Problem 5. For a given binary relation $Q(x,y)$ on a set A and $x \in A$, let

$$P(x) \iff (\text{for infinitely many } y)Q(x,y).$$

(5a) TRUE or FALSE: if $A = \mathbb{N}$ and $Q(x,y)$ is arithmetical, then $P(x)$ is also arithmetical. (Prove this or give a counterexample.)

(5b) TRUE or FALSE: if $A = \mathbb{N}^*$ is the universe of a non-standard model $\mathbf{N}^*$ of (true) arithmetic and $Q(x,y)$ is elementary in $\mathbf{N}^*$, then $P(x)$ is also elementary in $\mathbf{N}^*$. (Prove this or give a counterexample.)

Solution. This is FALSE for the relation

$$Q(x,y) \iff y \le x$$

which is elementary in $\mathbf{N}^*$ because

$$y \le x \iff \mathbf{N}^*, \{u := x, v := y\} \models (\exists z)[y + z = x].$$

However, clearly,

$$P(x) \iff \{y \in N^* \mid y \le x\} \text{ is infinite } \iff x \in N^* \setminus N;$$

the set $N^* \setminus N$ is not empty and has no least element; so if

$$P(x) \iff \mathbf{N}^*, \{u := x\} \models \phi$$

for some ϕ with just u free, then

$$(\star) \qquad \mathbf{N}^* \models (\exists u)\phi \wedge (\forall v)(\exists u)[\phi \wedge u < v].$$

However, for any ϕ,

$$\mathbf{N} \models (\exists u)\phi \to (\exists v)(\forall u)[\phi \to v \le u]$$

because *every non-empty set of numbers has a least element*, and so

$$\mathbf{N}^* \models (\exists u)\phi \to (\exists v)(\forall u)[\phi \to v \le u]$$

which contradicts $(\star)$.