f) \right) \quad (\text{with } C \text{ closed})$$

$$\iff (\exists \alpha)(\exists f) C'(x, \alpha, f) \quad (\text{with } C' \text{ closed})$$

$$\iff (\exists \alpha)S(x, \alpha) \quad (\text{where } S \text{ is } \aleph_1 \text{-Suslin})$$

which then implies that $Q$ is $\aleph_1$-Suslin by Problem 2A.9. By the same problem, it is enough to show that every $\Pi^1_2$ subset of $\mathcal{N} \times \mathcal{N}$ is $\aleph_1$-Suslin; and using a standard homeomorphism of $\mathcal{N} \times \mathcal{N}$ with $\mathcal{N}$, it is finally enough to show that every $\Pi^1_2$ subset of $\mathcal{N}$ is $\aleph_1$-Suslin.
Assume then that $\mathcal{P} \subseteq \mathcal{N}$ is $\Pi_1^1$ and fix a tree $T$ on $\omega \times \aleph_0$ by Problem 2B.3 so that by Problem 2B.2,

$$P(\alpha) \iff T(\alpha) \text{ is well founded} \iff T(\alpha) \text{ admits a rank function into } \aleph_1.$$ 

The idea of the proof is to define a tree $S$ on $\omega \times \aleph_1$ whose finite branches comprise all attempts to construct a rank function $\rho : T(\alpha) \to \aleph_1$, so that every infinite branch of $S(\alpha)$ will determine a rank function of $T(\alpha)$ and witness that $P(\alpha)$.

Let $u_0, u_1, u_2, \ldots$ be an enumeration of all finite sequences from $\omega$ such that

$$\text{length}(u_n) \leq n,$$

This is easy to arrange, for example by setting

$$u_i = ((i)_0, \ldots, (i)_{\lceil\log(i)\rceil - 1}).$$

For each $n$ then, $u_n = (s_0, \ldots, s_{k-1})$ with some $k \leq n$. We say that $u = (s_0, \ldots, s_{k-1})$ is $T$-compatible with $(t_0, \ldots, t_{m-1})$, if $k \leq m$ and $(t_0, s_0, t_{k-1}, s_{k-1}) \in T$,

and for any countable ordinals $\xi_0, \ldots, \xi_{n-1}$, we put

$$(t_0, \xi_0, \ldots, t_{n-1}, \xi_{n-1}) \in S \iff \text{for every } i, j < n, \text{ if } u_i, u_j \text{ are } T\text{-compatible with } (t_0, \ldots, t_{n-1}) \text{ and } u_i \text{ is a proper initial segment of } u_j, \text{ then } \xi_i > \xi_j.$$ 

Clearly $S$ is a tree on $\omega \times \aleph_1$. The claim is that

$$P(\alpha) \iff S(\alpha) \text{ is not well founded.}$$

Notice that for any fixed $\alpha$ and $u = (s_0, \ldots, s_{k-1}),$

$$u \text{ is } T\text{-compatible with } (\alpha(0), \ldots, \alpha(n-1)) \iff k \leq n \& (\alpha(0), s_0, \ldots, \alpha(k-1), s_{k-1}) \in T \iff k \leq n \& u \in T(\alpha).$$

Using the condition $\text{length}(u_n) \leq n$ we then have

$$(\xi_0, \ldots, \xi_{n-1}) \in S(\alpha) \iff (\alpha(0), \xi_0, \ldots, \alpha(n-1), \xi_{n-1}) \in S \iff \text{for every } i, j < n, \text{ if } u_i, u_j \text{ are in } T(\alpha) \text{ and } u_i \text{ is an initial segment of } u_j, \text{ then } \xi_i > \xi_j.$$ 

This observation implies immediately that if $(\xi_0, \xi_1, \ldots)$ is an infinite branch of $S(\alpha)$, then the mapping

$$u_i \mapsto \xi_i$$

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is a rank function on \( T(\alpha) \), so that \( T(\alpha) \) is well founded. Conversely, if \( T(\alpha) \) is well founded, let \( \rho \) be a rank function on \( T(\alpha) \), put \( \xi_i = \rho(u_i) \) and check immediately that \( (\xi_0, \xi_1, \ldots) \) is an infinite branch of \( S(\alpha) \), so that \( S(\alpha) \) is not well founded.

**Corollary 2B.5.** If a pointset \( P \) is \( \Sigma_1^2 \) and has more than \( \aleph_1 \) members, then \( P \) has a non-empty perfect subset.

**Proof** is immediate, by the Perfect Set Theorem 2A.5.

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### Additional problems for Section 2B

There are many applications of the theory of analytical pointsets to set theory. It is not our subject here, and there is a detailed exposition of the most fundamental of them (from scratch) in Chapter 8 of DST. We list just a few of them here, primarily to give a flavor of the sort of results that one can prove.

**Definability in set theory.** Suppose \( \mathcal{D} \) is a collection of transitive sets and classes (perhaps with only one member). A pointset \( P \subseteq \omega^m \times \mathbb{N}^n \) of type 0 or 1 is definable (in set theory, as a relation) absolutely for \( \mathcal{D} \), if there is a formula \( \varphi(v_1, \ldots, v_{m+n}) \) in the language of set theory such that

\[
\text{for all } M \in \mathcal{D} \text{ and all } x_1, \ldots, x_m, \alpha_1, \ldots, \alpha_n \in M, \\
P(x_1, \ldots, x_m, \alpha_1, \ldots, \alpha_n) \iff M \models \varphi[x_1, \ldots, x_m, \alpha_1, \ldots, \alpha_n].
\]

(Notice that if \( \mathcal{D} \) includes proper classes, then this definition must be understood properly, as a schema rather than a single proposition.)

Recall that ZFDC is Zermelo-Fraenkel set theory with the Axiom of Depended Choice.

**Problem 2B.6** (Mostowski’s Absoluteness Theorem). Prove that every \( \Sigma_1^1 \) pointset of type 0 or 1 is absolutely definable for the class \( \mathcal{D}_T \) of all transitive models of some sufficiently large, finite subset \( T \) of the axioms of ZFDC.

**Problem 2B.7** (Shoenfield’s Absoluteness Theorem). Prove that every \( \Sigma_1^2 \) pointset of type 0 or 1 is absolutely definable for the class \( \mathcal{D}_T \) of all uncountable, transitive models of some sufficiently large, finite subset \( T \) of the axioms of ZFDC.

**Problem 2B.8.** Prove that the pointset \( L \cap \mathcal{N} \) of constructible points in \( \mathcal{N} \) is \( \Sigma_2^1 \), and so is the restriction \( \leq_L \cap (\mathcal{N} \times \mathcal{N}) \) to \( \mathcal{N} \) of the canonical wellordering of \( L \).

**Problem 2B.9** (Shoenfield). Prove that if there exists a non-constructible \( \alpha \in \mathcal{N} \), then \( L \cap \mathcal{N} \) is not \( \Pi_2^1 \).

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2C. The Suslin-Kleene Theorem

Our main purpose in this section is to prove that every $\Delta^1_1$ pointset is uniformly Borel. Together with its converse Theorem 1I.11, this result yields simultaneous generalization of both

$$\Delta^1_1 = \mathcal{B} \text{ and } \Delta^1_1 \upharpoonright \omega = \text{HYP} \upharpoonright \omega,$$

the basic results of Suslin and Kleene in classical descriptive set theory and the theory of hyperarithmeticial sets. We will also derive some of the many consequences of this result in both classical and effective descriptive set theory.

The proof involves considerable computation, and it helps to break it up in three steps as follows:

(1) First we will give a proof of Lusin’s classical Strong Separation Theorem for $\Sigma^1_1$ subsets of $\mathcal{N}$, which introduces the main, set theoretic construction that is needed.

(2) Next we will “effectivise” the proof to get the Effective Strong Separation Theorem for $\Sigma^1_1$ subsets of $\mathcal{N}$. This part appeals to the Second Recursion Theorem.

(3) Finally, we will extend the Effective Strong Separation Theorem to $\Sigma^1_1$ subsets of any space $X$. This requires a second appeal to the Second Recursion Theorem—and then implies trivially the Suslin-Kleene Theorem.

The key to the construction is the following simple

**Lemma 2C.1.** Suppose that for all $i \in I, j \in J$, $A_i, B_j, C_{i,j} \subseteq X$,

$$A = \bigcup_{i \in I} A_i, \quad B = \bigcup_{j \in J} B_j,$$

and for all $i, j$, $C_{i,j}$ separates $A_i$ from $B_j$, i.e.,

$$A_i \subseteq C_{i,j}, \quad C_{i,j} \cap B_j = \emptyset.$$

Then the set

$$C = \bigcup_{i \in I} \bigcap_{j \in J} C_{i,j}$$

separates $A$ from $B$.

**Proof.** For any $i, j$, $A_i \subseteq C_{i,j}$ by the hypothesis, so $A_i \subseteq \bigcap_{j \in J} C_{i,j}$ and hence

$$A = \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} \bigcap_{j \in J} C_{i,j} = C.$$  

On the other hand, for any $i, j$, $B_j \subseteq X \setminus C_{i,j}$, hence

$$B = \bigcup_{j \in J} B_j \subseteq \bigcup_{j \in J} (X \setminus C_{i,j})$$

and since this holds for arbitrary $i$,

$$B \subseteq \bigcap_{i \in I} \bigcup_{j \in J} (X \setminus C_{i,j}) = X \setminus \bigcup_{i \in I} \bigcap_{j \in J} C_{i,j} = X \setminus C.$$  

2C. The Suslin-Kleene Theorem

2C.2. The Strong Separation Theorem for $\Sigma^1_1$ at $\mathcal{N}$. If $A, B$ are
disjoint, $\Sigma^1_1$-Suslin subsets of $\mathcal{N}$, then there exists a Borel set $C \subseteq \mathcal{N}$
which separates them,

\[ A \subseteq C, \quad C \cap B = \emptyset. \]

Proof. We give two proofs of the result—first a simple argument by
contradiction and then a constructive proof which defines by bar recursion
the required Borel set $C$.

Proof by contradiction. Suppose $T$ and $S$ are trees on $\omega \times \aleph_0$,

\[ A = \text{proj}[T], \quad B = \text{proj}[S], \quad A \cap B = \emptyset, \]

but $A$ cannot be separated from $B$ by a Borel set. Since

\[ A = \bigcup_{t \in \omega, \xi < \aleph_0} \text{proj}[T(t, \xi)], \quad B = \bigcup_{s \in \omega, \eta < \aleph_1} \text{proj}[S(s, \eta)], \]

there must exist by the Lemma some $t_0, \xi_0, \eta_0$ such that $\text{proj}[T(t_0, \xi_0)]$
and $\text{proj}[S(\xi_0, \eta_0)]$ cannot be separated. This implies that $t_0 = \xi_0$, otherwise
we could take

\[ C = \{ \alpha : \alpha(0) = t_0 \}, \]

which surely separates these two sets. Hence $(t_0, \xi_0) \in T$, $(t_0, \eta_0) \in S$
and $\text{proj}[T(t_0, \xi_0)]$, $\text{proj}[S(\xi_0, \eta_0)]$ cannot be separated by a Borel set.

Proceeding recursively, we find $t_0, t_1, t_2, \ldots, \xi_0, \xi_1, \xi_2, \ldots, \eta_0, \eta_1, \eta_2, \ldots$
such that for each $n,$

\[ u = (t_0, \xi_0, \ldots, t_{n-1}, \xi_{n-1}) \in T, \quad v = (t_0, \eta_0, \ldots, t_{n-1}, \eta_{n-1}) \in S, \]

and $\text{proj}[T_u]$, $\text{proj}[S_v]$ cannot be separated by a Borel set. However this is
absurd, since then $\alpha = (t_0, t_1, \ldots)$ is in both $A$ and $B$ and these sets were
assumed disjoint.

Constructive proof. Define the tree $J$ on $\omega \times \aleph_0 \times \aleph_0$
by

\[ ((t_0, \xi_0, \eta_0), \ldots, (t_{n-1}, \xi_{n-1}, \eta_{n-1})) \in J \]

\[ \iff (t_0, \xi_0, \ldots, t_{n-1}, \xi_{n-1}) \in T \& (t_0, \eta_0, \ldots, t_{n-1}, \eta_{n-1}) \in S. \]

We will omit the parentheses in writing the nodes of $J$ as we have been
doing for trees of pairs,

\[ (t_0, \xi_0, \eta_0, \ldots, t_{n-1}, \xi_{n-1}, \eta_{n-1}) = (t_0, \xi_0, \eta_0, \ldots, (t_{n-1}, \xi_{n-1}, \eta_{n-1}). \]

$J$ is a well founded tree;

because any infinite branch $(t_0, \xi_0, \eta_0, t_1, \xi_1, \eta_1, \ldots)$ in $J$ would determine
infinite branches $(t_0, \xi_0, t_1, \xi_1, \ldots)$ in $T$ and $(t_0, \eta_0, t_1, \eta_1, \ldots)$ in $S$ with
the same part $\alpha = (t_0, t_1, \ldots) \in \mathcal{N}$, so that $\alpha \in A \cap B$ contrary to hypothesis.

To simplify notation, assign to each sequence

\[ u = (t_0, \xi_0, \eta_0, \ldots, t_{n-1}, \xi_{n-1}, \eta_{n-1}) \]
from $\omega \times \aleph_0 \times \aleph_0$ the two sequences that it determines in $\omega \times \aleph_0$

$$\tau(u) = (t_0, \xi_0, \ldots, t_{n-1}, \xi_{n-1}), \quad \sigma(u) = (t_0, \eta_0, \ldots, t_{n-1}, \eta_{n-1}),$$

with the natural $\tau(\emptyset) = \sigma(\emptyset) = \emptyset$, so that

$$J = \{ u : \tau(u) \in T \text{ and } \sigma(u) \in S \}.$$

If $v$ is a sequence from $\omega \times \aleph_0$ put

$$A_v = \text{proj}[T_v], \quad B_v = \text{proj}[S_v].$$

We will define by bar recursion on the well founded tree $J$ a function

$$u \mapsto C_u$$

such that for each sequence $u$ in $\omega \times \aleph_0 \times \aleph_0$,

(a) $C_u$ is Borel,

(b) $C_u$ separates $A_{\tau(u)}$ from $B_{\sigma(u)}$.

This will complete the proof, since $A_{\tau(\emptyset)} = A_\emptyset = A$ and $B_{\sigma(\emptyset)} = B_\emptyset = B$, so $C = C_\emptyset$ will be the required set.

We have for each $u$

$$A_{\tau(u)} = \text{proj}[T_{\tau(u)}] = \bigcup_{t < \omega, \xi < \aleph_0} A_{\tau(u)} \cap (t, \xi),$$

and

$$B_{\sigma(u)} = \text{proj}[S_{\sigma(u)}] = \bigcup_{s < \omega, \eta < \aleph_0} B_{\sigma(u)} \cap (s, \eta).$$

It will be enough to define sets $D_{t,\xi,s,\eta}$ (depending on $u$) such that if the induction hypothesis of the bar recursion holds at $u$, then

(c) each $D_{t,\xi,s,\eta}$ is Borel, and

(d) $D_{t,\xi,s,\eta}$ separates $A_{\tau(u) \cap (t, \xi)}$ from $B_{\sigma(u) \cap (s, \eta)}$,

since then, by the Lemma, the set

$$C_u = \bigcup_{t < \omega, \xi < \aleph_0} \bigcap_{s < \omega, \eta < \aleph_0} D_{t,\xi,s,\eta}$$

is Borel and separates $A_{\tau(u)}$ from $B_{\sigma(u)}$.

If $t = s$ and $u \cap (t, \xi, \eta) \in J$, we can take

$$D_{t,\xi,s,\eta} = C_u \cap (t, \xi, \eta),$$

since by the induction hypothesis of the bar recursion we can assume that $C_u \cap (t, \xi, \eta)$ has been defined, it is Borel and it separates $A_{\tau(u) \cap (t, \xi)}$ from $B_{\sigma(u) \cap (s, \eta)}$. Hence it is enough to define $D_{t,\xi,s,\eta}$ when $t \neq s$ or $t = s$ but $u \cap (t, \xi, \eta) \notin J$.

If $t \neq s$, take

$$D_{t,\xi,s,\eta} = \{ \alpha : \alpha(n) = t \},$$

where $n$ is the length of the sequence $u$, so that

$$\alpha \in A_{\tau(u) \cap (t, \xi)} \implies \alpha(n) = t;$$
clearly $A_{\tau(u)}(t,\xi) \subseteq \{ \alpha : \alpha(n) = t \}$, while $B_{\sigma(s)}(s,\eta) \cap \{ \alpha : \alpha(n) = t \} = \emptyset$.

If $t = s$ but $u^\sim(t,\xi,\eta) \notin J$, there are two cases.

Case 1. $\tau(u)^\sim(t,\xi) \notin T$. In this case $A_{\tau(u)}(t,\xi) = \emptyset$ and we can take $D_{t,\xi,s,\eta} = \emptyset$.

Case 2. $\sigma(u)^\sim(s,\eta) \notin S$. In this case $B_{\sigma(s)}(s,\eta) = \emptyset$ and we can take $D_{t,\xi,s,\eta} = N$.

For the effective version of the Strong Separation Theorem we will need the following improved version of Theorem 1I.4. We assume here that for any $X, Z$, the partial function $\{ \varepsilon \} : X \rightarrow Z$ is defined as in (1H-11) from some $G^{X \times \omega} \subseteq N \times X \times \omega$ which is a good universal set for $\Sigma^0_1$ at $X \times \omega$,

\[ (2C-1) \quad \{ \varepsilon \}(x) = z \iff (\forall s)[G(\varepsilon, x, s) \leftrightarrow z \in N(Z, s)]; \]

we understand this to mean that $\{ \varepsilon \}(x)$ exactly when there is some $z$ (and hence a unique $z$) which satisfies the right-hand-side.

**Theorem 2C.3 (The $S^Y_X$-Theorem).** For any space $Y$ of type 0 or 1 and any $X$, there exists a recursive function

\[ S = S^Y_X : N \times Y \rightarrow N \]

such that for any space $Z$,

\[ (2C-2) \quad \{ \varepsilon \}(y, x) = \{ S(\varepsilon, y) \}(x) \quad (\varepsilon \in N, y \in Y, x \in X). \]

**Proof.** Suppose, for simplicity, that $Y = N \times \omega \times N$ and let

\[ P(\gamma, x, s) \iff G^{X \times \omega}((\gamma)0, (\gamma)1, (\gamma)2(0), (\gamma)3, x, s). \]

This is in $\Sigma^0_1$, so by the basic property of good universal sets, there is a recursive function $f^P(\gamma) (= S(\varepsilon_1, \gamma)$ with a recursive $\varepsilon_1$ in the notation of 1H.12) such that

\[ P(\gamma, x, s) \iff G^{X \times \omega}(f^P(\gamma), x, s); \]

and if we substitute $\gamma = (\varepsilon, \alpha, (\lambda) t, \beta)$ in this equivalence, we have

\[ G^{X \times \omega}(\varepsilon, \alpha, t, \beta, x, s) \iff G^{X \times \omega}(f^P(\gamma), x, s) \]

where $f^P(\gamma) = f^P((\varepsilon, \alpha, (\lambda) t, \beta))$ is a total, recursive function of $\varepsilon, \alpha, t, \beta$.

Tracing the definitions and showing (for once) all the embellishments which indicate what depends on what:

\[ \{ \varepsilon \}(y, x) = z \iff (\forall s)[G^{Y \times \omega}(\varepsilon, y, x, s) \leftrightarrow z \in N(Z, s)] \]

\[ \iff (\forall s)[G^{X \times \omega}(f^P(\gamma), x, s) \leftrightarrow z \in N(Z, s)] \]

\[ \iff \{ f^P(\gamma) \}(x) = z \]

as required, if we set $S(\varepsilon, \alpha, t, \beta) = f^P((\varepsilon, \alpha, (\lambda) t, \beta))$.  \[ \Box \]

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For the statement and the proof of the effective separation theorem, it is convenient to “identify” a tree \( T \) on \( \omega \times \omega \) with its code set, and then assign codes in \( N \) to trees: \( \tau \in N \) codes the tree \( T \) if

\[
\langle \langle t_0, \xi_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1} \rangle \rangle \in T^c \iff \tau(\langle \langle t_0, \xi_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1} \rangle \rangle) = 1.
\]

Similarly for a tree \( J \) on \( \omega \times \omega \times \omega \): \( \alpha \) codes \( J \), if

\[
\langle \langle t_0, \xi_0, \eta_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1}, \eta_{n-1} \rangle \rangle \in J^c \iff \alpha(\langle \langle t_0, \xi_0, \eta_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1}, \eta_{n-1} \rangle \rangle) = 1.
\]

It is clear that any code \( \tau \) (or \( \alpha \)) determines completely the tree \( T \) (or \( J \)).

As before, the projection \( \text{proj}[T] \) of tree \( T \) on \( \omega \times \omega \) is a subset of \( N \),

\[
\text{proj}[T] = \{ \alpha : (\exists f \in N)(\forall u)\left[\langle \langle \alpha(0), f(0) \rangle, \ldots, \langle \alpha(n-1), f(n-1) \rangle \rangle \in T^c \right]\}.
\]

**2C.4. The Effective Strong Separation Theorem for \( \Sigma_1^1 \) at \( N \).**

*There is a recursive function \( u : N \times N \to N \) such that whenever \( \tau \) and \( \sigma \) code respectively trees \( T \) and \( S \) on \( \omega \times \omega \) with

\[
\text{proj}[T] \cap \text{proj}[S] = \emptyset,
\]

then \( u(\tau, \sigma) \) is a Borel code of some set \( C \subseteq N \) which separates \( \text{proj}[T] \) from \( \text{proj}[S] \), i.e.,

\[
\text{proj}[T] \subseteq C, \quad C \cap \text{proj}[S] = \emptyset.
\]

**Proof.** Following closely the constructive proof of 2C.2, let us associate with any two trees of pairs \( T \) and \( S \) on \( \omega \times \omega \) the tree of triples \( J \),

\[
\langle \langle t_0, \xi_0, \eta_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1}, \eta_{n-1} \rangle \rangle \in J^c \iff \langle \langle t_0, \xi_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1} \rangle \rangle \in T^c
\]

\[
\& \langle \langle t_0, \eta_0 \rangle, \ldots, \langle t_{n-1}, \eta_{n-1} \rangle \rangle \in S^c,
\]

and fix recursive functions \( f(u) \) and \( h(u) \) such that if

\[
u = \langle \langle t_0, \xi_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1} \rangle \rangle,
\]

then

\[
f(u) = \langle \langle t_0, \xi_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1} \rangle \rangle,
\]

\[
h(u) = \langle \langle t_0, \eta_0 \rangle, \ldots, \langle t_{n-1}, \eta_{n-1} \rangle \rangle.
\]

Fix also a recursive function \( j(\tau, \sigma) \), such that if \( \tau \) and \( \sigma \) code trees \( T \) and \( S \), then \( j(\tau, \sigma) \) is a code of the tree \( J \) constructed from them, i.e.,

\[
j(\tau, \sigma)(u) = 1 \iff \tau(f(u)) = 1 \& \sigma(h(u)) = 1.
\]

Finally, let

\[
A_v = \text{proj}[T_v], \quad B_v = \text{proj}[S_v],
\]

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where \( v \) now varies over sequence codes, so that

\[
A = \text{proj}[T] = \text{proj}[T_1], \quad A = \text{proj}[S] = \text{proj}[S_1]
\]

recalling that \( 1 = \langle \rangle \) codes the empty sequence. We aim to define a recursive function

\[
\mathbf{u} : \mathcal{N} \times \mathcal{N} \times \omega \to \mathcal{N}
\]

such that whenever \( \tau \) and \( \sigma \) code trees \( T \) and \( S \) with \( \text{proj}[T] \cap \text{proj}[S] = \emptyset \) and \( u = \langle \langle t_0, \xi_0, \eta_0 \rangle, \ldots, \langle t_{n-1}, \xi_{n-1}, \eta_{n-1} \rangle \rangle \in J \), then \( \mathbf{u}(\tau, \sigma, u) \) is a Borel code of a set

\[
C_u = C(\tau, \sigma, u)
\]

which separates \( \text{proj}[T_{f(u)}] \) from \( \text{proj}[S_{h(u)}] \). The proof will be completed by taking

\[
\mathbf{u}(\tau, \sigma, u) = \mathbf{u}(\tau, \sigma, 1).
\]

The definition of \( \mathbf{u} \) will be by the Second Recursion Theorem, i.e.,

\[
\mathbf{u}(\tau, \sigma, u) = \{ \tau \}(\tau, \sigma, u)
\]

with some \( \tau \in \mathcal{N} \) chosen so that

\[
g(\tau, \tau, \sigma, u) \downarrow \Rightarrow \{ \tau \}(\tau, \tau, \sigma, u) = g(\tau, \tau, \sigma, u)
\]

for a suitable potentially recursive \( g \), the coded version of the function we used in the proof of Theorem 2C.2 by bar recursion. In fact \( g \) will be a total recursive function, so that \( \mathbf{u} \) will also be total.

We first prove a lemma which reduces the construction of a code for \( C(\tau, \sigma, u) \) to constructing codes for the sets \( D_{t,\xi,s,\eta} \) in the proof of 2C.2.

Lemma. There is a recursive function \( \mathbf{v}(\epsilon) \) such that if \( \{ \epsilon \}(t, \xi, s, \eta) \) is defined for all \( t, \xi, s, \eta \) and codes a Borel set \( D_{t,\xi,s,\eta} \), then \( \mathbf{v}(\epsilon) \) is a Borel code of the set

\[
\bigcup_{t,\xi,s,\eta} D_{t,\xi,s,\eta}.
\]

Proof. Recall the recursive function \( \mathbf{u}_2 \) and \( \mathbf{u}_3 \) of Lemma II.8 which construct codes for countable unions and intersections of Borel sets and define successively \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) by

\[
\begin{align*}
\mathbf{v}_1(\epsilon, t, \xi, s) & = \mathbf{u}_3(S(\epsilon, t, \xi, s)), \\
\mathbf{v}_2(\epsilon, t, \xi) & = \mathbf{u}_3(S(\mathbf{v}_1(\epsilon, t, \xi))), \quad \text{where} \ \{ \mathbf{v}_1 \}(\epsilon, t, \xi, s) = \mathbf{v}_1(\epsilon, t, \xi, s), \\
\mathbf{v}_3(\epsilon, t) & = \mathbf{u}_2(S(\mathbf{v}_2(\epsilon, t))), \quad \text{where} \ \{ \mathbf{v}_2 \}(\epsilon, t, \xi) = \mathbf{v}_2(\epsilon, t, \xi) \\
\mathbf{v}(\epsilon) & = \mathbf{u}_2(S(\mathbf{v}_3(\epsilon))),
\end{align*}
\]

where \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are recursive codes in \( \mathcal{N} \) of \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) as indicated. It follows that if

\[
\{ S(\epsilon, t, \xi, s) \}(\eta) = \{ \epsilon \}(t, \xi, s, \eta)
\]

codes \( D_{t,\xi,s,\eta} \) for each \( \eta \), then

\[
\mathbf{v}_1(\epsilon, t, \xi, s) = \mathbf{u}_3(S(\epsilon, t, \xi, s))
\]
codes $\bigcap_\eta D_{t, \xi, s, \eta}$; but
\[ v_1(\varepsilon, t, \xi, s) = \{ S(\check{v}_1, t, \xi) \}(s), \]
so that
\[ v_2(\varepsilon, t, \xi) = v_3(S(\check{v}_1, t, \xi)) \]

Continuing the same argument, we easily check that $v(\varepsilon)$ codes the required set. \(\square\) (Lemma)

Going back to the proof of the theorem, let us define (by cases) a partial function $d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta)$ so that the following hold. (After $\varepsilon$ is fixed by the Recursion Theorem, $d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta)$ will give a Borel code of the set $D_{t, \xi, s, \eta}$ in the proof of 2C.2.)

1. If $t = s$ and $j(\tau, \sigma)(u \ast (t, \xi, \eta)) = 1,$ then
\[ d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta) = \{ \varepsilon \}(\tau, \sigma, u \ast (t, \xi, \eta)). \]

2. If $t \neq s,$ then
\[ d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta) = d_2(u, t) \]
where $d_2(u, t)$ is a recursive function which gives a Borel code of the set $\{ \alpha : \alpha(\text{lh}(u)) = t \}$—this is easy to get.

3. If $t = s$ & $\tau(u \ast (t, \xi, \eta)) \neq 1,$ then
\[ d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta) = d_0, \]
where $d_0$ is a fixed (recursive) Borel code of $\emptyset$.

4. If $t = s$ & $\sigma(u \ast (t, \xi, \eta)) \neq 1,$ then
\[ d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta) = d_1 \]
where $d_1$ is a fixed (recursive) Borel code of $\mathcal{N}$.

It is clear that $d$ is recursive on its domain, so let $\hat{d} \in \mathcal{N}$ be recursive so that
\[
\begin{align*}
\quad & d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta) \\
\implies & d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta) = \{ \hat{d} \}(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta) \\
& = \{ S(\hat{d}, \varepsilon, \tau, \sigma, u) \}(t, \xi, s, \eta) 
\end{align*}
\]
and finally put
\[ g(\varepsilon, \tau, \sigma, u) = v(S(\hat{d}, \varepsilon, \tau, \sigma, u)). \]
Note that $g$ is a total recursive function.

Following the plan for the proof, choose a recursive $\check{\tau}$ by the Second Recursion Theorem so that
\[ g(\check{\tau}, \tau, \sigma, u) = \{ \check{\tau} \}(\tau, \sigma, u) \]
and let
\[ \check{u}(\tau, \sigma, u) = \{ \check{\tau} \}(\tau, \sigma, u). \]
To prove that $\overline{u}$ has the required properties, suppose $\tau$ and $\sigma$ code trees $T$ and $S$ so that $\text{proj}[T] \cap \text{proj}[S] = \emptyset$ and let $J$ be the tree of triples (with code $j(\tau, \sigma)$) we associated with $T$ and $S$ and which is now well founded. We check by bar induction on $J$ that if $u \in J$, then $\overline{u}(\tau, \sigma, u)$ codes a Borel set $C_u$ that separates $\text{proj}[T]$ from $\text{proj}[S]$ by looking over the steps in the constructive proof of 2C.2 and verifying that (for the fixed $T, S$), $d(\varepsilon, \tau, \sigma, u, t, \xi, s, \eta)$ codes the set $D_{t, \xi, s, \eta}$ and $\overline{u}(\tau, \sigma, u)$ codes the set $C_u$.

For the last part (3) of our plan, we need to verify first the effective version of the classical fact, that the inverse of a Borel set by a Borel measurable function is Borel:

**Problem 2C.5.** Prove that if $f : \mathcal{X} \to \mathcal{Y}$ is HYP-recursive, then the inverse image $f^{-1}[B]$ of every Borel subset of $\mathcal{Y}$ is a Borel subset of $\mathcal{X}$ uniformly in the codes; i.e., there exists a recursive $\overline{u} : \mathbb{N} \to \mathbb{N}$ such that

$$f^{-1}[B] = B^\mathcal{X}_{\overline{u}(\alpha)} \quad (\alpha \in \text{BC}).$$

**HINT:** By Problem 1D.4, the hypothesis insures that the relation $G^f(x, s) \iff f(x) \in N(\mathcal{Y}, s)$ is HYP. Use this at the basis of a definition by effective (transfinite) recursion of $\overline{u} : \mathbb{N} \to \mathbb{N}$. The proof is very much like that of Theorem 1I.11, with an appeal to Lemma 1I.8 at the induction step.

**2C.6. The Suslin-Kleene Theorem.** For every space $\mathcal{X}$, there is a recursive function $v : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that if $\alpha, \beta$ are $\Sigma^1_\tilde{1}$-codes of $A, B \subseteq \mathcal{X}$ respectively and $A \cap B = \emptyset$, then $v(\alpha, \beta)$ is a Borel code of some set $C \subseteq \mathcal{X}$ which separates $A$ from $B$, i.e.,

$$A \subseteq C, \quad C \cap B = \emptyset.$$

It follows that, for each $\mathcal{X}$, there is a recursive function $v^* : \mathbb{N} \to \mathbb{N}$ such that if $\alpha$ is a $\Delta^1_1$-code of $A \subseteq \mathcal{X}$, then $v^*(\alpha)$ is a Borel code of $A$.

**PROOF.** Take first $\mathcal{X} = \mathbb{N}$, let $G \subseteq \mathbb{N} \times \mathbb{N}$ be $\Sigma^1_\tilde{1}$ and universal for $\Sigma^1_\tilde{1}$ at $\mathbb{N}$ and by (3) of Theorem 1B.11 fix a recursive $Q$ such that

$$G(\alpha, \beta) \iff (\exists \gamma)(\forall t)Q(\alpha, \overline{\beta}(t), \overline{\gamma}(t)), \quad Q(\alpha, \overline{\beta}(t), \overline{\gamma}(t)) \& s < t \implies Q(\alpha, \overline{\beta}(s), \overline{\gamma}(s)).$$

For each $\alpha$, the set of sequence codes

$$T(\alpha) = \{ (\overline{\beta}(t), \overline{\gamma}(t)) : t \in \omega \& \neg Q(\alpha, \overline{\beta}(t), \overline{\gamma}(t)) \}$$

is clearly a tree and in fact

$$G_\alpha = \text{proj}[T(\alpha)];$$
moreover (easily) there is a recursive function $\theta$ such that for each $\alpha$, $\theta(\alpha)$ is a code of $T(\alpha)$. Letting $u : N \times N \to N$ be the recursive function of 2C.4, we can then take for the space $N$,

$$v_N(\alpha, \beta) = u(\theta(\alpha), \theta(\beta)).$$

For the general case now, fix a space $X$ and let

$$\pi : N \to X,$$

$$\pi^{-1} : X \to D \subseteq N$$

be the functions of Theorem 1F.12: the surjection $\pi$ is recursive and injective on the $\Pi^0_1$ set $D$ (called $A$ in 1F.12) and its inverse $\pi^{-1}$ is $\Sigma^0_2$-recursive, and hence HYP-recursive. It follows easily that there is a recursive function $g : N \to N$ such that

- if $A \subseteq X$ is $\Sigma^1_1$ with code $\alpha$, then $\pi^{-1}[A] \cap D \subseteq N$ is $\Sigma^1_1$ with code $g(\alpha)$.

In particular, if $A, B$ are disjoint $\Sigma^1_1$ subsets of $X$ with respective codes $\alpha, \beta$, then

$$A_1 = \pi^{-1}[A] \cap D, \quad B_1 = \pi^{-1}[B] \cap D$$

are disjoint $\Sigma^1_1$ subsets of $N$ with codes $g(\alpha), g(\beta)$ respectively, and so $v_N(g(\alpha), g(\beta))$ is a Borel code of some set $C \subseteq N$ which separates them. If $u : N \to N$ is the function supplied by Lemma 2C.5 for the HYP-recursive $\pi^{-1} : X \to N$, then

$$v(\alpha, \beta) = u(v_N(f(\alpha), f(\beta)))$$

is a Borel code of

$$(\pi^{-1})^{-1}[C] = \pi[C] \subseteq X$$

which separates $A = \pi[A_1]$ from $B = \pi[B]$ as required.

The second assertion about $\Delta^1_1$ follows immediately. \hfill \Box

We put together for the record some of the main corollaries of the Suslin-Kleene Theorem and its converse Theorem 11.11:

**Problem 2C.7.** (1) For every $\varepsilon \in N$, $\Delta^1_1(\varepsilon) = \text{HYP}(\varepsilon)$, and so $\Delta^1_1 = \text{HYP}$ and $\Delta^1_1 = B$.

(2) A function $f : X \to Y$ is $\Delta^1_1$-recursive if and only if it is HYP-recursive.

(3) A function $f : X \to Y$ is Borel measurable if and only if its graph $\{(x, y) : f(x) = y\}$ is $\Sigma^1_1$. 
2D. Basic structure theory for $\Pi^1_1$

Up until now, most of the results in this chapter depended directly on the fact that $\Sigma^1_1$ sets are $\aleph_0$-Suslin. Here we will use an effective version of this fact to derive a representation theorem for $\Pi^1_1$ sets which is the key to the structure properties of this pointclass.

The first lemma is a general version of a fact that we have already stated and used in various forms, e.g., in Section 1F.1.

**Lemma 2D.1.** (1) A pointset $P \subseteq X \times N^l$ $(l \geq 1)$ is $\Sigma^0_1$ if and only if there is a $\Sigma^0_1$ set $Q \subseteq X \times \omega^l$ such that

$$P(x, \alpha_1, \ldots, \alpha_l) \iff (\exists t) Q(x, \overline{\alpha}_1(t), \ldots, \overline{\alpha}_l(t))$$

and

$$[Q(x, \overline{\alpha}_1(t), \ldots, \overline{\alpha}_l(t)) \& t < s] \Rightarrow Q(x, \overline{\alpha}_1(s), \ldots, \overline{\alpha}_l(s)).$$

(2) A pointset $P \subseteq X$ is $\Pi^1_1$ if and only if there is a $\Sigma^0_1$ set $Q \subseteq X \times \omega$ such that

$$P(x) \iff (\forall \alpha)(\exists t) Q(x, \overline{\alpha}(t))$$

and

$$[Q(x, \overline{\alpha}(t)) \& t < s] \Rightarrow Q(x, \overline{\alpha}(s)).$$

Moreover, in both these results, if $X$ is of type 0 or 1, then $Q$ may be chosen to be recursive.

**Proof.** (2) follows immediately from (1).

To prove (1), take $l = 1$ for simplicity of notation and suppose by the Product Lemma 1B.2 that

$$P(x, \alpha) \iff (\exists u)(\exists v)\{x \in N(X, u) \& \alpha \in N(N, v) \& P^*(u, v)\}$$

with $P^*$ semirecursive, so there is a recursive $R$ such that

$$P(x, \alpha) \iff (\exists u)(\exists v)(\exists n)\{x \in N(X, u) \& \alpha \in N(N, v) \& R(u, v, n)\}.$$ 

By Problem 1A.8, there are recursive functions $g, h$ such that

$$\alpha \in N(N, v) \iff ((v)_1)_i \neq 0 \& (\forall i < g(v))[\alpha(i) = h(v, i)],$$

so that whenever $t \geq g(v)$, we easily have

$$\alpha \in N(N, v) \iff ((v)_1)_i \neq 0 \& (\forall i < g(v))[\overline{\alpha}(t)_i = h(v, i)].$$

Now put

$$Q(x, w) \iff \text{Seq}(w)$$

$$\& (\exists u \leq \text{lh}(w))(\exists v \leq \text{lh}(w))(\exists n \leq \text{lh}(w))$$

$$[x \in N(X, u) \& q(v) \leq \text{lh}(w) \& (\exists i < g(v))\{(w)_i = h(v, i)\} \& R(u, v, n)]$$

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and verify easily that
\[ P(x, \alpha) \iff (\exists t)Q(x, \pi(t)). \]

If \( X \) is \( f \) type 0 or 1, then \( Q \) is recursive since \( \{ (x, u) : x \in N(\mathcal{X}, u) \} \) is recursive by Theorem 1B.11.

Codes of countable wellorderings. With each \( \alpha \in \mathcal{N} \) we associate the binary relation on \( \omega \)
\[ \leq_{\alpha} = \{ (n, m) : \alpha(\langle n, m \rangle) = 1 \} \]
and we put
\[ \alpha \in \text{LO} \iff \leq_{\alpha} \text{ is a linear ordering} \iff (\forall n)(\forall m)[n \leq_{\alpha} m \implies (n \leq_{\alpha} n \& m \leq_{\alpha} m)] \]
\[ \& (\forall n)(\forall m)[(n \leq_{\alpha} m \& m \leq_{\alpha} n) \implies n = m] \]
\[ \& (\forall n)(\forall m)(\forall k)[(n \leq_{\alpha} m \& m \leq_{\alpha} k) \implies n \leq_{\alpha} k] \]
\[ \& (\forall n)(\forall m)[(n \leq_{\alpha} n \& m \leq_{\alpha} m) \implies (n \leq_{\alpha} m \vee m \leq_{\alpha} n)], \]
\[ \alpha \in \text{WO} \iff \leq_{\alpha} \text{ is a wellordering} \iff \alpha \in \text{LO} \& <_{\alpha} \text{ has no infinite descending chains} \iff \alpha \in \text{LO} \]
\[ \& (\forall \beta)[(\forall n)[\beta(n + 1) \leq_{\alpha} \beta(n)] \implies (\exists n)[\beta(n + 1) = \beta(n)]] \]
If \( \alpha \in \text{LO} \), let
\[ |\alpha| = \text{order type of } \leq_{\alpha}. \]
In particular, the mapping
\[ \alpha \mapsto |\alpha| \]
takes \text{WO} onto the set of countable ordinals and provides a coding for this set.

**Theorem 2D.2.** The set \( \text{WO} \) of ordinal codes is \( \Pi^1_1 \). Moreover, there are relations \( \leq_{\Pi} \leq_{\Sigma} \) in \( \Pi^1_1 \) and \( \Sigma^1_1 \) respectively, such that
\[ \beta \in \text{WO} \implies (\alpha \leq_{\Pi} \beta \iff \alpha \leq_{\Sigma} \beta \iff [\alpha \in \text{WO} \& |\alpha| \leq |\beta|]). \]

**Proof.** That \( \text{WO} \) is \( \Pi^1_1 \) is obvious from the formulas above. To prove the second assertion, take first
\[ \alpha \leq_{\Sigma} \beta \iff \alpha \in \text{LO} \& (\exists \gamma)[\gamma \text{ maps } \leq_{\alpha} \text{ into } \leq_{\beta} \text{ in a one-to-one order-preserving manner}] \]
\[ \iff \alpha \in \text{LO} \& (\exists \gamma)(\forall n)(\forall m)[n \leq_{\alpha} m \implies \gamma(n) <_{\beta} \gamma(m)]. \]
It is immediate that \( \leq_{\Sigma} = \Sigma^1_1 \) and for \( \beta \in \text{WO} \),
\[ \alpha \leq_{\Sigma} \beta \iff [\alpha \in \text{WO} \& |\alpha| \leq |\beta|]. \]
For the relation $\leq_\Pi$, take

$\alpha \leq_\Pi \beta \iff \alpha \in \text{WO} \& \text{there is no order-preserving map of } \leq_\beta$

onto a proper initial segment of $\leq_\alpha$

$\iff \alpha \in \text{WO}$

$\& (\forall \gamma)(\exists k)(\forall n)(\forall m)[n \leq_\beta m \iff [\gamma(n) \leq_\alpha \gamma(m) <_\alpha k]]$,

where of course we abbreviate

$s <_\alpha t \iff s \leq_\alpha t \& s \neq t$.

2D.3. The Representation Theorem for $\Pi^1_1$. A pointset $P \subseteq X$ is $\Pi^1_1$ if and only if there is a $\Delta^1_1$ function $f : X \rightarrow \mathcal{N}$ such that for all $x$, $f(x) \in \text{LO}$ and

\[ P(x) \iff f(x) \in \text{WO}. \]

In fact, we can choose $f : X \rightarrow \mathcal{N}$ so that for all $x$, $\leq_{f(x)}$ is a non-empty linear ordering, $(\ast)$ holds, and the relation

\[ R(x, n, m) \iff f(x)(n) = m \]

is arithmetical; and if $X$ is of type 0 or 1, then $(\ast)$ holds with a recursive $f$.

It follows that $P$ is $\Pi^1_1$ if and only if $(\ast)$ holds with a Borel $f$, or with a continuous $f$ if $X$ is of type 0 or 1.

Proof. This is an effective and “linear” version of the representation in Problem 2B.3 of $\kappa$-Suslin subsets of $\mathcal{N}$ in the form

\[ P(\alpha) \iff T(\alpha) \text{ is not wellfounded}, \]

where $T$ is a tree on $\omega \times \kappa$. We give a proof from scratch for subsets of an arbitrary space $X$.

Assume by Lemma 2D.1 that

\[ P(x) \iff (\forall \alpha)(\exists t)R(x, \overline{\alpha}(t)) \]

with $R$ semirecursive (or $R$ recursive if $X$ is of type 0 or 1) and such that

\[ R(x, \overline{\alpha}(t)) \& t < s \implies R(x, \overline{\alpha}(s)). \]

For each $x$, put

\[ T(x) = \{(u_0, \ldots, u_{t-1}) : \neg R(x, \langle u_0, \ldots, u_{t-1} \rangle)\} \]

so that $T(x)$ is a tree on $\omega$ and clearly

\[ P(x) \iff T(x) \text{ is wellfounded}. \]
What we need to do is to replace $T(x)$ by a linear ordering on $\omega$ which will be wellfounded precisely when $T(x)$ is. Put

$$(v_0, \ldots, v_{s-1}) >^x (u_0, \ldots, u_{t-1}) \iff (v_0, \ldots, v_{s-1}), (u_0, \ldots, u_{t-1}) \in T(x)$$

and

$$\{ v_0 > u_0 \lor [v_0 = u_0 \& v_1 > u_1] \lor [v_0 = u_0 \& v_1 = u_1 \& v_2 > u_2] \lor \cdots \lor [v_0 = u_0 \& v_1 = u_1 \& \cdots \& v_{s-1} = u_{s-1} \& s < t] \}$$

where $>$ on the right is the usual “greater than” in $\omega$.

It is immediate that if $(v_0, \ldots, v_{s-1}), (u_0, \ldots, u_{t-1})$ are both in $T(x)$ and $(v_0, \ldots, v_{s-1})$ is an initial segment of $(u_0, \ldots, u_{t-1})$, then $(v_0, \ldots, v_{s-1}) >^x (u_0, \ldots, u_{t-1})$; thus if $T(x)$ has an infinite branch, then $>^x$ has an infinite descending chain.

Assume now that $>^x$ has an infinite descending chain, say

$$v^0 >^x v^1 >^x v^2 >^x \cdots,$$

where

$$v^i = (v^i_0, v^i_1, \ldots, v^i_{s_i-1}),$$

and consider the following array:

$$v^0 = (v^0_0, v^0_1, v^0_2, \ldots, v^0_{s_0-1})$$
$$v^1 = (v^1_0, v^1_1, v^1_2, \ldots, v^1_{s_1-1})$$
$$\cdots \cdots$$
$$v^i = (v^i_0, v^i_1, v^i_2, \ldots, v^i_{s_i-1})$$
$$\cdots \cdots$$

The definition of $>^x$ implies immediately that

$$v^0_0 \geq v^1_0 \geq v^2_0 \geq \cdots,$$

i.e., the first column is a nonincreasing sequence of integers. Hence after a while they all are the same, say

$$v^0_0 = k_0 \quad \text{for } i \geq i_0.$$

Now the second column is nonincreasing below level $i_0$, so that for some $i_1, k_1$

$$v^1_0 = k_1 \quad \text{for } i \geq i_1.$$

Proceeding in the same way we find an infinite sequence

$$k_0, k_1, \ldots$$

such that for each $s$, $(k_0, \ldots, k_{s-1}) \in T(x)$, so $T(x)$ is not wellfounded. Thus we have shown,

$$P(x) \iff T(x) \text{ is wellfounded} \iff >^x \text{ has no infinite descending chains.}$$
Finally put

\[ u \leq^x v \iff (\exists t \leq u) (\exists s \leq v) \left[ \text{Seq}(u) \& \text{lh}(u) = t \& \text{Seq}(v) \& \text{lh}(v) = s \& u = v \lor ((v)_0, \ldots, (v)_{s-1}) >^x ((u)_0, \ldots, (u)_{t-1}) \right] \]

and notice that \( \leq^x \) is always a linear ordering, it is not empty (because the code 1 of the empty sequence is in its field), and

\[ P(x) \iff \leq^x \text{ is a wellordering.} \]

Moreover, the relation

\[ P(x, u, v) \iff u \leq^x v \]

is easily arithmetical for arbitrary \( X \) and recursive if \( X \) is of type 0 or 1.

The proof is completed by taking

\[ f(x)(n) = \begin{cases} 1, & \text{if } (n)_0 \leq^x (n)_1, \\ 0, & \text{otherwise}. \end{cases} \]

The classical consequence follows by observing that if \( f : X \rightarrow N \) and its (unfolded) graph \( \{(x, n, m) : f(x)(n) = m\} \) is arithmetical, then \( f \) is Borel.

The linear ordering \( \leq^x \) which we used in this proof is variously known in the literature as the Lusin-Sierpinski or the Kleene-Brouwer ordering. The (technical) observation that \( \leq^x \) is always a non-empty linear ordering insures that if \( P(x) \), then for all \( n \),

\[ \leq^x \left| f(x) \right| n < \left| \leq^x \right| n = \text{supremum}\left(\left| \leq^x \right| n + 1 : n \leq f(x) n\right); \]

this holds by definition if \( n \leq f(x) n \) and trivially if \( n \) is not in the field of \( \leq^x \), since in that case \( \left| \leq^x \right| n = 0 \), while \( \left| \leq^x \right| > 0 \). This is used in some places to simplify formulas.

**The Church-Kleene \( \omega_1 \)**. Let us prove here just one very useful corollary of this basic result. Put

\[ (2D-1) \quad \omega_1^{CK} = \text{supremum}\{\left| \alpha \right| : \alpha \in \text{WO and } \alpha \text{ is recursive}\}. \]

One may think of \( \omega_1^{CK} \) as an “effective analog” of the least uncountable ordinal \( \aleph_1 \): \( \omega_1^{CK} \) is the least ordinal which cannot be realized by a recursive wellordering with field in \( \omega \).

**2D.4. The Boundedness Theorem for \( \Pi^1_1 \)**. Suppose \( P \subseteq X \) and \( P \) satisfies the equivalence

\[ (\ast) \quad P(x) \iff f(x) \in \text{WO} \]

with some \( \Delta^1_1 \) function \( f \). Then \( P \) is \( \Delta^1_1 \) if and only if

\[ \text{supremum}\{|f(x)| : P(x)\} < \omega_1^{CK}. \]
Similarly, if \((*)\) holds with some Borel function \(f\), then \(P\) is Borel if and only if
\[
\sup\{|f(x)| : P(x)\} < \aleph_1.
\]

**Proof.** Assume first that for all \(x\), if \(P(x)\) then \(|f(x)| \leq |\alpha|\), where \(\alpha \in \text{WO}\) and \(\alpha\) is recursive. By Theorem 2D.2 then,
\[
P(x) \iff f(x) \leq \Sigma \alpha,
\]
so \(P\) is \(\Sigma^1_1\) and since it is evidently \(\Pi^1_1\), it is \(\Delta^1_1\).

Conversely, suppose \(\sup\{|f(x)| : P(x)\} \geq \omega_{CK}^1\). Let \(Q \subseteq \omega\) be any \(\Pi^1_1\) relation on \(\omega\), so by the Representation Theorem 2D.3 there is a recursive \(g : \omega \to \mathbb{N}\) and
\[
Q(n) \iff g(n) \in \text{WO}.
\]
Notice that for every \(n\), \(g(n) \in \mathcal{N}\) is recursive by (3) of Theorem 1D.11. Hence
\[
Q(n) \iff g(n) \in \text{WO} \land |g(n)| < \omega_{CK}^1
\]
\[
\iff (\exists x)\left(P(x) \land g(n) \leq \Sigma f(x)\right),
\]
which implies that if \(P\) is \(\Sigma^1_1\), then so is \(Q\). But \(Q\) was arbitrary \(\Pi^1_1\) on \(\omega\) and need not be \(\Sigma^1_1\), so \(P\) is not \(\Sigma^1_1\).

Proof of the boldface result is a bit simpler. ⊣

Put
\[
\delta_1^1 = \sup \{|\alpha| : \alpha \in \text{WO} \land \alpha \text{ is } \Delta^1_1\},
\]
where \(\alpha\) is \(\Delta^1_1\) if \(\{(n,m) : \alpha(n) = m\}\) is \(\Delta^1_1\).

**Problem 2D.5** (Spector). Prove that \(\delta_1^1 = \omega_{CK}^1\).

This result is rather surprising, as one might expect to get longer wellorderings of \(\omega\) in the complicated pointclass \(\Delta^1_1\) than one gets in \(\Delta^0_1\).

The next result is an effective version of one of the basic classical theorems about Borel sets.

**Theorem 2D.6.** For each \(\Delta^1_1\) pointset \(P \subseteq \mathcal{X}\), there is a recursive function \(\rho : \mathcal{N} \to \mathcal{X}\) and a \(\Pi^0_1\) set \(A \subseteq \mathcal{N}\), such that \(\rho\) is one-to-one on \(A\) and \(\rho[A] = P\).

Similarly, if \(P\) is \(\Delta^1_1\), then there is a continuous \(\rho : \mathcal{N} \to \mathcal{X}\) and a closed \(A \subseteq \mathcal{N}\) such that \(\rho\) is one-to-one on \(A\) and \(\rho[A] = P\).

**Proof.** Assume first, for simplicity, that \(\mathcal{X} = \mathcal{N}\).
By the special case for $\mathcal{N}$ of Theorem 2D.3 and the Boundedness Theorem 2D.4, fix a recursive $f : \mathcal{N} \to \mathcal{N}$ and a recursive $\beta \in \text{WO}$ such that

$$(*) \quad P(\alpha) \iff f(\alpha) \in \text{WO} \iff f(\alpha) \leq_{\Sigma} \beta.$$ 

Put

$$Q(\gamma, \alpha) \iff \gamma \text{ maps } \leq_{\alpha} \text{ onto an initial segment of } \leq_{\beta} \text{ in an order preserving fashion and } \gamma = 0 \text{ outside the field of } \leq_{\alpha}$$

$$\iff (\forall n)[\alpha(\langle n, n \rangle) \neq 1 \implies \gamma(n) = 0]$$

$$& [\alpha(\langle n, n \rangle) = 1 \implies \beta(\langle \gamma(n), \gamma(n) \rangle) = 1]$$

$$& (\forall n)(\forall m)[\alpha(\langle n, n \rangle) = 1 & \alpha(\langle m, m \rangle) = 1]$$

$$\implies [\alpha(\langle n, m \rangle) = 1 \iff \beta(\langle \gamma(n), \gamma(m) \rangle) = 1]$$

$$& (\forall n)(\forall m)[\alpha(\langle n, n \rangle) = 1 & \beta(\langle m, \gamma(n) \rangle) = 1]$$

$$\implies (\exists s)[\alpha(\langle s, n \rangle) = 1 & \gamma(s) = m].$$

Clearly, $Q$ is $\Pi^0_2$ and hence so is the pointset

$$Q^*(\gamma, \alpha) \iff Q(\gamma, f(\alpha))$$

$$\iff \gamma \text{ is an order preserving map of } \leq_{f(\alpha)} \text{ onto an initial segment of } \leq_{\beta}.$$ 

Moreover, easily

$$P(\alpha) \iff (\exists \gamma)Q^*(\gamma, \alpha)$$

$$\iff \text{there exists exactly one } \gamma \text{ such that } Q^*(\gamma, \alpha).$$

Bring $Q^*$ to normal form

$$Q^*(\gamma, \alpha) \iff (\forall n)(\exists m)R(\gamma, \alpha, n, m)$$

with $R$ recursive, and let

$$S(\delta, \gamma, \alpha) \iff (\forall n)[R(\gamma, \alpha, n, \delta(n)) & (\forall m < \delta(n))\neg R(\gamma, \alpha, n, m)].$$

Now $S$ is a $\Pi^0_1$ subset of $\mathcal{N} \times \mathcal{N} \times \mathcal{N}$ and the recursive map $(\delta, \gamma, \alpha) \mapsto \alpha$ takes $S$ onto $P$ and is one-to-one on $S$. The result follows because $\mathcal{N} \times \mathcal{N} \times \mathcal{N}$ is recursively homeomorphic with $\mathcal{N}$.

For the general case where $P \subseteq \mathcal{X}$ is a $\Delta^1_1$ subset of an arbitrary $\mathcal{X}$, let

$$\pi : \mathcal{N} \to \mathcal{X}$$

be a recursive surjection by Theorem 1F.12 which is injective on a $\Pi^0_1$ set $D \subseteq \mathcal{N}$ for which $\pi[D] = \mathcal{X}$, and apply the special case of the theorem to

$$P' = \pi^{-1}[P] \cap D \subseteq \mathcal{N};$$
we get a recursive $\rho : \mathcal{N} \to \mathcal{N}$ which is injective on some $\Pi^0_1$ set $A \subseteq \mathcal{N}$ such that $\rho[A] = P'$ and then the composition $\pi \circ \rho : \mathcal{N} \to \mathcal{X}$ is recursive, injective on $A$ and maps $A$ onto $P$.

The assertion about $\Delta^1_1$ sets follows by relativizing the argument or repeating it, starting with a continuous $f$ and some $\beta \in \text{WO}$ satisfying $(\ast)$. \dashleftarrow

This result is important, especially because we will prove later that every injective, recursive image of a $\Delta^1_1$ set is $\Delta^1_1$.

Additional problems for Section 2D

Problem 2D.7. Prove that a point $x_0$ is $\Delta^1_1$ if and only if there is a $\Pi^0_1$ singleton $\{\alpha_0\} \subseteq \mathcal{N}$ such that $x_0$ is recursive in $\alpha_0$.

It is not true that every $\Delta^1_1$ point $\alpha \in \mathcal{N}$ is a $\Pi^0_1$ (or even an arithmetical) singleton; this is due to Feferman, one of the first applications of forcing to recursion theory.

Problem 2D.8. Prove that for each countable ordinal $\xi$, the set

\[ I_\xi = \{ \alpha : \alpha \in \text{WO} \& |\alpha| \leq \xi \} \]  

is $\Delta^1_1$, uniformly in the coding for ordinals determined by WO and the canonical coding for $\Delta^1_1$, i.e., show that there is a potentially recursive function $u : \mathcal{N} \to \mathcal{N}$ whose domain of convergence contains WO and

\[ \beta \in \text{WO} \Rightarrow \left( u(\beta) \text{ is a } \Delta^1_1\text{-code of } \{ \alpha : \alpha \in \text{WO} \& |\alpha| \leq |\beta| \} \right). \]

Together with the Suslin-Kleene Theorem, this fact implies immediately that each $I_\xi$ is uniformly Borel. On the other hand, this fact can be proved directly and then it can be used to give new and interesting proofs of Suslin's Theorem and the Suslin-Kleene Theorem that are independent of the proofs we gave in Section 2C.6. We outline these arguments in the next three problems. (They are mostly derived from Spector [1955], but it is very likely that the first one was known to Suslin or Lusin.)

Problem 2D.9. Prove that each $I_\xi$ is a Borel set and use this to infer Suslin's Theorem, that $\Delta^1_1 = \mathcal{B}$.

Problem 2D.10. For any linear ordering $\leq \subseteq \omega \times \omega$, let

\[
\langle n, k \rangle \preceq \langle m, l \rangle \iff [n = m = 0 \& k \preceq l] \\
\quad \lor [n = 0 \& m > 0 \& (m - 1) \leq (m - 1)] \\
\quad \lor [n > 0 \& m = 0 \& (n - 1) < (m - 1)] \\
\quad \lor [n = m > 0 \& (n - 1) \leq (n - 1) \& (m - 1) \leq (m - 1) \& k \preceq l],
\]

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2E. Norms and the prewellordering property

where \( \leq^\omega \) is the usual ordering on \( \omega \) (and we understand the definition to mean that \( s \not\leq t \) unless \( s = \langle n, k \rangle \) and \( t = \langle m, l \rangle \) for some \( n, k, m, l \).)

Prove that

1. \( \leq' \) is a linear ordering;
2. if \( \leq \) is a wellordering, then \( \leq' \) is also a wellordering and
   \[ |\leq'| = 1 + |\cdot \omega > | \leq | \] (ordinal multiplication); and
3. there is a recursive function \( g : \mathcal{N} \to \mathcal{N} \) such that
   \[ \leq' = \leq_{g(\alpha)} . \]

Here \( | \leq | \) is the order type of \( \leq \), which is a countable ordinal ordinal if \( \leq \) is a wellordering; and the important fact in (2) is that \( | \leq' | > | \leq | \), i.e., the precise relation between these two ordinals is not important.

**Problem 2D.11.** With the notation of the preceding Problem 2D.10, prove that there is a recursive function \( h(\alpha) \) such that
\[
\alpha \in \text{WO} \implies \left( h(\alpha) \text{ is a Borel code of } \leq_{g(\alpha)} \right),
\]
and use this to prove the Suslin-Kleene Theorem 2C.6 for \( \Delta^\mathbb{N}_1 \) subsets of any \( X \). HINT: The proof is by effective recursion on \( | \leq_{g(\alpha)} | \), and the key fact is that \( \leq_{g(\alpha)} \) is defined so we can tell whether a point \( \langle n, k \rangle \) in its field is “a notation” for 0, a successor ordinal or a limit ordinal. This means that if
\[
\rho : \text{Field}(\leq_{g(\alpha)}) \to \xi = | \leq_{g(\alpha)} |
\]
is the unique surjection of \( \text{Field}(\leq_{g(\alpha)}) \) onto an ordinal (its rank), then for \( \langle n, k \rangle \in \text{Field}(\leq_{g(\alpha)}) \):
1. \( \rho(\langle 0, 0 \rangle) = 0 \);
2. \( \rho(\langle n, k + 1 \rangle) = \rho(\langle n, k \rangle) + 1 \);
3. \( \rho(\langle n + 1, 0 \rangle) = \sup_{m,k} \{ \rho(\langle m, k \rangle) : m = 0 \lor m - 1 <_\alpha n \} \).

2E. Norms and the prewellordering property

The Representation Theorem 2D.3 implies a simple “structural property” of \( \Pi^1_1 \), which has many consequences for arbitrary pointclasses.

A **norm** on a pointset \( P \) is any function
\[
\varphi : P \to \text{Ordinals}
\]
which assigns ordinals to the elements of \( P \) and so determines a relation \( \leq^\varphi \) on \( P \)
\[
x \leq^\varphi y \iff \varphi(x) \leq \varphi(y)
\]
2. Structure theory for pointclasses

which is a prewellordering, i.e., reflexive, transitive, well founded and also connected,

\((\forall x, y \in P)[x \leq^{\varphi} y \lor y \leq^{\varphi} x]\).

Conversely, if \(\leq\) is a prewellordering on \(P\), then \(\leq = \leq^{\varphi}\) for some norm \(\varphi\); and \(\varphi\) is uniquely determined if we insist that it be regular, i.e., that \(\varphi\) maps \(P\) onto some ordinal \(\lambda\). Two norms \(\varphi\) and \(\psi\) on \(P\) are equivalent if \(\leq^{\varphi} = \leq^{\psi}\), i.e.,

\[\varphi(x) \leq \varphi(y) \iff \psi(x) \leq \psi(y).\]

Clearly, every norm is equivalent to a unique regular norm.

There are many trivial norms on a set, e.g., the constant 0 function or the bijection of \(P\) with an ordinal given by any wellordering of \(P\). The concept becomes non-trivial if we impose definability conditions on a norm, which then allow us to interpret \(\varphi\) as a complexity measure on \(P\).

Let \(\Gamma\) be a pointclass and \(P \subseteq X\) any pointset. A function \(\varphi : P \to \lambda\) is a \(\Gamma\)-norm if there exist relations \(\leq^{\varphi}\Gamma\), \(\leq^{\varphi}\neg\Gamma\) in \(\Gamma\) and \(\neg\Gamma\) respectively such that for every \(y\),

\((\ast)\quad P(y) \Rightarrow (\forall x)[(P(x) \& \varphi(x) \leq \varphi(y)] \iff x \leq^{\varphi}_{\Gamma} y \iff x \leq^{\varphi}_{\neg\Gamma} y\).

A pointclass \(\Gamma\) is normed or has the prewellordering property if every pointset \(P\) in \(\Gamma\) admits a \(\Gamma\)-norm.

It is important for the applications that the definition of \(\Gamma\)-norm be precisely that given by (\(\ast\)). Notice that if \(\Gamma\) satisfies some minimum closure properties and \(P \in \Gamma\), then (\(\ast\)) is stronger than simply requiring that the associated prewellordering \(\leq^{\varphi}\) be in \(\Gamma\) but weaker than insisting that \(\leq^{\varphi}\) be in \(\Gamma \cap \neg\Gamma\).

In addition to the prewellordering \(\leq^{\varphi}\), there are two other relations that are naturally associated with a norm \(\varphi\). Put

\[x \leq^{\varphi}_{\psi} y \iff P(x) \& \neg P(y) \lor \varphi(x) \leq \varphi(y),\]

\[x <^{\varphi}_{\psi} y \iff P(x) \& \neg P(y) \lor \varphi(x) < \varphi(y).\]

The meaning of these relations becomes clear if we extend the norm \(\varphi\) on \(P \subseteq X\) to all of \(X\) by

\[\varphi(x) = \infty, \quad \text{if } \neg P(x),\]

where \(\infty\) is assumed larger than all the ordinals. Then obviously, with this extended \(\varphi\),

\[x \leq^{\varphi}_{\psi} y \iff P(x) \& \varphi(x) \leq \varphi(y),\]

\[x <^{\varphi}_{\psi} y \iff P(x) \& \varphi(x) < \varphi(y).\]

**Problem 2E.1.** Suppose \(\Gamma\) is a \(\Sigma\)-pointclass and \(\varphi\) is a norm on some \(P\) in \(\Gamma\); prove that \(\varphi\) is a \(\Gamma\)-norm if and only if both \(\leq^{\varphi}_{\psi}\) and \(<^{\varphi}_{\psi}\) are in \(\Gamma\).

**Theorem 2E.2.** Every \(\Pi_{1}^1(z)\) and \(\Pi_{1}^1\) are all normed.
2E. Norms and the prewellordering property

Proof. Given \( P \) in \( \Pi^1_1 \), choose a \( \Delta^1_1 \) function \( f \) by 2D.3 such that
\[
P(x) \iff f(x) \in \text{WO}
\]
and for \( x \in P \), put
\[
\varphi(x) = |f(x)|.
\]
Using the notation of 2D.2, we can take
\[
x \leq^*_\Pi y \iff f(x) \leq f(y),
\]
\[
x \leq^*_\Sigma y \iff f(x) \leq \Sigma f(y)
\]
and verify easily that they satisfy (*)

The same proof works for \( \Pi^1_1(z) \) and \( \Pi^1_\sim \), taking a Borel \( f \) for the boldface version.

\[\dashv\]

2E.3. The \( \exists^N \)-norm-transfer theorem. If \( \Gamma \) is a \( \Sigma \)-pointclass and \( P \subseteq X \times N \) is in \( \Gamma \) and admits a \( \Gamma \)-norm, then \( \exists^N P \) admits an \( \exists^N \forall^N \Gamma \)-norm.

Hence, if \( \Gamma \) is normed and closed under \( \forall^N \), then \( \exists^N \Gamma \) is normed. In particular, \( \Sigma^1_2 \), every \( \Sigma^1_2(z) \) and \( \Sigma^1_3 \) are all normed.

Proof. It is enough to establish the first assertion. Suppose that
\[
Q(x) \iff (\exists \alpha)P(x, \alpha)
\]
with \( P \) in \( \Gamma \), let \( \varphi \) be a \( \Gamma \)-norm on \( P \) and define \( \psi \) on \( Q \) by
\[
\psi(x) = \inf \{ \varphi(x, \alpha) : P(x, \alpha) \}.
\]
Proof that \( \psi \) is an \( \exists^N \forall^N \Gamma \)-norm is immediate from the equivalences
\[
x \leq^*_\psi y \iff (\exists \alpha)(\forall \beta)[(x, \alpha) \leq^*_\psi (y, \beta)],
\]
\[
x <^*_\psi y \iff (\exists \alpha)(\forall \beta)[(x, \alpha) <^*_\psi (y, \beta)].
\]

This result is typical of the kind of abstract setting in which the notion of a \( \Gamma \)-norm proves useful.

We will study many consequences of the prewellordering property in the next two sections. Here we concentrate on just a few facts which are simple, useful and indicate the strength of the hypothesis.

2E.4. The Easy Uniformization Theorem. Suppose \( \Gamma \) is a \( \Sigma \)-pointclass, \( \mathcal{Y} \) is a space of type 0, \( P \subseteq X \times \mathcal{Y} \) is in \( \Gamma \) and \( P \) admits a \( \Gamma \)-norm. Then \( P \) can be uniformized by some \( P^* \) in \( \forall^w \Gamma \).

In particular, if \( \Gamma \) is also closed under \( \forall^w \), then every \( P \subseteq X \times \mathcal{Y} \) in \( \Gamma \) with \( \mathcal{Y} \) of type 0 can be uniformized by some \( P^* \) in \( \Gamma \).
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Proof. It is enough to prove the result with \( \mathcal{Y} = \omega \). Assume then that \( P \subseteq \mathcal{X} \times \omega \) is in \( \Gamma \), let \( \varphi \) be a \( \Gamma \)-norm on \( P \) and put

\[
P^*(x,n) \iff P(x,n) \& (\forall m)[(x,n) \leq^\varphi (x,m)]
\]

\[
\& (\forall m)[(x,n) <^\varphi (x,m) \lor n \leq m],
\]

or in other words

\[
P^*(x,n) \iff P(x,n) \& \varphi(x,n) = \infimum\{\varphi(x,m) : P(x,m)\}
\]

\[
\& n = \infimum\{m : P(x,m) \& \varphi(x,m) = \varphi(x,n)\},
\]

Clearly \( P^* \) is in \( \forall^* \Gamma \) and

\[
P^*(x,n) \& P^*(x,n') \implies P(x,n) \& P(x,n')
\]

\[
\& \varphi(x,n) = \varphi(x,n') \& n \leq n' \& n' \leq n \implies n = n',
\]

so \( P^* \) is the graph of a function. If \( (\exists n)P(x,n) \), let

\[
\xi = \infimum\{\varphi(x,n) : P(x,n)\},
\]

\[
n = \infimum\{m : P(x,m) \& \varphi(x,m) = \xi\},
\]

and verify easily that \( P^*(x,n) \). Thus \( P^* \) uniformizes \( P \).

\( \Box \)

The problem of uniformizing subsets of \( \mathcal{X} \times \mathcal{Y} \) for arbitrary spaces \( \mathcal{Y} \) is much harder and cannot be settled using only the prewellordering property.

Additional problems for Section 2E

Theorem 2E.4 is most often used in the form of the following easy corollary.

Problem 2E.5 (The \( \Delta \)-Selection Principle). Suppose \( \Gamma \) is a normed \( \Sigma \)-pointclass closed under \( \forall^\omega \), let \( P \subseteq \mathcal{X} \times \mathcal{Y} \) be in \( \Gamma \) with \( \mathcal{Y} \) of type 0, assume that \( A \subseteq \mathcal{X} \) is in \( \Delta = \Gamma \cap \neg \Gamma \) and

\[
(\forall x \in A)(\exists y)P(x,y).
\]

Prove that there is a \( \Delta \)-recursive function \( f : \mathcal{X} \to \mathcal{Y} \) such that

\[
(\forall x \in A)P(x,f(x)).
\]

Problem 2E.6. Let \( \Gamma \) be a \( \Sigma \)-pointclass. Check that a norm \( \varphi \) on some \( P \) in \( \Gamma \) is a \( \Gamma \)-norm if and only if the unique regular norm \( \psi \) which is equivalent to \( \varphi \) is a \( \Gamma \)-norm. Prove also that if \( \varphi \) is a \( \Gamma \)-norm, then there are relations \( <^\varphi_\Gamma \), \( <^\varphi_{\neg \Gamma} \) in \( \Gamma \) and \( \neg \Gamma \) respectively such that for every \( y \),

\[
P(y) \implies (\forall x)[P(x) \& \varphi(x) < \varphi(y)] \iff x <^\varphi_\Gamma y \iff x <^\varphi_{\neg \Gamma} y.
\]

Problem 2E.7. Prove that if \( \Gamma \) is a normed \( \Sigma \)-pointclass, then the associated boldface class \( \Gamma^* \) is also normed.
Problem 2E.8. Prove that for \( n \geq 2 \), the pointclasses \( \Sigma^0_n \), \( \Sigma^{\tilde{0}}_n \) are normed. Prove also that every \( \Sigma^1 \) (or \( \Sigma^{\tilde{1}} \)) pointset of type 0 or 1 admits a \( \Sigma^1 \) (or \( \Sigma^{\tilde{1}} \)) norm. Show that the latter result fails for sets of reals.

Problem 2E.9. Suppose \( \Gamma \) is a \( \Sigma \)-pointclass closed under \( \Delta^1 \) substitutions. Prove that if every pointset of type 1 in \( \Gamma \) admits a \( \Gamma \)-norm, then \( \Gamma \) is normed.

**Hint:** Use Theorem 1F.12.

Recall the reduction and separation properties for pointclasses defined in 1F.31 and 1F.33.

Problem 2E.10. Prove that every normed \( \Sigma \)-pointclass has the reduction property. Infer from Problem 1F.36 that if \( \Gamma \) is a normed and \( \omega \)-parametrized \( \Sigma^* \)-pointclass, then the class \( \neg \Gamma \) of its complements has the separation property and is not normed.

For example, \( \Sigma^1, \Pi^1 \) and their relativized and boldface versions are not normed by Theorem 2E.2.

It follows from these simple facts that the Kleene pointclasses which are normed from what we know now are exactly those circled in Diagram 8. The circle around \( \Sigma^0_1 \) is dotted, since only \( \Sigma^0_1 \) pointsets of type 0 or 1 admit \( \Sigma^0_1 \)-norms, and the diagram for the boldface classes is identical. Whether the higher analytical pointclasses (and which ones) are normed is a difficult question, independent of ZFC. We will discuss it briefly later on.

Many structure results for pointclasses have uniform versions which are easy to establish using the methods of Section 2C, especially the \( SYX \)-Theorem 2C.3. We record one fact of this type as an example.

Problem 2E.11. Prove that if \( \Gamma \) is an \( \omega \)-parametrized \( \Sigma \)-pointclass with the reduction property, then \( \Gamma \) has the uniform reduction property: i.e., for each \( X \), there are recursive functions \( u_1(\alpha, \beta), u_2(\alpha, \beta) \) such that whenever \( \alpha, \beta \) code respectively subsets \( P, Q \) of \( X \) in \( \Gamma \), then \( u_1(\alpha, \beta), u_2(\alpha, \beta) \) code respectively \( \Gamma \) subsets \( P^*, Q^* \) of \( X \) in \( \Gamma \) which reduce the pair \( P, Q \).
2F. Spector pointclasses

The consequences of the prewellordering property in the preceding section depended on several side conditions on a pointclass \( \Gamma \), e.g., closure under various operations or parametrization. Here we will bundle the most important of these hypotheses into the basic notion of a Spector pointclass. The simplest Spector pointclasses are \( \Pi^1_1 \) and \( \Sigma^1_2 \)—in fact \( \Pi^1_1 \) is the least Spector pointclass, and many of its structure properties follow from the Spector pointclass axioms; they also hold then of several richer Spector pointclasses that we will construct.

A Spector pointclass is any normed \( \Sigma^* \)-pointclass which is closed under \( \forall \omega \), i.e.,

(1) \( \Gamma \) contains \( \Sigma^0_1 \) and is closed under trivial substitutions, \&, \lor, \exists^\leq, \forall^\leq, \exists^\omega \) and \( \forall^\omega \);

(2) \( \Gamma \) has the substitution property (and so it is closed under recursive substitutions);

(3) \( \Gamma \) is \( \omega \)-parametrized; and

(4) \( \Gamma \) is normed.

All the analytical pointclasses \( \Sigma^1_n, \Pi^1_n \) satisfy (1) – (3), and so to prove that one of them is a Spector pointclass we need only verify the prewellordering property. It is also trivial to check that every relativization \( \Gamma(z) \) of a Spector pointclass \( \Gamma \) is a Spector pointclass, see Problem 2F.4. Thus \( \Pi^1_1, \Sigma^1_2, \Pi^1_1(z), \Sigma^1_2(z) \) are Spector pointclasses—and they are the only ones we know at this time.

We prove first a strong closure property of Spector pointclasses which, in particular, implies that \( \Pi^1_1 \) is the least one. It will follow from the following simple lemma which gives a "structural" characterization of the Suslin quantifier:

Lemma 2F.1. Suppose \( Q, P \subseteq X \times \omega \) and \( P \) admits a norm \( \varphi \) such that

\[
P(x, w) \iff Q(x, w) \lor (\forall s)[(x, w * \langle s \rangle) <^* \varphi (x, w)];
\]

then

\[
P(x, w) \iff (Su)Q(x, w * u) \iff (\forall \alpha)(\exists t)Q(x, w * \overline{\alpha}(t)).
\]

Proof. First we prove by induction on \( \varphi(x, w) \) that

\[
P(x, w) \implies (\forall \alpha)(\exists t)Q(x, w * \overline{\alpha}(t)).
\]

Assuming this for all \( (x, u) \in P \) with \( \varphi(x, w) < \varphi(x, u) \) and supposing that \( P(x, w) \) holds, we have by the hypothesis

\[
Q(x, w) \lor (\forall s)[(x, w * \langle s \rangle) <^* \varphi (x, w)].
\]
If $Q(x, w)$ holds, then $(\forall \alpha)Q(x, w * \pi(0))$ since $w * \pi(0) = w$. Otherwise, we have

$$(\forall s)[(x, w * \langle s \rangle) < \varphi(x, w)],$$

so that for each $s$, $P(x, w * \langle s \rangle)$ and $\varphi(x, w * \langle s \rangle) < \varphi(x, w)$. By the induction hypothesis then,

$$(\forall s)((\forall \alpha)(\exists t)Q(x, w * \langle s \rangle * \alpha(t)))$$

from which $(\forall \alpha)(\exists t)Q(x, w * \pi(t))$ follows immediately.

Conversely, if we assume $\neg P(x, w)$, then $\neg Q(x, w)$ and there exists some $s_0$ such that $\neg (x, w * \langle s_0 \rangle) < \varphi(x, w)$. This implies $\neg P(x, w * \langle s_0 \rangle)$ since $P(x, w * \langle s_0 \rangle)$ and $\neg P(x, w)$ together imply $(x, w * \langle s_0 \rangle) < \varphi(x, w)$. Again, $\neg Q(x, w * \langle s_0 \rangle)$ and so for some $s_1$, $\neg (x, w * \langle s_0, s_1 \rangle) < \varphi(x, w * \langle s_0 \rangle)$. Continuing in this way, we determine some $\alpha = (s_0, s_1, \ldots) \in \mathcal{N}$ such that $(\forall t)\neg Q(x, w * \pi(t))$.

The lemma is (perhaps) a bit peculiar, especially as it assumes no definability hypotheses on the norm $\varphi$, but it yields the following

**Theorem 2F.2.** Every Spector pointclass is closed under the Suslin quantifier.

It follows that $\Pi^1_1$ is the smallest Spector pointclass and $\Sigma^1_2$ is the smallest Spector pointclass closed under $\exists^* N$.

**Proof.** The second claim follows immediately from the first and the fact that $\Sigma^1_2$ is a Spector pointclass.

To prove the first, suppose

$$P(x) \iff (Su)Q(x, u)$$

with $Q \in \Gamma$. It is enough to find some $R^* \subseteq X \times \omega$ in $\Gamma$ which admits a norm $\varphi$ so that

$$R^*(x, w) \iff Q(x, w) \lor (\forall s)[(x, w * \langle s \rangle) < \varphi(x, w)],$$

since we then have by the lemma

$$(Su)Q(x, w * u) \iff R^*(x, w)$$

and so $P(x) \iff R^*(x, 1)$.

Let $G \subseteq \mathcal{N} \times X \times \omega$ be a good universal set for $\Gamma$ at $X \times \omega$ and let $\psi : G \to \text{Ordinals}$ be a $\Gamma$-norm on $G$. Set

$$R(\alpha, x, w) \iff Q(x, w) \lor (\forall s)[(\alpha, x, w * \langle s \rangle) < \varphi(\alpha, x, w)].$$

Now $R$ is in $\Gamma$, so by (2) of the Good Parametrization Theorem 1H.13 (the Second Recursion Theorem for relations), there is a recursive $\pi$ so that

$$R(\pi, x, w) \iff G(\pi, x, w).$$
Put
\[ R^*(x, w) \iff R(\overline{z}, x, w) \]
and on \( R^* \) put the norm
\[ \varphi(x, w) = \psi(\overline{z}, x, w) . \]

Computing,
\[
R^*(x, w) \iff R(\overline{z}, x, w) \\
\iff Q(x, w) \lor (\forall s)(\overline{z}, x, w \ast s) <^*_\varphi (\overline{z}, x, w)] \\
\iff Q(x, w) \lor (\forall s)(x, w \ast s) <^*_\varphi (x, w)]
\]
so that \( R^* \) has the required property. \( \dashv \)

This theorem is interesting partly because it gives an intrinsic structural characterization of \( \Pi^1_1 \). Of course, \( \Pi^1_1 \) can be easily characterized by its closure properties, e.g., it is the smallest \( \Sigma^* \)-pointclass closed under \( \forall^\omega \) and \( \forall^N \). But nothing very deep can be proved in general about \( \Sigma^* \)-pointclasses closed under \( \forall^\omega \) and \( \forall^N \). We will see that Spector pointclasses have a rich structure theory, much of it giving new results even when we specialize it to \( \Pi^1_1 \).

The next result is a strengthening of part of Problem 1H.23, but we will give a full proof:

**2F.3. The \( S \)-norm theorem.** Suppose \( \Gamma \) is a \( \Sigma^* \)-pointclass such that
\[ \neg \Gamma \subseteq S\Gamma; \text{ then } S\Gamma \text{ is a Spector pointclass and} \]
(2F-1) \[ \Gamma \subseteq S\Gamma \subseteq S\neg S\Gamma . \]

In particular, the following pointclasses are Spector:
(2F-2) \[ SS_1 \subseteq SS_1 \subseteq SSS_1, \ldots . \]

**Proof.** There are several facts that need to be checked.

(1) \( S\Gamma \) is closed under \( S \).

**Proof.** Notice that for any \( P(u, v) \),
\[ (\exists \alpha)(\forall \beta)(\forall s)P(\alpha(t), \beta(s)) \iff (\exists \gamma)(\forall s)P(\gamma_0(t), (\gamma)_t+1(s)) \]
where by Problem 1D.15, for any \( \delta \), \( (\delta)_t = \delta(i, t) \). This is basically obvious: for the direction \( \Rightarrow \), choose \( \alpha \) which satisfies the rest, choose for any \( t \) a \( \beta \) which satisfies the matrix and check that the RHS is satisfied by any \( \gamma \) such that \( (\gamma)_0 = \alpha \) and for every \( t \), \( (\gamma)_t+1 = \beta_t \); and for the direction \( \Leftarrow \) let \( \gamma \) be the point in \( N \) guaranteed by the RHS, take \( \alpha = (\gamma)_0 \) and for any \( t \), take \( \beta = (\gamma)_t+1 \).

Suppose now that
\[ P(x) \iff (\forall \alpha)(\exists t)(\exists \beta)(\exists s)R(x, \alpha(t), \beta(s)) \]
with \( R(x, u, v) \) in \( \Gamma \) and monotone upward in the sequence variables \( u, v \), which we can insure using the closure of \( \Gamma \) under bounded quantification.

By the equivalence above, we have then that

\[
P(x) \iff (\forall \gamma)(\exists i)(\exists s) R(x, (\gamma)_0(t), (\gamma)_{t+1}(s)) \iff (\forall \gamma)(\exists i) R^*(x, \gamma(i))
\]

where

\[
R^*(x, w) \iff Seq(w) & (\exists u, v, t, s \leq w) \left( R(x, u, v) & (\exists \gamma)[w = \tau(\text{lh}(w)) & u = (\gamma)_0(t) & v = (\gamma)_{t+1}(s)] \right).
\]

This is easy to check: for the \((\Rightarrow)\) direction, given \( \gamma \) and the \( s, t \) which satisfy the LHS, we choose \( t \) large enough so that \((\gamma)_0(t), (\gamma)_{t+1}(s) \leq \gamma(i)\), which then insures \( R^*(x, \gamma(i)) \), and the converse is also simple. Finally, the second line in the definition of \( R^*(x, w) \) defines a relation in \( \Gamma \), despite the occurrence of \( (\exists \gamma) \) in it, because any \( \gamma \) that satisfies it must have \( \gamma(j) = (w)_j \) for every \( j < \text{lh}(w) \), and so whether such a \( \gamma \) exists can be checked by examining \( w, u, v, s \) and \( t \).

(2) \( \mathcal{S} \Gamma \) is closed under \( \forall \omega \) and \( \exists \omega \).

Proof. For \( \forall \omega \) check that

\[
(\forall t)(\forall \beta)(\exists s) P(x, \beta(s), t) \iff (\forall \alpha)(\exists t)(\forall \beta)(\exists s) Q(x, \beta(s), \alpha(t))
\]

where \( Q(x, u, v) \iff P(x, u, (v)_0) \). And similarly, for \( \exists \omega \),

\[
(\exists t)(\forall \beta)(\exists s) P(x, \beta(s), t) \iff (\forall \alpha)(\exists t)(\forall \beta)(\exists s) Q'(x, \beta(s), \alpha(t))
\]

where \( Q'(x, u, v) \iff P(x, u, \text{lh}(v)) \).

(3) \( \mathcal{S} \Gamma \) is closed under trivial substitutions, it has the substitution property and it is \( \omega \)-parametrized.

These are all simple, by the standard “propagation” arguments we used in Section 1F—they use no special properties of \( \mathcal{S} \).

(4) \( \mathcal{S} \Gamma \) is closed under \( \& \), \( \lor \), and bounded quantification of both kinds.

Proof. For conjunction, use the equivalence

\[
(\forall \alpha)(\exists t) P(x, \alpha(t)) \& (\forall \alpha)(\exists t) Q(x, \alpha(t)) \iff (\forall \alpha)(\exists t)[P(x, \alpha(t)) \& Q(x, \alpha(t))]
\]

which holds if \( P(x, u) \) and \( Q(x, v) \) are monotone in their sequence code arguments \( u, v \); and we may insure this by appealing to the closure of \( \Gamma \) under bounded quantification. Closure under \( \lor \) is “easier” (as we do not need to appeal to monotonicity), and bounded quantification can be expressed using unbounded quantification, the recursive relation \( s \leq t \) (which is in \( \Gamma \)) and trivial substitutions.
2. Structure theory for pointclasses

At this point it only remains to prove the prewellordering property for $\mathcal{SG}$, and we already know that it is a $\Sigma^*$-pointclass; so we can appeal to the Kleene calculus for $\mathcal{SG}$-recursion, especially the Second Recursion Theorem, (4) of Theorem 1H.16. We will also use the hypothesis that $\neg\Gamma \subseteq \mathcal{SG}$, which we have not yet needed.

(5) $\mathcal{SG}$ is normed.

Proof. Suppose

$$P(x, w) \iff (Su)R(x, w * u) \iff (\forall \alpha)(\exists t)R(x, w * \pi(t))$$

with $R \in \Gamma$ and (upward) monotone in its second argument, and put

$$T(x, w) = \{\langle u_0, \ldots, u_{n-1} \rangle : \neg R(x, w * \langle u_0, \ldots, u_{n-1} \rangle)\}.$$  

This is a tree (by the monotonicity of $R$) and

$$P(x, w) \iff T(x, w) \text{ is well founded;}$$

so we set

$$\varphi(x, w) = |T(x, w)| \quad (P(x, w)),$$

and it suffices to prove that $\varphi$ is a $\mathcal{SG}$-norm. We note that it satisfies the following two conditions for $x, w$ such that $P(x, w)$:

(2F-3) $R(x, w) \implies \varphi(x, w) = 0$,  
(2F-4) $\neg R(x, w) \implies \varphi(x, w) = \sup\{\varphi(x, w * \langle s \rangle) + 1 : s \in \omega\}$. 

The first of these holds because $R(x, w) \implies T(x, w) = \emptyset$, by the assumed monotonicity of $R$, and the second by the definition of the canonical rank function on a tree and the easy fact that

$$P(x, w) \& \neg R(x, w) \implies (\forall s)P(x, w * \langle s \rangle).$$

Skipping the subscript to simplify notation, let

$$\{\varepsilon\}(x, y, u, v) = \{\varepsilon\}_{\mathcal{SG}}(x, y, u, v)$$

be defined by (1H-11) from a good universal set for $\mathcal{SG}$ at $\mathcal{X} \times \mathcal{Y} \times \omega \times \omega$, and by the Second Recursion Theorem fix a recursive $\varepsilon$ such that

$$\{\varepsilon\}(x, y, u, v) = \begin{cases} 1, & \text{if } R(x, u), \\ 0, & \text{ow. if } R(x, v), \\ 1, & \text{ow. if } (\forall s)(\exists t)\left(\{\varepsilon\}(x, y, u * \langle s \rangle, v * \langle t \rangle) = 1\right), \\ 0, & \text{ow. if } (\exists t)(\forall s)\left(\{\varepsilon\}(x, y, u * \langle s \rangle, v * \langle t \rangle) = 0\right), \\ \uparrow & \text{otherwise.} \end{cases}$$

We use the hypothesis that $\neg R \in \mathcal{SG}$ here to distinguish among the cases, but, other than that, it is obvious that the partial function defined by the cases construct is $\mathcal{SG}$-recursive.
Lemma. For all $x, y, u, v$ and the norm $\varphi$ defined above, if either $P(x, u)$ or $P(y, v)$, then

$$\varphi(x, u) \leq \varphi(y, v) \implies \{\varpi\}(x, y, u, v) = 1,$$

$$\varphi(y, v) < \varphi(x, u) \implies \{\varpi\}(x, y, u, v) = 0.$$  

Proof is by induction on $\min(\varphi(x, u), \varphi(y, v))$, simultaneously for both implications, and it is quite direct, using (2F-3), (2F-4). In the consideration of cases, we assume that $\varphi(x, u) = \infty$ if $\neg P(x, u)$.

Case 1, $\varphi(x, u) = 0$. This can only happen if $R(x, u)$, and in this case $\{\varpi\}(x, y, u, v) = 1$ by the definition.

Case 2, $\varphi(x, u) > 0$ & $\varphi(y, v) = 0$. Now $\neg R(x, u)$ & $R(y, v)$, and so $\{\varpi\}(x, y, u, v) = 0$ by the definition.

Case 3, neither of the two preliminary cases applies, $P(x, u)$ holds, and $\varphi(x, u) \leq \varphi(y, v)$. In this case, using (2F-4), we have that

$$(\forall s)(\exists t)[\varphi(x, u \ast (s)) \leq \varphi(y, v \ast (t))],$$

and the induction hypothesis with the definition give $\{\varpi\}(x, y, u, v) = 1$.

Case 4, neither of the two preliminary cases applies, $P(y, v)$ holds, and $\varphi(y, v) < \varphi(x, u)$. Same argument as in Case 3.

Finally we claim that

$$(x, u) \leq_{\varphi} (y, v) \iff P(x, u) \& \{\varpi\}(x, y, u, v) = 1,$$

$$(y, v) <_{\varphi} (x, u) \iff P(y, v) \& \{\varpi\}(x, y, u, v) = 0,$$

which will complete the proof since the relations on the right are in $S\Gamma$.

The implications from left-to-right follow immediately from the lemma, and those from right-to-left are easy, by contradiction. For example: if $P(x, u) \& \{\varpi\}(x, y, u, v) = 1$, then either $(x, u) \leq_{\varphi} (y, v)$ or $(y, v) <_{\varphi} (x, u)$ must hold, since $P(x, u)$, and the second cannot hold because it implies (by the lemma) that $\{\varpi\}(x, y, u, v) = 0$.

(6) $\Gamma \subseteq S\Gamma \subseteq S\neg S\Gamma$.

Proof. $\Gamma \subseteq S\Gamma$ by “vacuous quantification”, for any $\Gamma$; and we cannot have $\Gamma = S\Gamma$, because with the hypothesis this gives $\neg \Gamma \subseteq \Gamma$ which violates the $\omega$-parametrization assumption about $\Gamma$. The second inclusion follows immediately from the assumed $\neg \Gamma \subseteq S\Gamma$, which gives $\Gamma \subseteq \neg \neg S\Gamma$.

The point of the second inclusion becomes clear if we write it in the form

$$\neg (\neg \neg S\Gamma) \subseteq S(\neg \neg S\Gamma),$$

which then allows us to apply the main part of the theorem to $\neg \neg S\Gamma$ and iterate: starting with any $\Gamma$ which satisfies the hypotheses, we get an infinite sequence of distinct Spector pointclasses

$$S\Gamma \subseteq \neg S\neg S\Gamma = S\neg S\neg S\neg \Gamma = SS\neg \Gamma \subseteq \neg S\neg \neg S\neg \neg \Gamma = \neg \neg \neg S\neg \neg \neg \Gamma \cdot \cdot \cdot$$
This is exactly the sequence in (2F-2), for the important special case \( \Gamma = \Sigma^1_1 \), where, in fact, all these pointclasses are contained in \( \Delta^1_2 \) by Problem 1F.25.

**Additional problems for Section 2F**

**Problem 2F.4.** Prove the following elementary facts about a Spector pointclass \( \Gamma \):

1. Every relativization \( \Gamma(z) \) of \( \Gamma \) is Spector.
2. Every (total) \( \Delta^1_1 \)-recursive function \( f : X \to Y \) is \( \Gamma \)-recursive—and hence \( \Delta \)-recursive, with \( \Delta = \Gamma \cap \neg \Gamma \).
3. The boldface version \( \Gamma \) of \( \Gamma \) contains \( \Pi^1_1 \) and is closed under Borel substitutions and countable unions and intersections; it is \( \mathcal{N} \)-parametrized in the sense of Section 1F.8; and it is normed. (See 1F-1 for the definition of \( \Gamma \)).
4. The pointclass \( \neg \Gamma \) of complements of \( \Gamma \) sets is closed under the operation \( A \).

**Resolvents.** If \( \varphi : P \to \lambda \) is a regular norm, we call \( |\varphi| = \lambda \) the length of \( \varphi \), and for \( \xi < |\varphi| \), we set

\[
P^\xi = \{ x : \varphi(x) \leq \xi \}.
\]

This is the \( \xi \)'th resolvent of \( P \) relative to \( \varphi \), and clearly

\[
P = \bigcup_{\xi < |\varphi|} P^\xi.
\]

**Problem 2F.5.** Let \( \Gamma \) be a Spector pointclass and let \( \varphi : P \to |\varphi| \) be a regular \( \Gamma \)-norm on a pointset \( P \) in \( \Gamma \), where \( P \) is of type 0. Prove that for every \( \xi < |\varphi| \), the resolvent \( P^\xi \) is in \( \Delta \).

Similarly, if \( \varphi : P \to |\varphi| \) is a regular \( \Gamma \)-norm on some \( P \) in \( \Gamma \), then for every \( \xi < |\varphi| \), the resolvent \( P^\xi \) is in \( \Delta = \Gamma \cap \neg \Gamma \). In particular,

\[
P = \bigcup_{\xi < |\varphi|} P^\xi,
\]

with each \( P^\xi \) in \( \Delta \).

**The ordinals of a pointclass.** This result is more useful if we can get an estimate on the length \( |\varphi| \) of a \( \Gamma \)-norm. Given a pointclass \( \Gamma \) (which need not be a Spector pointclass), put

\[
\delta = \text{supremum}\{| < | : < \text{ is a prewllordering of } \omega, < \text{ in } \Delta}\},
\]

\[
\tilde{\delta} = \text{supremum}\{| < | : < \text{ is a prewllordering of } \mathcal{N}, < \text{ in } \Delta \}.
\]

Clearly, \( \delta \) is a countable ordinal, but \( \tilde{\delta} \) may well be uncountable—the only obvious bound is

\[
\tilde{\delta} < (2^\aleph_0)^+ = \text{least cardinal} > 2^\aleph_0.
\]
The results in the next problem hold under hypotheses much weaker than \( \Gamma \) being a Spector pointclass.

**Problem 2F.6.** Prove that if \( \Gamma \) is a Spector pointclass, then:

1. \( \delta = \sup \{ \leq | \leq \text{ is a wellordering on } \omega \text{ with } < \text{ in } \Delta \} \).
2. \( \delta \) is a limit ordinal and for every \( \Gamma \)-norm \( \varphi \) on a pointset \( P \) of type 0 in \( \Gamma \), \( |\varphi| \leq \delta \).
3. For every \( \Gamma \)-norm \( \varphi \) on a pointset \( P \) in \( \Gamma \), \( |\varphi| \leq \delta \).
4. \( \delta \) is an ordinal of cofinality \( > \omega \) and every pointset in \( \Gamma \) is the union of \( \delta \) sets in \( \Delta \).

Infer that every \( \Pi^1_1 \) pointset \( P \subseteq \mathcal{X} \) is the union of \( \aleph_1 \) Borel sets.

The last part of this problem is a classical result about \( \Pi^1_1 \) which we have not stated up until now.

The traditional notation for \( \delta \) and \( \delta \) when \( \Gamma \) is \( \Sigma^1_n \) (or \( \Pi^1_n \)), is \( \delta^1_n \) and \( \delta^1_n \). Similarly, for the relativized class \( \Sigma^1_n(z) \) (or \( \Pi^1_n(z) \)), its ordinal is \( \delta^1_n(z) \).

From the Kunen-Martin Theorem (2G.2 in DST) which we have not covered in these notes,\[ \delta^1_1 = \aleph_1, \quad \delta^1_2 \leq \aleph_2. \]

This is about all that can be proved about these ordinals in classical set theory, except for 2D.5, that\[ \delta^1_1 = \omega_{CK}^1 = \text{least nonrecursive ordinal}. \]

The next problem gives an interesting generalization of the Boundedness Theorem 2D.4 to arbitrary \( \Pi^1_1 \)-norms.

**2F.7. The Bounded Norm Theorem for \( \Pi^1_1 \).** Suppose \( P \subseteq \mathcal{X} \) is \( \Pi^1_1 \) and \( \varphi : P \to \text{Ordinals} \) is a regular, \( \Pi^1_1 \)-norm on \( P \). Prove that \( P \) is Borel if and only if \( |\varphi| < \aleph_1 \).

**HINT:** Use Problem 2F.5.

This result is often useful in conjunction with the following, very general formulation of the Boundedness Theorem 2D.4:

**2F.8. The Covering Lemma.** Let \( \Gamma \) be a Spector pointclass, let \( \varphi \) be a regular \( \Gamma \)-norm on some \( P \subseteq \mathcal{X} \) in \( \Gamma \setminus \Delta \), let \( Q \) be in \( \neg \Gamma \) and assume that either \( \mathcal{X} \) is of type 0 or \( \Gamma \) is closed under \( \forall \). Prove that\[ Q \subseteq P \implies \text{for some } \xi < |\varphi|, \, Q \subseteq P^\xi = \{ x \in P : \varphi(x) \leq \xi \}. \]

Similarly, let \( \Gamma \) be a Spector pointclass closed under \( \forall \), let \( \varphi \) be a regular \( \Gamma \)-norm on some \( P \subseteq \mathcal{X} \) in \( \Gamma \setminus \Delta \) and let \( Q \) be in \( \neg \Gamma \). Prove again that\[ Q \subseteq P \implies \text{for some } \xi < |\varphi|, \, Q \subseteq P^\xi. \]

In particular, if \( \Gamma \) is a Spector pointclass closed under \( \forall \), \( G \subseteq \mathcal{N} \times \mathcal{X} \) is a good universal set for \( \Gamma \) at \( \mathcal{X} \) and \( \varphi : G \to \text{Ordinals} \) is a \( \Gamma \)-norm on \( G \),
then a pointset $P \subseteq \mathcal{X}$ is in $\Delta$ if and only if there are $\varepsilon, \varepsilon_0 \in \mathcal{N}$ and some $x_0 \in \mathcal{X}$ such that $G(\varepsilon, x_0)$ and

$$P = \{ x \in \mathcal{X} : G(\varepsilon, x) & \varphi(\varepsilon, x) \leq \varphi(\varepsilon_0, x_0) \}$$

**Hint:** See the proof of 2D.4.

The next result is a simple but interesting extension of the $\Delta$-Selection Principle.

**Problem 2F.9** (The Principle of $\Gamma$-Dependent Choices). Let $\Gamma$ be a Spector pointclass, suppose $P \subseteq \mathcal{X} \times \mathcal{Y} \times \mathcal{Y}$ is in $\Gamma$, $\mathcal{Y}$ is of type 0, and

$$(\forall x)(\forall y)(\exists y') P(x, y, y').$$

Prove that for each fixed $y_0 \in \mathcal{Y}$, there is a function $f : \mathcal{X} \times \omega \rightarrow \mathcal{Y}$ which is $\Delta$-recursive and such that

$$f(x, 0) = y_0,$$

$$(\forall n)P(x, f(x, n), f(x, n + 1)).$$

Another simple but interesting application of the $\Delta$-Selection Principle comes up in the next result. This is essentially a representation theorem for $\Delta$ sets which happen to be open.

**Problem 2F.10.** Let $\Gamma$ be a Spector pointclass closed under $\forall^\mathcal{N}$, let $P \subseteq \mathcal{X}$ be a pointset in $\Delta$ which is open. Prove that there is some $\varepsilon$ in $\Delta$ such that

$$P = \bigcup_n N(\mathcal{X}, \varepsilon(n))$$

and for each $n$,

$$\overline{N}(\mathcal{X}, \varepsilon(n)) = \text{the closure of } N(\mathcal{X}, s) \subseteq P.$$
In particular, under these hypotheses, $P$ is semirecursive in some $\varepsilon \in \Delta$.

HINT: Check that the set

$$Q(x, s) \iff P(x) \& x \in N_s \& (\forall y)[y \in N_s \implies P(y)]$$

is in $\Gamma$ and $(\forall x \in P)(\exists s)Q(x, s)$, so by 2E.5, there is a $\Delta$-recursive function $f : \mathcal{X} \to \omega$ such that $(\forall x)Q(x, f(x))$. Now check that

$$A = \{s : (\exists x \in P)[f(x) = s]\}$$

is in $\neg \Gamma$ and use the separation property for $\neg \Gamma$ to separate it from a suitable $B$.

The last problem is an interesting effective generalization of the "classical" fact that well founded $\Sigma^1_1$ relations have countable rank whose proof uses Kleene’s recursion theorem.

**Problem 2F.11.** Let $\Gamma$ be a Spector pointclass closed under $\forall \mathcal{N}$, suppose $\prec$ is a (strict) well founded relation on a space $\mathcal{X}$ which is in $\neg \Gamma$, let $G \subseteq \mathcal{N} \times \mathcal{X}$ be a a good universal set for $\Gamma$ at $\mathcal{X}$, and let $\varphi : G \to \text{Ordinals}$ be a regular $\Gamma$-norm on $G$. Then there exists a recursive function

$$f : \mathcal{X} \to \mathcal{N} \times \mathcal{X}$$

which is order-preserving from $\prec$ into $\leq_\varphi$, i.e.,

$$x \prec y \implies f(x), f(y) \in G \& \varphi(f(x)) < \varphi(f(y)).$$

Infer that

(1) $| \prec | < |\varphi|.$
(2) If $\mathcal{X}$ is uncountable, then $|\varphi| = \delta$.

HINT: Choose a recursive $\pi \in \mathcal{N}$ such that

$$G(\pi, x) \iff (\forall y)[y \prec x \implies (\pi, y) <^\varphi (\pi, x)]$$

and check that for every $x,$

$$G(\pi, x) \& (\forall y \prec x)[(\pi, y) <^\varphi (\pi, x)].$$

The required function for the main part is $f(x) = (\pi, x).$ For the rest, relativize, appeal to the Covering Lemma 2F.8, and for (2) use Problem 1G.6.

### 2G. The Parametrization Theorem for $\Delta \cap \mathcal{X}$

The theorem in the section title is a fundamental result of the effective theory which yields easily intuitive and strong versions of several classical results about $\Pi^1_1$.

Recall that a partial function $f : \mathcal{X} \to \mathcal{Y}$ is $\Gamma$-recursive if it is potentially $\Gamma$-recursive and $\text{Domain}(f)$ is in $\Gamma$. These partial functions are very useful when $\Gamma$ is a Spector pointclass. We collect for reference some of their
elementary properties, most of them already established under weaker hypotheses on $\Gamma$.

**Problem 2G.1.** Suppose $\Gamma$ is a Spector pointclass and $f : \mathcal{X} \rightarrow \mathcal{Y}$. Prove that the following are equivalent:

1. $f$ is $\Gamma$-recursive.
2. The nbhd diagram $\{(x, s) : f(x) \downarrow \land f(x) \in N_s\}$ of $f$ is in $\Gamma$.
3. The graph $\{(x, y) : f(x) \downarrow \land f(x) = y\}$ of $f$ is in $\Gamma$.

**Problem 2G.2.** Suppose $\Gamma$ is a Spector pointclass and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a $\Gamma$-recursive partial function. Prove the following:

1. The relations $P(x) \iff f(x) \downarrow$, $Q(x, s) \iff f(x) \downarrow \land f(x) \notin N_s$, $R(x, y) \iff f(x) = y$ ($\iff f(x) \downarrow \land f(x) = y$), $S(x, y) \iff f(x) \neq y$ ($\iff f(x) \downarrow \land f(x) \neq y$) are all in $\Gamma$.
2. If $Q \subseteq \mathcal{Y}$ is in $\Gamma$ and
   
   $$R(x) \iff f(x) \downarrow \land Q(f(x))$$

   then $R$ is in $\Gamma$.
3. For every $x \in \mathcal{X}$, if $f(x) \downarrow$, then $f(x) \in \Delta(x)$, i.e., $f(x)$ is $\Delta(x)$-recursive. In fact, for any $z \in \mathcal{Z}$, if $x \in \Delta(z)$ and $f(x) \downarrow$, then $f(x) \in \Delta(z)$.
4. If $A \in \Delta$ and $f(x) \downarrow$ for every $x \in A$, then there is a $\Delta$-recursive total function $g : \mathcal{X} \rightarrow \mathcal{Y}$ such that $f \upharpoonright A = g \upharpoonright A$.

**Hint:** Some of the proofs require appealing to the Substitution Property.

We have been using the abbreviation

$$y \in \Delta \iff y \text{ is } \Delta\text{-recursive} \quad (\iff \mathcal{U}(y) \text{ is in } \Delta),$$

and similarly for $\Delta(x)$. It is also convenient for any pointclass $\Lambda$ to put

$$\Lambda \cap \mathcal{X} = \{x \in \mathcal{X} : x \text{ is } \Lambda\text{-recursive}\}.$$

For example $\Sigma^0_1 \cap \mathbb{R} = \Delta^0_1 \cap \mathbb{R} = \text{the set of recursive real numbers}$.

The following basic result of the section is due to Kleene for $\Pi^1_1 \cap \mathcal{N}$:

**Theorem 2G.3** (The Parametrization Theorem for $\Delta(x) \cap \mathcal{Y}$). Let $\Gamma$ be a Spector pointclass.

1. For every product space $\mathcal{Y}$, there is a $\Gamma$-recursive partial function

   $$d : \omega \rightarrow \mathcal{Y}$$

   which enumerates $\Delta \cap \mathcal{Y}$, i.e.,

   $$y \in \Delta \iff \text{for some } i, d(i) \downarrow \land d(i) = y.$$
(2) Similarly, for any $\mathcal{X}, \mathcal{Y}$ there is a $\Gamma$-recursive partial function
\[ d : \omega \times \mathcal{X} \rightarrow \mathcal{Y} \]
which enumerates $\Delta(x) \cap \mathcal{Y}$, i.e.,
\[ y \in \Delta(x) \iff \text{for some } i, d(i, x) \downarrow \& d(i, x) = y. \]

(3) It follows that $\Delta \cap \mathcal{Y} \in \Gamma$ and $\Delta(x) \cap \mathcal{Y} \in \Gamma(x)$.

**Proof.** Take first the case $\mathcal{Y} = \mathcal{N}$. We prove (2), (1) being simpler.

Choose a set $G \subseteq \omega \times \mathcal{X} \times \omega \times \omega$ which is universal for $\Gamma$ at $\mathcal{X} \times \omega \times \omega$, and let $G^* \subseteq G$ be in $\Gamma$ and uniformize $G$ by the Easy Uniformization Theorem 2E.4. Here we are thinking of $G$ as a subset of $\omega \times \mathcal{X} \times \omega \times \omega$, i.e., we uniformize only on the last variable. Now put
\[ d(i, x) \downarrow \iff (\forall n)(\exists m) G^*(i, x, n, m) \]
and if $d(i, x) \downarrow$, let
\[ d(i, x) = \alpha \]
where for all $n, m$
\[ \alpha(n) = m \iff G^*(i, x, n, m), \]
so that $d$ is (easily) $\Gamma$-recursive. From this it follows that each $d(i, x)$ is in $\Delta(x)$ by 2G.2. Conversely, if $\alpha \in \Delta(x)$, choose $i$ so that
\[ \alpha(n) = m \iff G(i, x, n, m) \]
so that
\[ \alpha(n) = m \iff G^*(i, x, n, m) \]
and hence $d(i, x) \downarrow \& d(i, x) = \alpha$.

For arbitrary $\mathcal{Y}$ we appeal as usual to Theorem 1F.12, which supplies us a recursive $\pi : \mathcal{N} \rightarrow \mathcal{Y}$, a $\Pi^0_1$ set $A \subseteq \mathcal{N}$ on which $\pi$ is injective and a $\Delta^1_1$-recursive (and hence $\Gamma$-recursive) inverse $\pi^{-1} : \mathcal{X} \rightarrow A$ of $\pi$. We set
\[ d^*(i, x) = \pi(d(i, x)) \quad (i \in \omega, x \in \mathcal{X}). \]
This is $\Gamma$-recursive, every value $\pi(d(i, x))$ of it is in $\Delta(x)$ by Problem 2G.2, and for every $y \in \Delta(x)$, $\alpha = \pi^{-1}(y)$ is a $\Delta(x)$-recursive point in $A$, so $\alpha = d(i, x)$ for some $i$ and then $d^*(i, x) = y$.

(3) follows immediately, since
\[ y \in \Delta(x) \iff (\exists i \in \omega)[d^*(i, x) = y]. \]

The *upper classification* of $\Delta \cap \mathcal{Y}$ in (3) is best possible for perfect $\mathcal{Y}$, cf. Theorem 2G.17.

One, immediate corollary is the extension of the $\Delta$-selection result in Problem 2E.5 to a principle of “restricted selection” from arbitrary spaces:
Corollary 2G.4 (The Strong $\Delta$-Selection Principle). If $P \subseteq X \times Y$ is a pointset in some Spector pointclass $\Gamma$, then there exists a $\Gamma$-recursive partial function $f : X \to Y$ such that

1. $f(x) | \iff (\exists y \in \Delta(x)) P(x, y)$,
2. $(\exists y \in \Delta(x)) P(x, y) \iff P(x, f(x))$.

Proof. Put $Q(x, i) \iff d(i, x) \downarrow \& P(x, d(i, x))$ where $d$ parametrizes $\Delta(x) \cap Y$ by 2G.3 and let $Q^* \subseteq Q$ uniformize $Q$ in $\Gamma$ by 2E.4. Now $Q^*$ is the graph of a $\Gamma$-recursive partial function $g : X \to \omega$ by 2G.1 and the partial function we need is $f(x) = d(g(x), x)$. $\dashv$

The next corollary of 2G.3 is specific for $\Pi^1_1$:

2G.5. The Effective Perfect Set Theorem. If $P \subseteq X$ is $\Sigma^1_1$ and has at least one non-$\Delta^1_1$ member, then $P$ has a non-empty perfect subset (Harrison for $\mathcal{N}$).

Proof. If $P$ is $\Sigma^1_1$ and has at least one member not in $\Delta^1_1$, then the pointset $P \setminus (\Delta^1_1 \cap X)$ is non-empty, it is also $\Sigma^1_1$ by (3) Theorem 2G.3 applied to $\Pi^1_1$, and it has no $\Delta^1_1$ members; and so it has a non-empty perfect subset by Problem 2A.7. $\dashv$

The relativized version of this theorem says that if $P$ is $\Sigma^1_1(\epsilon)$ and has a member not in $\Delta^1_1(\epsilon)$, it then has a non-empty perfect subset. It strengthens Suslin’s classical Perfect Set Theorem 2A.5 and provides an “effective explanation” for it: if $P$ is uncountable, it must have some member which is not $\Delta^1_1$ in its $\Sigma^1_1$-code, and this suffices to insure that $P$ has a non-empty perfect subset.

The next corollary of Theorem 2G.3 is also due to Kleene for $\Pi^1_1$ on $\mathcal{N}$:

2G.6. The Theorem on Restricted Quantification. Let $\Gamma$ be a Spector pointclass, assume that $Q \subseteq X \times Y$ is in $\Gamma$ and put

$P(x) \iff (\exists y \in \Delta) Q(x, y)$.

Then $P$ is in $\Gamma$.

Similarly, if $Q \subseteq X \times Z \times Y$ is in $\Gamma$ and

$P(x, z) \iff (\exists y \in \Delta(z)) Q(x, z, y)$

then $P$ is in $\Gamma$.

Proof. Taking the second case,

$P(x, z) \iff (\exists i) \{d(i, z) \downarrow \& Q(x, z, d(i, z))\}$,

so $P$ is in $\Gamma$ by (ii) of 2G.2. $\dashv$

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2G. The Parametrization Theorem for $\Delta \cap \mathcal{X}$

There are many applications of this fact, including the following very powerful method for uniformizing Borel sets by Borel sets when this is possible:

2G.7. The $\Delta$-Uniformization Criterion. Let $\Gamma$ be a Spector pointclass closed under $\forall^N$, let $P \subseteq \mathcal{X} \times \mathcal{Y}$ be in $\Delta$ and assume that each section $P_x = \{y : P(x,y)\}$ is either $\emptyset$ or contains some points in $\Delta(x) \cap \mathcal{Y}$, i.e.,

$(\exists y)P(x,y) \iff (\exists y \in \Delta(x))P(x,y)$.

Then the projection $\exists^\mathcal{Y}P$ is in $\Delta$ and $P$ can be uniformized by some $P^*$ in $\Delta$.

Conversely, if $P \subseteq \mathcal{X} \times \mathcal{Y}$ is in $\Delta$ and can be uniformized by some $P^*$ in $\Delta$, then every non-empty section $P_x$ has some point in $\Delta(x)$.

Proof. Assume $(\ast)$ and let $Q = \exists^\mathcal{Y}P$, i.e.,

\[
Q(x) \iff (\exists y)P(x, y) \iff (\exists y \in \Delta(x))P(x, y).
\]

Clearly $Q$ is in $\Delta$ by closure of $\neg \Gamma$ under $\forall^N$ and 2G.6. Now put

\[
R(x, i) \iff P(x, d(i, x))
\]

where $d$ parametrizes $\Delta(x) \cap \mathcal{Y}$ by Theorem 2G.3. By the $\Delta$-Selection Principle 2E.5, since $(\forall x \in Q)(\exists i)R(x, i)$, there is a $\Delta$-recursive $g : \mathcal{X} \to \omega$ such that $(\forall x \in Q)R(x, g(x))$. Put

\[
P^*(x, y) \iff Q(x) \& d(g(x), x) = y.
\]

It is immediate that $P^*$ uniformizes $P$ and it is $\Delta$-recursive follows by 2G.2, since

\[
P^*(x, y) \iff Q(x) \& (\exists i)[d(i, x) \downarrow \& d(i, x) = y \& g(x) = i],
\]

\[
\neg P^*(x, y) \iff \neg Q(x) \vee (\exists i)[d(i, x) \downarrow \& d(i, x) \neq y \& g(x) = i].
\]

For the converse, suppose $P^* \subseteq P$ is in $\Delta$ and uniformizes $P$ and assume that $(\exists y)P(x,y)$; then there is a unique $y^*$ such that $P^*(x,y^*)$ and

\[
y^* \in N_s \iff (\exists y)[P^*(x,y) \& y \in N_s]
\]

\[
\iff (\forall y)[P^*(x,y) \implies y \in N_s],
\]

so $y^* \in \Delta(x)$.

Several classical uniformization results for Borel sets follow immediately from this theorem, cf. Problems 2G.14, 2G.15. One of the most important is the following, whose proof appeals also to the Effective Perfect Set Theorem 2G.5:
2G.8. Borel sets with countable sections. If \( P \subseteq X \times Y \) is \( \Delta^1_1 \) and every section \( P_x = \{ y : P(x, y) \} \) is countable, then \( P \) can be uniformized by some \( P^* \) in \( \Delta^1_1 \).

It follows that every Borel set with countable sections can be uniformized by a Borel set (Lusin, Novikov).

Proof. If \( P \in \Delta^1_1 \), then every non-empty section \( P_x \) is \( \Delta^1_1 \) and since it is countable it must be contained in \( \Delta^1_1 \cap Y \) by the relativized version of the Effective Perfect Set Theorem 2G.5. By 2G.6 then, the projection \( \exists^YP \) is in \( \Delta^1_1 \) and it can be uniformized by some \( P^* \in \Delta^1_1 \).

For the classical, second claim, we appeal to the relativized version of the first claim and Suslin’s Theorem: If \( P \) is Borel, it is then \( \Delta^1_1(\epsilon) \) for some \( \epsilon \in N \); and so it can be uniformized by some \( P^* \in \Delta^1_1(\epsilon) \); which is Borel by Suslin’s Theorem.

There are many not-so-easy extensions of this fact, including the Arsenin-Kunugui Theorem: If \( P \subseteq X \times Y \) is Borel and every section \( P_x \) is \( \sigma \)-compact, then \( P \) can be uniformized by a Borel set, cf. Section 4F of DST.

We end this section with three more effective refinements and generalizations of basic classical facts about \( \Pi^1_1 \).

Theorem 2G.9 (Injective images of \( \Delta \)-sets). Let \( \Gamma \) be a Spector point-class closed under \( \forall^N \), suppose \( P \subseteq X \) is in \( \Delta \) and assume that \( f : X \to Y \) is \( \Delta \)-recursive and one-to-one on \( P \); it follows that the image \( f[P] \) is in \( \Delta \) and that there is a \( \Delta \)-recursive function \( g : Y \to X \) which agrees with the inverse function \( f^{-1} \) on \( f[P] \).

Proof. If \( P(x) \& f(x) = y \), then \( x \) is the unique point in \( P \) whose image is \( y \); hence

\[
s \in U(x) \iff x \in N_s \iff (\exists x')[f(x') = y \& P(x') \& x' \in N_s] \iff (\forall x')[f(x') \neq y \lor \neg P(x') \lor x' \in N_s]
\]

and \( U(x) \) is in \( \Delta(y) \), i.e., \( x \in \Delta(y) \). Hence

\[
y \in f[P] \iff (\exists x)[P(x) \& y = f(x)] \iff (\exists x \in \Delta(y))[P(x) \& y = f(x)]
\]

and \( f[P] \) is in \( \Delta \) by closure of \( \neg \Gamma \) under \( \exists^N \) and 2G.6.

To compute the inverse function, notice that

\[
(\forall y \in f[P])[(\exists x \in \Delta(y))[f(x) = y]]
\]

and apply the strong \( \Delta \)-Selection Principle, 2G.4.

\[\Box\]
2G. The Parametrization Theorem for $\Delta \cap X$

This is a lightface version of a basic classical result about Borel measurable functions which follows easily from its relativized version; here $f : \mathcal{X} \to \mathcal{Y}$ is $\Delta$-measurable if every inverse image $f^{-1}[N(Y, s)]$ is a $\Delta$-subset of $\mathcal{X}$.

**Corollary 2G.10.** Let $\Gamma$ be a Spector pointclass closed under $\forall^\mathcal{Y}$, let $P \subseteq \mathcal{X}$ be in $\Delta$ and assume that $f : \mathcal{X} \to \mathcal{Y}$ is $\Delta$-measurable and one-to-one on $P$. It follows that $f[P]$ is in $\Delta$ and there is a $\Delta$-measurable function $g : \mathcal{Y} \to \mathcal{X}$ which agrees with the inverse $f^{-1}$ on $f[P]$.\

**Proof.** If $P$ is in $\Delta$, then $P$ is in $\Gamma(\varepsilon_0)$ and in $\neg\Gamma(\varepsilon_1)$ for some $\varepsilon_0$, $\varepsilon_1$ in $\mathcal{N}$, and so $P$ is in $\Delta(\varepsilon)$ for some $\varepsilon$, say with $(\varepsilon)_0 = \varepsilon_0$, $(\varepsilon)_1 = \varepsilon_1$. Similarly, if $f$ is $\Delta$-measurable, then $f$ is $\Delta(\varepsilon')$-recursive for any $\varepsilon'$ such that $\{(x, s) : f(x) \in N_s\}$ is in $\Delta(\varepsilon')$. Thus we can find some $\varepsilon^*$ such that $P$ is in $\Delta(\varepsilon^*)$ and $f$ is $\Delta(\varepsilon^*)$-recursive and apply 2G.9 to $\Gamma^* = \Gamma(\varepsilon^*)$; it follows that $f[P]$ is in $\Delta(\varepsilon^*) \subseteq \Delta$ and similarly for the inverse. ⊣

It is also worth putting down for the record the characterizations of HYP and $\mathcal{B}$ which follow from 2G.9 and 2D.6.

**2G.11. Borel sets as injective images of closed sets.** A set $P \subseteq \mathcal{X}$ is $\Delta^1_1$ if and only if $P$ is the recursive, injective image of some $\Pi^0_1$ set $A \subseteq \mathcal{N}$; and $P$ is Borel if and only if it is the continuous, injective image of some closed $A \subseteq \mathcal{N}$.

This elegant characterization of $\mathcal{B}$ is sometimes said to be Luzin’s favorite.

**Additional problems for Section 2G**

**Problem 2G.12.** Prove that there is a $\Pi^0_1$ set $A \subseteq \mathcal{N}$, such that $A \neq \emptyset$ but $A$ has no $\Delta^1_1$-recursive member; similarly, for each $x$, there is a $\Pi^0_1(x)$ set $A \subseteq \mathcal{N}$, $A \neq \emptyset$, such that $A$ has no $\Delta^1_1(x)$-recursive member (Kleene).\n
Infer that not every $\Pi^0_1$ set $A \subseteq \mathcal{N}$ is a recursive image of $\mathcal{N}$.

**HINT:** Use the Restricted Quantification Theorem 2G.6

**Problem 2G.13.** Prove that there is a $\Pi^0_1$ set $P \subseteq \mathcal{N} \times \mathcal{N}$ which cannot be uniformized by any $\Sigma^1_1$ set.

**Problem 2G.14.** Let $\Gamma$ be a Spector pointclass closed under $\forall^\mathcal{Y}$, let $P \subseteq \mathcal{X} \times \mathcal{Y}$ be in $\Delta$ and assume that for each $x$, the section $P_x$ has at least one isolated point. Prove that $P$ can be uniformized by some $P^*$ in $\Delta$. Infer the same result for $P$ in $\Delta$, with $P^*$ in $\Delta$.

**Problem 2G.15.** Prove that if $P \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is a *connected* Borel set, then the projection $Q = \{x \in \mathbb{R}^n : (\exists y \in \mathbb{R}^m) P(x, y)\}$
is Borel and $P$ can be uniformized by a Borel set.

**Problem 2G.16.** Let $\Gamma$ be a Spector pointclass and let $d: \omega \to X$ be a $\Gamma$-recursive partial function which parametrizes $\Delta \cap X$ by Theorem 2G.3. Prove that there is a $\Gamma$-recursive partial function $c: X \to \omega$ such that

$$c(x) | \iff x \in \Delta \text{ and } (\forall x \in \Delta)[d(c(x)) = x].$$

**HINT:** Use the Easy Uniformization Theorem 2E.4 or 2G.4.

**2G.17. Lower classification of $\Delta^1_1, \Delta^1_2$ at $N$.** If $\Gamma$ is a Spector pointclass closed under either $\forall N$ or $\exists N$, then for every perfect space $X$, the set $\Delta \cap X$ is not in $\neg \Gamma$.

In particular, for perfect $X$, $\Delta^1_1 \cap X$ is not $\Sigma^1_1$ and $\Delta^1_2 \cap X$ is not $\Pi^1_2$.

**Proof.** It is enough to prove the result for $N$, since every perfect $X$ is $\Delta^1_1$-isomorphic—and hence $\Delta$-isomorphic—with $N$.

**Case 1.** $\Gamma$ is closed under $\forall N$. Using the function $c$ of Problem 2G.16, let

$$j \in J \iff (\exists \alpha)[\alpha \in \Delta \& c(\alpha) = j]$$

$$\iff (\exists i)[d(i) \downarrow \& c(d(i)) = j],$$

so $J$ is in $\Gamma$. Also

$$j \notin J \iff (\forall \alpha)[\alpha \notin \Delta \lor [c(\alpha) \& c(\alpha) \neq j]],$$

so that if $\Delta \cap N$ were in $\Delta$, then $J$ would be in $\Delta$, and then the function

$$\alpha(j) = \begin{cases} d(j)(j) + 1, & \text{if } j \in J, \\ 0, & \text{if } j \notin J, \end{cases}$$

would be in $\Delta$ and different from all $d(i)$.

**Case 2.** $\Gamma$ is closed under $\exists N$. Let

$$i \in I \iff d(i) \downarrow$$

and let $\varphi$ be a $\Gamma$-norm on $I$. Put

$$P(\alpha) \iff (\forall i)[\alpha(i) \leq 1] \& (\forall i)[\alpha(i) = 0 \implies i \in I]$$

$$\& (\forall i)(\forall j)[(\alpha(j) = 0 \& i <^\varphi j) \implies \alpha(i) = 0].$$

Clearly $P$ is in $\Gamma$ and

$$P(\alpha) \iff (\forall i)[\alpha(i) \leq 1]$$

$$\& \left[ \{i : \alpha(i) = 0\} = I \lor (\exists j) [j \in I \& \{i : \alpha(i) = 0\} = \{i : \varphi(i) < \varphi(j)\}] \right].$$

Since $I \notin \Delta$, or else we get a contradiction as before, we have

$$i \notin I \iff (\exists \alpha)[\alpha \notin \Delta \& P(\alpha) \& \alpha(i) \neq 0]$$

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which proves $I \in \Delta$ and yields a contradiction. ⊣

The next result is a converse to 2G.6 for the case $\Gamma = \Pi^1_1$. We start with an easy problem and a lemma.

**Problem 2G.18** ($\Delta$-transitivity). Suppose $\Gamma$ is a Spector pointclass and $x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{Z}$. Prove that

$$ (y \in \Delta(x) \& z \in \Delta(x, y)) \implies z \in \Delta(x). $$

**Lemma 2G.19.** There is a $\Pi^0_1$ relation $S(\alpha, \beta, \gamma)$, such that whenever $\beta \in \text{WO}$ and $\alpha \in \text{LO}$,

$$ \alpha \in \text{WO} \& |\alpha| \leq |\beta| \iff (\exists \gamma)S(\alpha, \beta, \gamma) \iff (\exists \gamma \in \Delta^1_1(\alpha, \beta))S(\alpha, \beta, \gamma). $$

**Proof.** The notation is that of 2D. As in the proof of 2D.2, put

$$ Q(\alpha, \beta, \gamma) \iff \gamma \text{ maps } \leq_\alpha \text{ onto an initial segment of } \leq_\beta \text{ in an order-preserving fashion and } \gamma = 0 \text{ outside the field of } \leq_\alpha, $$

where we allow “initial segment” to include all of $\leq_\alpha$. It is clear that $Q$ is $\Pi^0_2$, so we can put it in normal form

$$ Q(\alpha, \beta, \gamma) \iff (\forall n)(\exists m)R(\alpha, \beta, \gamma, n, m) $$

with $R$ recursive. Put further,

$$ Q^*(\alpha, \beta, \gamma, \delta) \iff (\forall n)R(\alpha, \beta, \gamma, n, \delta(n)) $$

and notice that

$$ Q(\alpha, \beta, \gamma) \iff (\exists \delta)Q^*(\alpha, \beta, \gamma, \delta) \iff (\exists \delta \in \Delta^1_1(\alpha, \beta, \gamma))Q^*(\alpha, \beta, \gamma, \delta), $$

since if $(\exists \delta)Q^*(\alpha, \beta, \gamma, \delta)$, we can choose

$$ \delta(n) = \text{ least } mR(\alpha, \beta, \gamma, n, m), $$

and this $\delta$ is clearly in $\Delta^1_1(\alpha, \beta, \gamma)$. Moreover, if $\beta \in \text{WO}$ and $\alpha \in \text{LO}$, then

$$ \alpha \in \text{WO} \& |\alpha| \leq |\beta| \implies \text{ there is a unique } \gamma \text{ such that } Q(\alpha, \beta, \gamma) \implies (\exists \gamma \in \Delta^1_1(\alpha, \beta))Q(\alpha, \beta, \gamma), $$

since the unique $\gamma$ such that $Q(\alpha, \beta, \gamma)$ is surely in $\Delta^1_1(\alpha, \beta)$. Thus we have, for $\beta \in \text{WO}$ and $\alpha \in \text{LO}$,

$$ \alpha \in \text{WO} \& |\alpha| \leq |\beta| \iff (\exists \gamma)(\exists \delta)Q^*(\alpha, \beta, \gamma, \delta) \iff (\exists \gamma \in \Delta^1_1(\alpha, \beta))(\exists \delta \in \Delta^1_1(\alpha, \beta, \gamma))Q^*(\alpha, \beta, \gamma, \delta) $$

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2. Structure theory for pointclasses

with $Q^*$ in $\Pi_1^0$. By Problem 2G.18, $\delta \in \Delta_1^1(\alpha, \beta)$ in this equivalence, which then yields

$\alpha \in \text{WO} & |\alpha| \leq |\beta| \iff (\exists \gamma)(Q^*(\alpha, \beta, \gamma, \delta))$

$\implies (\exists \gamma \in \Delta_1^1(\alpha, \beta))(\exists \delta \in \Delta_1^1(\alpha, \beta))Q^*(\alpha, \beta, \gamma, \delta)$.

Finally, take $S(\alpha, \beta, \gamma) \iff Q^*(\alpha, \beta, (\gamma_0, \gamma_1))$ and verify easily that the lemma holds with this $S$. ⊣

2G.20. The Spector-Gandy Theorem. For every $\Pi_1^1$ set $P \subseteq X$, there is a $\Pi_0^1$ set $R \subseteq X \times N$ such that

$P(x) \iff (\exists \alpha \in \Delta_1^1(x))R(x, \alpha)$.

Proof. Suppose first that $P$ is $\Delta_1^1$. By 2D.6 there is a $\Pi_0^1$ set $A \subseteq N$ and a recursive $\pi : N \rightarrow X$ which is injective on $A$ and $\pi[A] = P$. Hence,

$P(x) \iff (\exists \alpha)[\alpha \in A \& \pi(\alpha) = x]$

$\iff (\exists \alpha \in \Delta_1^1(x))\{\alpha \in A \& \pi(\alpha) = x\}$,

where the second equivalence holds because if $\pi(\alpha) = x$ and $\alpha \in A$, then $\alpha$ is the unique point in $N$ satisfying these conditions and it is easily $\Delta_1^1(x)$. Thus for a $\Delta_1^1$ set $P$ we have the stronger representation:

$P(x) \iff (\exists \alpha \in \Delta_1^1(x))R(x, \alpha)$

$\iff (\exists \alpha \in A \& \pi(\alpha) = x)R(x, \alpha)$

$\iff (\exists \alpha \in \Delta_1^1(x))R(x, \alpha)$,

where $R$ is some $\Pi_1^0$ set.

To prove the result for $\Pi_1^1$ pointsets of type 0 or 1 we will use (3) of Theorem 2G.3, by which $\{(\alpha, x) : \alpha \in \Delta_1^1(x)\} \in \Pi_1^1$. Hence, for $X$ of type 0 or 1, there is a recursive function $g : N \times X \rightarrow N$ such that for each $\alpha, x$, $g(\alpha, x) \in \text{LO}$ and

$\alpha \in \Delta_1^1(x) \iff g(\alpha, x) \in \text{WO}$.

For each $x$, let

$\omega_1^x = \text{supremum}\{|\beta| : \beta \text{ is recursive in } x, \beta \in \text{WO}\}$;

by the relativized version of 2D.4, easily, for each $x$

$\supremum\{|g(\alpha, x) : \alpha \in \Delta_1^1(x)\} = \omega_1^x$,

or else $\{\alpha : \alpha \in \Delta_1^1(x)\}$ would be $\Delta_1^1(x)$, contradicting 2G.17.

Suppose now $P \subseteq X$ is $\Pi_1^1$, with $X$ of type 0 or 1, so there is a recursive $f : X \rightarrow N$ such that for each $x$, $f(x) \in \text{LO}$ and

$P(x) \iff f(x) \in \text{WO}$.
From (2G-1) we get immediately

\[ P(x) \iff (\exists \alpha \in \Delta^1_1(x))[f(x) \in \text{WO} \land |f(x)| \leq |g(\alpha, x)|], \]

since for each \( x \), \( f(x) \) is recursive in \( x \). We claim

\[ (2G-2) \quad P(x) \iff (\exists \alpha \in \Delta^1_1(x))(\exists \gamma \in \Delta^1_1(x))S(f(x), g(\alpha, x), \gamma), \]

where \( S \) is the \( \Pi^0_1 \) set of the lemma.

To prove direction \( (\implies) \) of (2G-2), assume \( P(x) \); then \( f(x) \in \text{WO} \) and \( f(x) \) is recursive in \( x \), so by (2G-1) there is some \( \alpha \in \Delta^1_1(x) \) such that \( |f(x)| \leq |g(\alpha, x)| \). By the lemma then, there is some \( \gamma \in \Delta^1_1(f(x), g(\alpha, x)) \)

\[ S(f(x), g(\alpha, x), \gamma); \]

but clearly, \( \gamma \in \Delta^1_1(x) \) since \( f(x) \) is recursive in \( x \) and \( g(\alpha, x) \) is recursive in \( (\alpha, x) \) and hence \( \Delta^1_1(x) \)-recursive.

To prove direction \( (\impliedby) \) of (2G-2), suppose there is an \( \alpha \in \Delta^1_1(x) \) and some \( \gamma \) such that \( S(f(x), g(\alpha, x), \gamma) \). Now \( g(\alpha, x) \in \text{WO} \) and \( f(x) \in \text{LO}, \)

so by the lemma we have \( f(x) \in \text{WO} \), i.e., \( P(x) \).

This completes the proof of (2G-2). From (2G-2) we get the theorem for any pointset of type 0 or 1 by a trivial contraction of quantifiers.

Finally, suppose \( P \subseteq X \) where \( X \) is arbitrary, so by Theorem 1F.12 there is a recursive \( \pi : \mathcal{N} \rightarrow X \) and a \( \Pi^0_1 \) set \( A \subseteq \mathcal{N} \) on which \( \pi \) is injective, and for a \( \Pi^1_1 \) set \( P \subseteq X \) put

\[ Q(\alpha) \iff P(\pi(\alpha)). \]

Now \( Q \) is \( \Pi^1_1 \), and so by the fact just proved, there is a \( \Pi^0_1 \) relation \( R(\alpha, \beta) \) such that

\[ Q(\alpha) \iff (\exists \beta \in \Delta^1_1(\alpha))R(\alpha, \beta), \]

and with this we compute using the same “transitivity” facts and that

\[ (\alpha \in A \land \pi(\alpha) = x) \implies \alpha \in \Delta^1_1(x), \]

which holds because \( \pi \) is injective on \( A \):

\[ P(x) \iff (\exists \alpha)[\alpha \in A \land \pi(\alpha) = x \land Q(\alpha)] \]

\[ \iff (\exists \alpha \in \Delta^1_1(x))[\alpha \in A \land \pi(\alpha) = x \land (\exists \beta \in \Delta^1_1(\alpha))R(\alpha, \beta)] \]

\[ \iff (\exists \alpha \in \Delta^1_1(x))(\exists \beta \in \Delta^1_1(\alpha))[\alpha \in A \land \pi(\alpha) = x \land R(\alpha, \beta)] \]

\[ \iff (\exists \alpha \in \Delta^1_1(x))((\exists \beta \in \Delta^1_1(x))[\alpha \in A \land \pi(\alpha) = x \land R(\alpha, \beta)]. \]

The argument is concluded by contracting quantifiers and using the facts that membership in \( A \) and equality on \( X \) are \( \Pi^1_1 \).

The Specter-Gandy Theorem does not have many applications but it is undoubtedly one of the jewels of the effective theory. It gives a very elegant characterization of \( \Pi^1_2 \) in terms of a (restricted) existential quantifier which
is particularly significant in the case of relations on $\omega$: $P \subseteq \omega$ is $\Pi^1_1$ if and only if there is a $\Pi^0_1$ set $R \subseteq \omega \times N$ such that

$$P(n) \iff (\exists \alpha \in \Delta^1_1) R(n, \alpha).$$

This corollary says in effect that the collection of $\Delta^1_1$ points in Baire space somehow “determines” the collection of $\Pi^1_1$ relations on $\omega$.

2H. The problem of $\Delta^1_1$-isomorphism

By Theorem 1G.5, every perfect Polish space is $\Delta^1_1$-isomorphic to $N$ and by Problem 1G.6 every uncountable space $X$ is $\Delta^1_1(p(X))$-isomorphic with $N$, where $p(X)$ is the characteristic function of the local size relation

$$P_X(s) \iff N(X, s) \text{ is uncountable.}$$

This implies that $X$ is $\Delta^1_1$-isomorphic with $N$ if $P_X$ is a $\Delta^1_1$ relation, but in general, this is not true:

**Problem 2H.1.** Show that for every $X$, the local size relation $P_X$ is $\Sigma^1_1$.

In this section we will define a class of spaces $N^T$, one for each recursive tree $T$ on $\omega$, such that

1. every space $X$ is $\Delta^1_1$-isomorphic with some $N^T$; and
2. for some choices of $T$, $X = N^T$ is not $\Delta^1_1$-isomorphic with $N$, and hence $P_X$ is not $\Delta^1_1$.  

By varying $T$, in fact, we can locate among the $N^T$’s many new recursive Polish spaces which interesting effective properties.

2H.2. The spaces $N^T$. For any non-empty, recursive tree $T$ on $\omega$, let

$$N^T = T \cup [T],$$

where $[T] = \{ \alpha \in N : (\forall t)[(\alpha(0), \ldots, \alpha(t-1)) \in T] \}$ is, as usual, the body of $T$—and it may be empty. We picture this in the “plane” $N \times \omega^{<\omega}$ in Figure ??, and we could use this picture to put a topology on it, but it is more useful to put a metric on $N^T$ by injecting it in $N$. We define the function $\rho : N^T \to N$ as follows:

$$\rho(x) = \begin{cases} (x(0) + 1, \ldots, x(n-1) + 1, 0, 0, \ldots) & \text{if } x = (x(0), \ldots, x(n-1)) \in T, \\ (x(0) + 1, x(1) + 1, \ldots) & \text{if } x \in [T]. \end{cases}$$

So $\rho(u)$ is a sequence with an initial segment of non-zero values followed by a tail of all 0’s if $u \in T$, and if $x \in [T]$, then $\rho(x)(t) > 0$ for every $t$.

---

* The results in this section are due to Vassilis Gregoriades.
Lemma 2H.3. For every non-empty, recursive tree \( T \) on \( \omega \):

1. The map \( \rho : \mathcal{N}^T \to \mathcal{N} \) is one-to-one.
2. The image \( \rho[\mathcal{N}^T] \) of \( \rho \) is a \( \Pi^0_1 \) subset of \( \mathcal{N} \), and so is the image \( \rho[[T]] \) of the body of \( T \).
3. With the distance function it inherits from \( \mathcal{N} \), \( \rho[\mathcal{N}^T] \) is a recursive metric space with dense subset the image \( \rho[T] \) of the tree \( T \).

Proof. (1) is trivial. To prove (2) first for the image \( \rho[[T]] \) of the body of \( T \), check easily that

\[
\alpha \in \rho[[T]] \iff (\forall m) (\alpha(m) > 0 \& \langle \alpha(0) - 1, \ldots, \alpha(m) - 1 \rangle \in T^c).
\]

For the full image the corresponding equivalence is just a bit more complex:

\[
\alpha \in \rho[\mathcal{N}^T] \iff (\forall m) (\alpha(m) > 0 \Rightarrow [ (\forall i < m) (\alpha(i) > 0) \& \langle \alpha(0) - 1, \ldots, \alpha(m) - 1 \rangle \in T^c ]).
\]

This is also easy to check, taking cases on whether \( \alpha = \rho(u) \) for some \( u \in T \) or \( \alpha = \rho(x) \) for some \( x \in [T] \) for the direction \( \Rightarrow \), and on whether \( \alpha(t) = 0 \) for all \( t \), or \( (\alpha(t) > 0 \text{ for all } t, \text{ or otherwise}) \) for the direction \( \Leftarrow \). (For example, the image \( \rho(\emptyset) \) of the empty sequence is the constant function \( 0 \).)

(3) It is clear that as a subspace of \( \mathcal{N} \), \( \rho[\mathcal{N}^T] \) is complete because it is closed, and it is separable, because the image \( \rho(T) \) of the tree \( T \) is clearly dense in it,

\[
\rho(x) = (x(0) + 1, x(1) + 1, \ldots) = \lim_n (x(0) + 1, x(1) + 1, \ldots, x(n) + 1, 0, 0, \ldots) = \lim_n \rho((x(0), \ldots, x(n))) \quad (x \in [T]).
\]

Fix a recursive enumeration of \( \rho(T) \), e.g.,

\[
(2H-1) \quad \mathbf{r}_s = \begin{cases} 
\rho((s_0, \ldots, s_{n-1})) & \text{if } s = \langle s_0, \ldots, s_{n-1} \rangle \in T^c, \\
\rho(\emptyset), & \text{otherwise}, 
\end{cases}
\]

where \( T^c = \{ \langle s_0, \ldots, s_{n-1} \rangle : (s_0, \ldots, s_{n-1}) \in T \} \) is the code set of \( T \), and let \( d' \) be the restriction to \( \rho[\mathcal{N}^T] \) of the distance function on \( \mathcal{N} \). The values of \( d'(\alpha, \beta) \) are all rational numbers; and so to prove that the pair \( (d', \{ r_s \}) \) is a recursive presentation of \( \rho[\mathcal{N}^T] \), it is enough by Problem 1A.1 to check that the function

\[
(i, j) \mapsto d'(r_i, r_j)
\]

on \( \omega^2 \) to \( \mathbb{Q} \) is recursive, which it is. \( \dashv \)
We now use $\rho$ to carry to $N^T$ the distance function of $N$:

$$d^T(x, y) = d^N(\rho(x), \rho(y)).$$

This makes $N^T$ into a metric space and $\rho : N^T \mapsto \rho[N^T]$ an isometry, so all the properties of $\rho[N^T]$ in the Lemma are transferred over to $N^T$. In particular, the function

$$r^T(i) = \rho^{-1}(r(i))$$

enumerates a dense subset of $N^T$, so that $(d^T, r^T)$ is a recursive presentation of it.

We collect for reference some of these properties of $N^T$ in one basic result:

**Theorem 2H.4.** For every non-empty, recursive tree $T$ on $\omega$, the space $N^T$ has the following properties.

1. Every point of $T$ is an isolated point of $N^T$.
2. If $N(T, u) = \{x \in N^T : u \subseteq x\}$ for every $u \in T$, then the family
   
   $$\{N(T, u) : u \in T\} \cup \{\{u\} : u \in T\}$$

   forms a basis for the topology of $N^T$. In particular, $T$ is dense in $N^T$.
3. The function $r^T : \omega \to N^T$

   $$r_s = \begin{cases} ((s)_0, \ldots, (s)_{\lh(s)} - 1), & \text{if } s \in T^c, \\ \emptyset, & \text{otherwise} \end{cases}$$

   is a recursive presentation of $(N^T, d^T)$. In particular, $N^T$ is a recursive Polish space.
4. The tree $T$ is a $\Sigma^0_1$ subset of $N^T$ and so $[T]$ is a $\Pi^0_1$ subset of $N^T$.

**Proof.** (1) – (3) are basically immediate from the definitions, and (4) follows because if $s \in T^c$, then (by chasing the definitions using the isometry $\rho$), $r_s$ is the unique point in the ball about it with radius $< 1/(\lh(s) + 1)$,

$$x \in T \iff (\exists s)(\exists n)[s \in T^c \& \lh(s) = n \& d^T(r_s, x) < \frac{1}{n + 1}].$$

**Theorem 2H.5.** Every space $\mathcal{X}$ is $\Delta^1_1$-isomorphic with some $N^T$.

**Proof.** We may assume that $\mathcal{X}$ is infinite, the result being completely trivial for finite $\mathcal{X}$.

We fix a compatible pair $(d, r)$ and we also assume without loss of generality that $r : \omega \mapsto \mathcal{X}$ is one-to-one, cf. Problem 1D.30. Set

$$P = \mathcal{X} \setminus \{r(2i) : i \in \omega\};$$

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this is $\Pi^0_1$, hence $\Delta^1_1$, and so by Theorem 2G.11 there is a recursive function $\pi: \mathcal{N} \to \mathcal{X}$ and a $\Pi^0_1$ set $A \subseteq \mathcal{N}$ such that $\pi$ is injective on $A$ and $\pi[A] = P$. Also $A \neq \emptyset$, since $P \neq \emptyset$. Put $A$ in normal form,

$$\alpha \in A \iff (\forall t) R(\pi(t))$$

with $R(u)$ recursive and monotone downwards, i.e.,

$$R(u) & v \sqsubseteq u = \Rightarrow R(v),$$

and let

$$T = \{(u_0, \ldots, u_{t-1}) : R(\langle u_0, \ldots, u_{t-1} \rangle)\} \neq \emptyset.$$

We claim that $\mathcal{X}$ is $\Delta^1_1$-isomorphic with $\mathcal{N}^T$. To see this, we construct a recursive enumeration

$$T^c = t_0, t_1, \ldots,$$

of the code set of $T$ without repetitions and we define $f: \mathcal{N}^T \to \mathcal{X}$ by

$$f(x) = \begin{cases} r(2i) & \text{if } x = (u_0, \ldots, u_{n-1}) \in T \& t_i = \langle u_0, \ldots, u_{n-1} \rangle, \\ \pi(x) & \text{otherwise, i.e., if } x \in [T] = A. \end{cases}$$

This is clearly a $\Delta^1_1$-recursive bijection.

**2H.6. Kleene trees and spaces.** A recursive tree $T$ in $\omega$ is a **Kleene tree** if its body $[T]$ is non-empty and does not contain $\Delta^1_1$-members. Such trees exist by Problem 2G.12: if $A \subseteq \mathcal{N}$ is $\Pi^0_1$, not empty and with no $\Delta^1_1$ members, put $A$ in normal form as in (2H-2) and define $T$ by (2H-3).

A space $\mathcal{N}^T$ is a **Kleene space** if $T$ is a Kleene tree. Notice that every Kleene space is uncountable.

To analyze these spaces we will need one more result relating definability in $\mathcal{N}^T$ with definability in $\mathcal{N}$:

**Lemma 2H.7.** For every recursive tree $T$, there are recursive relations $R^T \subseteq \mathcal{N}^T \times \omega$ and $Q^T \subseteq \mathcal{N} \times \omega$ such that

$$x \in N(\mathcal{N}^T, s) \iff Q^T(x, s),$$

$$y \in N(\mathcal{N}, t) \iff R^T(y, t)$$

for all $(x, s), (y, t) \in [T] \times \omega$.

**Proof.** Let (temporarily) $\mathcal{X}^T$ be $\rho[N^T]$ as a metric subspace of $\mathcal{N}$. For (2H-4) first, we compute for arbitrary $x \in \mathcal{N}^T$:

$$x \in N(\mathcal{N}^T, s) \iff \rho(x) \in N(\mathcal{X}^T, s)$$

$$\iff \rho(x) \in N(\mathcal{N}, s) \& \rho(x) \in \rho[N^T],$$

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because $\rho : N^T \to \mathcal{X}$ is an isometry which carries the recursive presentation $(d^T, r^T)$ of $N^T$ to the recursive presentation of $\mathcal{X}$. If $x \in [T]$, then the condition $\rho(x) \in \rho[N^T]$ holds, so

$$x \in N(N^T, s) \iff \rho(x) \in N(N, s) \quad (x \in [T]),$$

and so (2H-4) holds for $x \in [T]$ with

$$Q^T(x, s) \iff \rho(x) \in N(N, s)$$

which is a recursive relation because $\rho$ is recursive. A similar argument yields $R^T$, using now that $\rho^{-1}$ is recursive on $\rho[N[T]]$. \(\diamondsuit\)

**Theorem 2H.8.** For every recursive tree $T$, every $x \in [T]$ and every $\Sigma_0$-pointclass $\Gamma$,

$$x \text{ is a } \Gamma\text{-point of } N^T \iff x \text{ is a } \Gamma\text{-point of } N.$$  

**Proof.** Consider the recursive relations $R^T \subseteq N^T \times \omega$ and $Q^T \subseteq N \times \omega$ in Lemma 2H.7. There are semirecursive $R^*, Q^* \subseteq \omega^2$ such that

$$R^T(x, t) \iff (\exists s)[x \in N(N^T, s) \land R^*(t, s)]$$

for all $(x, t) \in N^T \times \omega$ and

$$Q^T(x, t) \iff (\exists s)[x \in N(N, s) \land Q^*(t, s)]$$

for all $(x, t) \in N \times \omega$. For all $x \in N$ we have that

$$x \in N(N, t) \iff R^T(x, t)$$

$$\iff (\exists s)[x \in N(N^T, s) \land R^*(t, s)];$$

so if $x \in [T]$ and $\{s : x \in N(N^T, s)\} \in \Gamma$, then $\{t : x \in N(N, t)\} \in \Gamma$.

For the converse direction we use the relations $Q^T$ and $Q^*$.

\(\diamondsuit\)

**Theorem 2H.9.** For every Kleene tree $T$,

$$\Delta^1_1 \cap N^T = \text{the scattered part of } N^T$$

$$= \text{the set of isolated points of } N^T = T$$

and it is $\Sigma^0_1$.

It follows that there is no $\Delta^1_1$-injection $f : N \to N^T$, and so $N^T$ is not $\Delta^1_1$-isomorphic with $N$.

**Proof.** Round Robin style:

$x$ is isolated $\implies x$ is in the scattered part

$\implies (\exists s)[x \in N(N^T, s) \land N(N^T, s) \text{ is countable}]$

$\implies x \in \Delta^1_1$ (by the Perfect Set Theorem)

$\implies x \notin [T]$ (because $T$ is Kleene)

$\implies x \in T \implies x$ is isolated.
2I. Computation of parameters

The further inference that $\Delta_1^1 \cap \mathcal{N}^T = T$ is $\Sigma_1^0$ follows by Theorem 2H.4.

For the second claim, if there were a $\Delta_1^1$-injection $f : \mathcal{N} \rightarrow \mathcal{N}^T$, then (using the injectiveness of $f$) we would have that

$$\alpha \in \Delta_1^1 \iff f(\alpha) \in \Delta_1^1 \iff f(\alpha) \in T$$

for all $\alpha \in \mathcal{N}$, and so the set of $\Delta_1^1$ points of $\mathcal{N}$ would be $\Delta_1^1$ contradicting the Lower Classification of $\Delta_1^1 \cap \mathcal{N}$, Theorem 2G.17).

The property of Kleene spaces in this theorem actually characterizes them, by appealing to the following problem which also solves the $\Delta_1^1$-isomorphism problem for countable spaces:

Problem 2H.10. Prove that if $A \subseteq \mathcal{X}$ is $\Delta_1^1$ and countably infinite, then there is a $\Delta_1^1$-recursive injection $f : \omega \rightarrow \mathcal{X}$ which enumerates it,

$$f : \omega \rightarrow \mathcal{X}, \quad A = f[\omega] = \{f(0), f(1), \ldots\}.$$  

Infer that every countably infinite space $\mathcal{X}$ is $\Delta_1^1$-isomorphic with $\omega$, which is recursively isomorphic with some $\mathcal{N}^T$.

HINT: Use Problem 2G.16 and the Effective Perfect Set Theorem.

Theorem 2H.11. The set of $\Delta_1^1$-points of an uncountable space $\mathcal{X}$ is in $\Delta_1^1$ if and only if $\mathcal{X}$ is $\Delta_1^1$-isomorphic to a Kleene space.

Proof. The right-to-left-hand direction has been proved in Theorem 2H.9. For the converse, let $\mathcal{X}$ be such that the set $\Delta_1^1 \cap \mathcal{X}$ of $\Delta_1^1$-points of $\mathcal{X}$ is in $\Delta_1^1$, set $D = \Delta_1^1 \cap \mathcal{X}$ and $C = \mathcal{X} \setminus D$. Notice that $C$ is uncountable because $\mathcal{X}$ is uncountable. Since $D$ is a countable $\Delta_1^1$ set from Problem 2H.10 there exists a $\Delta_1^1$ sequence $(x_n)_{n \in \omega}$ which enumerates $D$ without repetitions.

Now we deal with $C$. Using Theorem 1F.12 we obtain a recursive tree $T$ on $\omega$ and a recursive function $\pi : \mathcal{N} \rightarrow \mathcal{X}$ which is injective on $[T]$ and $\pi[[T]] = C$. From its definition the set $C$ has no $\Delta_1^1$ members; hence $[T]$ has no $\Delta_1^1$ members as well. Since $C \neq \emptyset$ we have that $T$ is a Kleene tree. We define $f : \mathcal{N}^T \rightarrow \mathcal{X}$ by

$$f(y) = \begin{cases} x_n, & \text{ if } y = (y_0, \ldots, y_{m-1}) \in T \land \langle y_0, \ldots, y_{m-1} \rangle = n, \\ \pi(y), & \text{ if } y \in [T]. \end{cases}$$

Since $T$ is a semirecursive subset of $\mathcal{N}^T$ it is clear that the function $f$ is $\Delta_1^1$. Moreover $f$ is bijective from the properties of $\pi$ and $(x_n)_{n \in \omega}$. ⊣

2I. Computation of parameters

Every uncountable Polish space is Borel isomorphic with $\mathcal{N}$ by Problem 1G.6, but this basic result of the classical theory does not “effectivise
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fully": every uncountable recursive $\mathcal{X}$ is $\Delta^1_1(\varepsilon)$-isomorphic with $\mathcal{X}$ for some $\varepsilon$ which is recursive in some $\Sigma^1_1$ subset of $\omega$ but cannot always be chosen to be $\Delta^1_1$, by Theorem 2H.9. In this section we will consider two additional, important classical results whose effective versions require the introduction of a non-trivial parameter, specifically:

- The Cantor-Bendixson Theorem 1E.5, and
- The $G_\delta$ Theorem: a subspace $A \subseteq \mathcal{X}$ of a Polish space is Polish if and only if $A$ is a $G_\delta$ ($\Pi^1_2$) subset of $\mathcal{X}$.

We will also prove along the way some important results of the effective theory, including Kleene’s Basis Theorem for $\Sigma^1_1$, 2I.5.

2I.1. Spector’s $W$. As it happens, the required parameter in these (and many other results) is recursive in some $\Sigma^1_1$ subset of $\omega$, and so it is convenient to introduce a standard way of referring to this level of complexity. We will use the set of codes of recursive wellorderings of $\omega$ used by Spector in his analysis of $\Pi^1_1$ on $\omega$:

(2I-1) $W = \{ e \in \omega : \{ e \} : \omega \to \omega \text{ is total and } e \in \text{WO} \}$.

Problem 2I.2. Prove that Spector’s $W$ is $\Pi^1_1$ and complete for $\Pi^1_1$ relations on $\omega$, i.e., for every $\Pi^1_1$ set $P \subseteq \omega^n$ there is a total recursive function $f : \omega \to \omega$ such that

$$P(\vec{x}) \iff f(\vec{x}) \in W.$$  

It will also be useful to have notation which makes it easy to “manipulate” trees on $\omega$. For $s \in \text{Seq}$, let

$$u[s] = ((s)_0, \ldots, (s)_{lh(s) - 1})$$

be the finite sequence coded by $s$. It is clear that the relation $Q \subseteq \omega^3$ defined by

$$Q(s, t, i) \iff s, t \in \text{Seq} \land i < \min\{lh(s), lh(t)\} \land u[s](i) = u[t](i)$$

is recursive. We can use it to compute easily the complexity level of many relations involving finite sequences: for example the relation

$$L(s, t) \iff s, t \in \text{Seq} \land u[s] \leq_{\text{lex}} u[t]$$

is also recursive.

Lemma 2I.3. For every recursive tree $T$ with non-empty body, the sequence

$$s \mapsto \alpha_s = \begin{cases} 
\text{the leftmost infinite branch of } [T_{u[s]}], & \text{if } [T_{u[s]}] \neq \emptyset, \\
\text{the leftmost infinite branch of } [T], & \text{otherwise},
\end{cases}$$

is $W$-recursive.
In particular the leftmost infinite branch of a recursive tree with non-empty body is recursive in Spector’s \( W \).

**Proof.** The relation

\[
P(s) \iff [T_u[s]] \neq \emptyset
\]

is obviously \( \Sigma^1_1 \) and so it is recursive in Spector’s \( W \). Now if \( P(s) \), then

\[
u[t] \subseteq \alpha_s \iff u[t] \text{ is an initial segment of } u[s] \text{ or}
\]

\[
\begin{align*}
& (u[t] \text{ extends } u[s] \text{ and } [T_u[t]] \neq \emptyset \text{ and for every } u = u[k] \text{ which extends } u[s] \text{ and is } \leq_{\text{lex}} u[t] \\
& \quad \text{we have that } [T_u[k]] = \emptyset) \\
\end{align*}
\]

\[
\iff u[t] \subseteq u[s] \vee \left( u[s] \subseteq u[t] \& P(t) \& (\forall i < \text{lh}(t)) [i \geq \text{lh}(s) \rightarrow (\forall n < (t)_i)[\neg P((t)_0, \ldots, (t)_{i-1}, n)]) \right).
\]

The right-hand-side of the last equivalence clearly defines a relation \( R(s,t) \) which is recursive in \( W \) and for all \( t, s \) we have that

\[
u[t] \subseteq \alpha_s \iff [P(s) \& R(s, t)] \vee [\neg P(s) \& R(0, t)],
\]

so that the relation \( u[t] \subseteq \alpha_s \) is \( W \)-recursive. From Problem 1A.8 it follows that the sequence \( (\alpha_s)_{s \in \omega} \) is \( W \)-recursive.

\[\square\]

**Problem 2I.4.** Show that a subset \( P \subseteq X \) of a recursive Polish space is \( \Sigma^1_1 \) exactly when it is the recursive image of a \( \Pi^0_1 \) subset of the Baire space; but not every \( \Sigma^1_1 \) set \( \subseteq X \) is a recursive image of \( N \).

**Remark.** This is a fundamental difference from the classical theory, where \( \Sigma^1_1 \) is the family of sets which are the continuous image of Baire space.

**2I.5. Kleene’s Basis Theorem.** Every non-empty \( \Sigma^1_1 \) subset of a space \( X \) has a member which is recursive in Spector’s \( W \).

In fact, if \( A \subseteq X \) is \( \Sigma^1_1 \), then there is a \( W \)-recursive function \( f : \omega \rightarrow X \) such that for every \( s \)

\[
(2I-2) \quad (N(X, s \cap A) \neq \emptyset \Rightarrow f(s) \in (N(X, s) \cap A)).
\]

**Proof.** We prove the stronger, second claim.

Suppose \( A \) is \( \Sigma^1_1 \) and (without loss of generality) not empty. By Problem 2I.4, there exists a recursive function \( f_1 : N \rightarrow X \) and a \( \Pi^0_1 \) set \( B_1 \) such that \( f_1[B_1] = A \). We set

\[
\alpha \in B \iff \alpha^* \in B_1 \& f_1(\alpha^*) \in N(X, \alpha(0)).
\]
Now $B$ is also $\Pi^0_1$, and easily

\[(s) \ast \alpha \in B \implies f_1(\alpha) \in (A \cap N(X, s)),\]
\[x \in (A \cap N(X, s)) \implies (\exists \alpha)[(s) \ast \alpha \in B \land x = f_1(\alpha)].\]

By Lemma 2A.1, $B$ is the body of a non-empty recursive tree $T$, and by these two implications,

\[(\exists x)[x \in (A \cap N(X, s))] \implies (\exists \alpha)[\alpha \in B \land f_1(\alpha) \in (A \cap N_s) = \implies (\exists \alpha)[\alpha \in B \land f_1(\alpha) \in (A \cap N_s)].\]

So to complete the proof we only need set $f(s) = f_1(\alpha)$ where the map $(s \mapsto \alpha)$ is that of Lemma 2I.3 for the tree $T$.

\[\downarrow\]

21.6. The Effective Cantor-Bendixson Theorem. For every recursive Polish space $X$ and every closed, $\Sigma^1_1$ set $A \subseteq X$:

1. the perfect kernel $A_p$ of $A$ is $\Sigma^1_1$, and
2. if it is not empty, then the scattered part $A_{sc}$ of $A$ can be enumerated by a W-recursive function $f: \omega \to X$.

Proof. (1) Recall that $A_p$ comprises the condensation points of $A$, so

\[x \in A_p \iff (\forall s)[x \in N_s \implies (N_s \cap A) \text{ is uncountable}]
\[\iff (\forall s)[x \in N_s \implies (\exists y)[y \in A \land y \notin \Delta^1_1 \cap X]],\]

where the second equivalence comes from the Effective Perfect Set Theorem 2G.5. The equivalence witnesses the fact that $A_p$ is $\Sigma^1_1$.

(2) We will use the $\Pi^1_1$-recursive partial function $d: \omega \to X$ which parametrizes $\Delta^1_1 \cap X$, i.e.,

\[x \in \Delta^1_1 \cap X \iff (\exists i)[d(i) = x].\]

Put

\[Q(s, i) \iff (N_s \cap A) \text{ is countable} \& d(i) \downarrow \& d(i) \in (N_s \cap A).\]

(i) The relation $Q \subseteq \omega^2$ is $\Pi^1_1$. This is because by the Effective Perfect Set Theorem 2G.5

\[(N_s \cap A) \text{ is countable} \iff (\forall x)[x \in (N_s \cap A) \implies x \in \Delta^1_1]\]

and $\Delta^1_1 \cap X$ is $\Pi^1_1$, by (3) of Theorem 2G.3.
(ii) $A_{sc} = \{ d(i) : (\exists s)Q(s, i) \}$. Because, for one inclusion,

$$x \in A_{sc} \implies (\exists s)[x \in (N_s \cap A) \& (N_s \cap A) \text{ is countable}]$$

$$\implies (\exists s)[(N_s \cap A) \text{ is countable} \& (\exists i)[x = d(i) \in (N_s \cap A)]]$$

$$\implies (\exists s)(\exists i)[Q(s, i) \& x = d(i)],$$

and the converse inclusion is equally simple.

(iii) If $x_0$ is some fixed member of $A_{sc}$ and we set

$$g(s, i) = \begin{cases} 
    d(i) & \text{if } Q(s, i), \\
    x_0, & \text{otherwise},
\end{cases}$$

then $f$ is W-recursive and $A_{sc} = f[\omega \times \omega]$. The second claim follows immediately from (ii), and for the first, we compute:

$$g(s, i) \in N_t \iff [Q(s, i) \& d(i) \in N_t] \lor [\neg Q(s, i) \& x_0 \in N_t].$$

Now (2) follows if we contract the variables in $g$:

$$f(j) = g((j)_0, (j)_1).$$

The representation of the Cantor-Bendixson decomposition is basically optimal, by the following result whose proof we will not include in these notes:

**Theorem 2I.7** (Kreisel). There is a $\Pi^0_1$ set $A \subseteq \mathcal{N}$ such that

1. the perfect kernel $A_p$ of $A$ is not $\Pi^0_1$, and so
2. the scattered part $A_{sc}$ is not $\Sigma^0_1$.

**2I.8. $G_\delta$ subsets of Polish spaces.** We have already mentioned in the beginning of this section the classical $G_\delta$ Theorem which characterizes those (topological) subspaces of a Polish space $\mathcal{X}$ which are Polish. In this section we will formulate and prove its recursive version, which is a bit complex and requires several steps. The relevant results are Theorems 2I.14 and 2I.16.

**Theorem 2I.9.** Suppose $A \subseteq \mathcal{X}$ is a closed, infinite, $\Sigma^0_1$ subset of a space $\mathcal{X}$ and $(d, r)$ is a compatible pair for $\mathcal{X}$.

1. There is a W-recursive injection $r^A : \omega \rightarrow A$ whose image $r^A[\omega]$ is dense in $A$, i.e.,

$$\text{for } (N_s \cap A) \neq \emptyset \implies (\exists i)[r^A(i) \in (N_s \cap A)].$$

2. There is a W-recursive real number $\bar{a} \neq 0$ such that the relations

$$P^A(i, j, t) \iff \bar{a} \cdot d(r^A(i), r^A(j)) < q_t$$

$$Q^A(i, j, t) \iff \bar{a} \cdot d(r^A(i), r^A(j)) \leq q_t,$$

are W-recursive.
It follows that as a subspace of $\mathcal{X}$, every closed, infinite $\Sigma^1_1$ subset of $\mathcal{X}$ is a $W$-recursive Polish space. In particular the perfect kernel of every uncountable $\Sigma^1_1$ closed subset of $\mathcal{X}$ is a perfect $W$-recursive Polish space.

**Proof.** (1) The strong conclusion in Kleene’s Basis Theorem 2I.5 gives us a $W$-recursively enumerable set $f[\omega]$ which is dense in $A$; this is infinite, and so Problem 1D.30 (relativized to $W$) gives us a $W$-recursive injective enumeration $r^A \mapsto [\omega]$ of it.

(2) We use the method of Problem 1D.28. Since the set of $W$-recursive real numbers cannot be recursively enumerated, there exists a $W$-recursive $b \neq 0$ such that $q_t \cdot d(r^A(i), r^A(j)) \neq b$ for all $t, i, j \in \omega$. By setting $\bar{a} = b^{-1}$ we have that $\bar{a} \cdot d(r^A(i), r^A(j))$ is either $0$ or irrational. It follows that $\bar{a}d(r^A(i), r^A(j)) < q_t \iff \bar{a}d(r^A(i), r^A(j)) \leq q_t$ for all $i \neq j$ and all $t$. Hence the relations $P^A$ and $Q^A$ coincide whenever $i \neq j$ and since they are clearly in $\Sigma^1_0(W)$ and $\Pi^0_1(W)$ respectively, they are $W$-recursive.

The conclusions follow because the metric $\bar{a}d$ is equivalent to $d$ and $A$ is complete because it is closed, and because by Theorem 2I.6, the perfect kernel of a $\Sigma^1_1$ set is $\Sigma^1_1$.

The theorem says, in part, that $\Pi^0_1$ subspaces of $\mathcal{X}$ are $W$-recursive Polish spaces. To prove this for $\Pi^0_2$ subspaces, we need a little bit of metric topology.

For any metric space $(\mathcal{X}, d)$ and any non-empty $A \subseteq \mathcal{X}$, set

$$d(x, A) = \inf \{d(x, y) : y \in A\}.$$ 

The map $x \mapsto d(x, A)$ is continuous, in fact Lipschitz:

**Problem 2I.10.** Show that for every metric space $(\mathcal{X}, d)$ and every non-empty $A \subseteq \mathcal{X}$,

$$|d(x, A) - d(y, A)| \leq d(x, y) \quad (x, y \in \mathcal{X}).$$

**Hint:** Take cases on whether $d(x, A) \geq d(y, A)$ or $d(x, A) < d(y, A)$.

We now examine this function from the definability point of view:

**Lemma 2I.11.** For every recursively presented metric space $(\mathcal{X}, d, r)$ and every $\Sigma^1_1$ set $A \subseteq \mathcal{X}$, the function

$$f(x) = d(x, A) \quad (x \in \mathcal{X}, f(x) \in \mathbb{R})$$

is $W$-recursive.

**Proof.** By (1) of Theorem 2I.9, there exists a $W$-recursive function $r^A : \omega \to \mathcal{X}$ such that the set $\{r^A(i) : i \in \omega\}$ is dense in $A$. This gives that

$$d(x, A) = \inf \{d(x, r^A(i)) : i \in \omega\}.$$
which implies immediately that
\[ d(x, A) < q_t \iff (\exists i)[d(x, r^A(i)) < q_t], \]
and so the relation
\[ P_1(x, t) \iff d(x, A) < q_t \]  
(2I-3)
is W-semirecursive. A little less obvious is that
\[ d(x, A) > q_t \iff (\exists s)[x \in N_s \& (\forall y)(\forall i)[y \in N_s \implies d(y, r^A(i)) > q_t]]. \]
The direction (\(\Leftarrow\)) of this is immediate, and the direction (\(\Rightarrow\)) is proved by checking the contrapositive: if for all \(s\)
\[ x \in N_s \implies (\exists y, i)[y \in N_s \& d(y, r^A(i)) \leq q_t] \]
we can then apply this to a sequence \(N_{s_n}\) with radii tending to 0 and obtain a sequence \(y_n\) converging to \(x\) such that \(d(y_n, A) \leq q_t\), which in the limit and by the continuity of \(x \mapsto d(x, A)\) gives \(d(x, A) \leq q_t\). We can rewrite (2I-4) in the form
\[ d(x, A) > q_t \iff (\exists s)[x \in N_s \& (\forall y)(\forall z)(\forall i)[y \in N_s \& z = r^A(i) \implies d(y, z) > q_t]]. \]
which makes it obvious that the relation
\[ P_2(x, t) \iff d(x, A) > q_t \]  
(2I-5)
is W-semirecursive, because the second conjunct within the brackets is \(\Pi_1^1\). Finally, the fact that both \(P_1\) and \(P_2\) in (2I-3) and (2I-5) are W-semirecursive implies that \(x \mapsto d(x, A)\) is W-recursive by (the relativization to W) of Problem 1D.20.

We will need the following “uniform” version of this theorem, which is proved by the same method:

**Problem 2I.12.** Suppose that \((\mathcal{X}, d, r)\) is a recursively presented metric space, \(A \subseteq \omega \times \mathcal{X}\) is \(\Sigma_1^1\) and for each \(n\), \(A_n = \{x \in \mathcal{X} : A(n, x)\}\). Prove that the function
\[ f(x, n) = d(x, A_n) \quad (x \in \mathcal{X}, n \in \omega) \]
is W-recursive.

**Remark.** We cannot replace \(\omega \times \mathcal{X}\) by an arbitrary \(Z \times \mathcal{X}\) in this problem, because the corresponding \(f : X \times Z\) may even fail to be continuous.
21.13. \(\varepsilon\)-recursive subspaces. An injection \(f : \mathcal{X} \hookrightarrow \mathcal{Y}\) is a recursive embedding of \(\mathcal{X}\) into \(\mathcal{Y}\) if it is recursive and its inverse \(f^{-1} : \mathcal{Y} \rightarrow \mathcal{X}\) is a potentially recursive partial function with domain of convergence \(f[\mathcal{X}]\). A subset \(A\) of \(\mathcal{X}\) is a recursive (Polish) subspace of \(\mathcal{X}\), if there exists a pair \((d_A, r_A)\) such that the metric space \((A, d_A, r_A)\) is recursively presented and the identity \(\text{id} : A \hookrightarrow \mathcal{X}\) is a recursive embedding.

We naturally define \(\varepsilon\)-recursive embeddings and \(\varepsilon\)-recursive subspaces by relativization: a set \(A \subseteq \mathcal{X}\) is an \(\varepsilon\)-recursive subspace of \(\mathcal{X}\) if there exists a pair \((d_A, r_A)\) such that the metric space \((A, d_A, r_A)\) is \(\varepsilon\)-recursively presented and the identity \(\text{id} : A \hookrightarrow \mathcal{X}\) is an \(\varepsilon\)-recursive embedding.

It is easy to verify that the topology of an \(\varepsilon\)-recursive subspace of \(\mathcal{X}\) is the one inherited from \(\mathcal{X}\), so an \(\varepsilon\)-recursive subspace is Polish with the relative topology. Conversely if \(\mathcal{X}\) is a recursive Polish space and \(A \subseteq \mathcal{X}\) is Polish with the relative topology, then there is an \(\varepsilon\) such that \(A\) is an \(\varepsilon\)-recursive subspace of \(\mathcal{X}\).

**Theorem 21.14.** Every \(\Pi^0_2\) subset of a recursive Polish space is a \(W\)-recursive subspace.

It follows that \(G_4\) subsets of Polish spaces are Polish.

**Proof.** Suppose \(A\) is infinite (to avoid trivialities) and a \(\Pi^0_2\) subset of \(\mathcal{X}\), so that for some \(\Pi^0_2\) set \(F \subseteq \omega \times \mathcal{X}\) is \(\Pi^0_1\),

\[ x \in A \iff \forall n \exists F(x, n). \]

Fix a compatible pair \((d, r)\) for \(\mathcal{X}\), let \(F_n = \{ x \in \mathcal{X} : F(n, x) \}\) be the \(n\)-th section of \(F\), and notice that \(d(x, F_n) > 0\) for all \(x \in A\) and all \(n \in \omega\). Now set

\[ d^A(x, y) = d(x, y) + \sum_{n \in \omega} 2^{-n} \cdot |d(x, F_n)^{-1} - d(y, F_n)^{-1}| \quad (x, y \in A). \]

**Lemma.** The metric space \((A, d^A)\) is complete.

**Proof.** To see this let \((x_i)_{i \in \omega}\) be a \(d^A\)-Cauchy sequence in \(A\). Then clearly \((x_i)_{i \in \omega}\) is \(d\)-Cauchy so there is some \(x \in \mathcal{X}\) such that \(d(x_i, x) \to 0\), and since \(x \mapsto d(x, F_n)\) is continuous, \(d(x_i, F_n) \to d(x, F_n)\) for all \(n \in \omega\).

We claim that \(x \in A\), and to prove it by contradiction, suppose \(x \in F_n\) for some \(n\) and so \(d(x, F_n) = 0\) and \(d(x_i, F_n) \to 0\). Therefore the number \(|d(x_i, F_n)^{-1} - d(x_j, F_n)^{-1}|\) would get arbitrarily large for large \(i\) and \(j\). Since \(d^A(x_i, x_j) \geq 2^{-n} \cdot |d(x_i, F_n)^{-1} - d(x_j, F_n)^{-1}|\) the sequence \((x_i)_{i \in \omega}\) would not be \(d^A\)-Cauchy, a contradiction. Therefore \(x \in A\). From this it follows easily that \(d^A(x_i, x) \to 0\), and so the metric space \((A, d^A)\) is complete.

From Theorem 21.9 there exists an injective \(W\)-recursive function \(r^A : \omega \to \mathcal{X}\) such that the set \(\{r^A(i) : i \in \omega\}\) is dense in \(A\). From Problems
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The function

\[ f(x, y) = \sum_{n \in \omega} 2^{-n} \cdot |d(x, F_n)^{-1} - d(y, F_n)^{-1}| \]

is W-recursive as a composition of W-recursive functions. It follows that the function \((i, j) \mapsto d^A(r^A(i), r^A(j))\) is W-recursive as well. We now use once more the method of Problem 1D.28 to find a W-recursive real number \(\bar{a}\) such that the relations

\[ P^A(i, j, k) \iff \bar{a} \cdot d^A(r^A(i), r^A(j)) < q_k \]
\[ Q^A(i, j, k) \iff \bar{a} \cdot d^A(r^A(i), r^A(j)) \leq q_k, \]

are W-recursive; which implies that \(d^A = \bar{a} \cdot d^A\) is a complete metric on \(A\), and hence \((A, d^A, r^A)\) is W-recursively presented.

It is now not difficult to check by the same methods that \((A, d^A, r^A)\) is a W-recursive subspace of \(X\), and we will omit the details.

\[ \text{Problem 2I.15.} \]

Find a \(\Pi_0^1\) subset of a recursive Polish space \(X\) which cannot be turned into an \(\varepsilon\)-recursive subspace \(X\) for any \(\varepsilon \in \Delta_1^1\).

\[ \text{Theorem 2I.16.} \]

Every recursive subspace of a recursive Polish space \(X\) is a \(\Pi_0^2\) subset of \(X\).

It follows that every subset of a Polish space \(X\) which is Polish with the relative topology is \(G_\delta\) in \(X\).

\[ \text{Proof.} \]

Let \(A \subseteq X\) be a recursive subspace and suppose that \((d^A, r^A)\) and \((d, r)\) are compatible with \(A\) and \(X\) respectively. The hypothesis that \(A\) is a recursive subspace of \(X\) implies the following:

1. The function \((i \mapsto r^A(i))\) is recursive in \(X\).
2. The partial function \(f : X \times X \to \mathbb{R} \)

\[ f(x, y) = d^A(x, y) \quad (x, y \in A) \]

is potentially recursive with domain of convergence \(A \times A\).
3. The topology on \((A, d^A)\) is that induced by \(X\).

By (3) of Theorem 1H.4, there is a potentially recursive extension

\[ \overline{f} : X \times X \to \mathbb{R} \]

of \(d^A\) whose domain of convergence is \(\Pi_0^1\).

We now claim that for all \(x \in X\),

\[ (2I-6) \quad x \in A \iff (\forall i)[\overline{f}(x, r^A(i)) \leq \varepsilon] \]
\[ &\quad \& (\forall k)(\exists i)[\overline{f}(x, r^A(i)) < 2^{-k} \& d(x, r^A(i)) < 2^{-k}] \]
\[ &\quad \& (\forall i)(\forall j)[d^A(r^A(i), r^A(j)) \leq \overline{f}(x, r^A(i)) + \overline{f}(x, r^A(j))]. \]

This will complete the proof since the expression on the right is \(\Pi_0^2\).
Proof of \( \Rightarrow \). Assume \( x \in A \). We need to verify the right-hand-side of the implication with \( f \) replaced by \( d^A \), since \( f \) coincides with \( d^A \) on arguments in \( A \). The first conjunct on the right is trivially true. To verify the second conjunct, we use the fact that \( r^A[\omega] \) is dense in \( A \) to define a function \( g : \omega \to \omega \) such that
\[
\lim_k r^A(g(k)) = x \quad (\text{in } (A, d^A)).
\]
Since the topology on \( (A, d^A) \) is that induced by \( X \), this implies that
\[
\lim_k r^A(g(k)) = x \quad (\text{in } (X, d)).
\]
and then these two implications together imply the second conjunct. The third conjunct is just the triangle inequality for \( d^A \) when \( x \in A \).

Proof of \( \Leftarrow \). Assume the right-hand-side of (2I-6) holds for some \( x \in X \), and for each \( k \) choose \( g(k) \) such that
\[
f(x, r^A(g(k))) < 2^{-k} \quad \text{and} \quad d(x, r^A(g(k))) < 2^{-k}.
\]
The sequence \( \{r^A(g(k))\}_k \) lies in \( A \) and the third condition on the right-hand-side implies that it is Cauchy in \( (A, d^A) \), because
\[
d^A(r^A(g(k)), r^A(g(k + 1))) \leq d^A(r^A(g(k)), r^A(g(k + 1))) + f(x, r^A(g(k + 1))) < 2^{-k} + 2^{-k-1} < 2^{-k+1}.
\]
So there is some point \( y \in A \) such that
\[
\lim_k r^A(g(k)) = y \quad (\text{in } (A, d^A)).
\]
Since the topology on \( (A, d^A) \) is that induced by \( X \), again, this implies that
\[
\lim_k r^A(g(k)) = y \quad (\text{in } (X, d)).
\]
On the other hand, the second condition on the right also implies that
\[
\lim_k r^A(g(k)) = x \quad (\text{in } (X, d));
\]
and so \( x = y \in A \). \( \dashv \)

**Additional problems for Section 2I**

**Problem 2I.17.** Prove that if \( f \subseteq \mathcal{X} \) is non-empty, countable and \( \Sigma^1_1 \), then there is a \( \Delta^1_1 \)-recursive (total) function \( f : \omega \to \mathcal{X} \) such that
\[
A \subseteq f[\omega] = \{f(0), f(1), \ldots\}.
\]
**Hint:** Use Theorem 2G.3 which enumerates \( \Delta^1_1 \cap \mathcal{X} \) and its \( \Pi^1_1 \)-recursive “inverse” \( c : \mathcal{X} \to \omega \) of Problem 2G.16.
Problem 2I.18. Prove that if $A \subseteq \mathcal{X}$ is non-empty and the difference
\[ A = B \setminus C \]
of two countable $\Sigma^1_1$ sets, then there is a $\Delta^1_1(W)$-recursive (total) function $g: \omega \to \mathcal{X}$ which enumerates $A$, i.e.,
\[ A = g[\omega] = \{g(0), g(1), \ldots \}. \]

Problem 2I.19. (1) Prove that Problem 2I.17 cannot be strengthened to say that every non-empty, countable $\Sigma^1_1$ set $A \subseteq \mathcal{X}$ can be enumerated by a $\Delta^1_1$-recursive function $f: \omega \to \mathcal{X}$.

(2) Give an example of a closed, countable $\Sigma^1_1$ set pointset $A$ whose scattered part cannot be enumerated by a $\Delta^1_1$-recursive function.

HINT: For both parts, look for counterexamples in $\mathcal{X} = \omega$. 

Informal notes, full of errors, December 8, 2012, 16:24
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