Mathematics 114L, Spring 2017
Yiannis N. Moschovakis
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Name (last name first): ______________________________

Signature: ______________________________

There are eight problems or problem parts, each of them worth 15 points for a total of 120 points with 100 points being your goal. The problems vary widely in difficulty: don't get stuck in a difficult one, do first those that you can do very easily.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

Solved!

The last page is blank, to use for scratch.

Problem (1a) __________

Problem (1b) __________

Problem (1c) __________

Problem (2a) __________

Problem (2b) __________

Problem (2c) __________

Problem 3 __________

Problem 4 __________

Total: ______________________

Problem 1. Consider the following proposition about an arbitrary structure $\mathbf{A}$:

($\star$) *If* $R(x,y)$ *is an elementary relation on* $\mathbf{A}$ *and*

$$P(x) \iff \text{the set } \{y \mid R(x,y)\} \text{ is infinite,}$$

then $P(x)$ *is also elementary on* $\mathbf{A}$.

Prove (in outline) that ($\star$) is true for each of the following structures.

(1a) $\mathbf{A} = \mathbf{N} = (\mathbb{N}, 0, 1, S, +, \cdot)$ is the standard structure of arithmetic.

Solution.

$$P(x) \iff (\forall u)(\exists y)[u \leq y \wedge R(x,y)].$$

(1b) $\mathbf{A} = (\mathbb{Z}, \leq)$, the rational integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ with their usual ordering.

Solution.

$$P(x) \iff (\forall u)(\exists y)[u \leq y \wedge R(x,y)] \vee (\forall u)(\exists y)[y \leq u \wedge R(x,y)].$$

To prove the implication ($\Leftarrow$) of this equivalence, we take cases on the two disjuncts on the right. If the first disjunct $(\forall u)(\exists y)[u \leq y \wedge R(x,y)]$ is true, then the set $\{y \geq 0 \mid R(x,y)\}$ has no maximum and so it is infinite; and if the second disjunct $(\forall u)(\exists y)[y \leq u \wedge R(x,y)]$ is true, then the set $\{y \leq 0 \mid R(x,y)\}$ has no minimum, and so its is again infinite.

To prove the converse implication ($\Rightarrow$) of the claimed equivalence, we assume that for some fixed x, both disjuncts on the the right hand side are false. It follows that there are numbers u_r, u_l such that

$$\text{for all } y, R(x,y) \Longrightarrow [y \leq u_r \wedge y \geq u_l],$$

so that the set $\{y \mid R(x,y)\}$ is contained in the interval

$$[u_l, u_l + 1, \dots, u_r - 1, u_r]$$

and is finite.

(Problem 1 continuing from the previous page.)

(1c) $\mathbf{A}$ is a structure which admits tuple coding.

Solution. It is sufficient to verify that the negation of $P(x)$ is elementary. To check this, we let $\gamma(\vec{a}, i)$ be the assumed, $\mathbf{A}$-elementary tuple coding (with $\vec{a} = a_1, \dots, a_n$ varying over n-tuples) and we compute:

$$\begin{aligned} \neg P(x) \iff & \text{ the set } \{y \mid R(x,y)\} \text{ is a subset of some finite tuple} \\ & \iff (\exists \vec{a})(\exists k \in \mathbb{N}')(\forall y)[R(x,y) \rightarrow (\exists i \leq k)[y = \gamma(\vec{a}, i)]]. \end{aligned}$$

Problem 2. Consider a structure

$$\mathbf{A} = (A, R),$$

where A is an infinite set and $R(x)$ is a unary relation on A which is infinite and co-infinite, i.e., it is true of infinitely many x and also false for infinitely many x. (A specific example would be $(\mathbb{N}, R)$ with $R(x) \iff x$ is even.)

(2a) What are the literals in the vocabulary (R) of $\mathbf{A}$?

Solution. The literals are prime or negations of prime formulas, i.e., formulas of the form $\mathbf{tt}, \mathbf{ff}, u = v, R(u), \neg\mathbf{tt}, \neg\mathbf{ff}, u \neq v, \neg R(v)$.

(2b) State the Quantifier Elimination test and use it to prove that to show that $\mathbf{A}$ admits effective quantifier elimination it is sufficient to "eliminate the quantifier" effectively from formulas in one of the following three forms, where $m, k \geq 0$ and x does not occur in the quantifier-free formula ϕ:

$$\text{(1)} \qquad \exists x\Big[(x = z_1 \wedge \cdots \wedge x = z_k) \wedge (x \neq w_1 \wedge \cdots \wedge x \neq w_m)\Big] \wedge \phi$$

$$\text{(2)} \qquad \exists x\Big[(x = z_1 \wedge \cdots \wedge x = z_k) \wedge (x \neq w_1 \wedge \cdots \wedge x \neq w_m) \wedge R(x)\Big] \wedge \phi$$

$$\text{(3)} \qquad \exists x\Big[(x = z_1 \wedge \cdots \wedge x = z_k) \wedge (x \neq w_1 \wedge \cdots \wedge x \neq w_m) \wedge \neg R(x)\Big] \wedge \phi$$

Solution. The Quantifier Elimination Test requires the elimination of the quantifier $\exists x$ from these three forms plus the form

$$\exists x\Big[(x = z_1 \wedge \cdots \wedge x = z_k) \wedge (x \neq w_1 \wedge \cdots \wedge x \neq w_m) \wedge R(x) \wedge \neg R(x)\Big] \wedge \phi$$

in which both $R(x)$ and $\neg R(x)$ occur among the conjuncts that involve the variable x; but this is false, and so the form is simply equivalent to $\mathbf{ff}$ in this case.

(Problem 2 continuing from the previous page.)

(2c) Outline a proof that the structure (A, R) admits effective quantifier elimination.

Solution. If an equation $x = z_i$ occurs in any of the forms (1) – (3), we eliminate the quantifier by replacing x by z_i throughout. We need to deal then with the following three forms in which no equation of the form $x = z_i$ occurs:

$$\exists x \Big[(x \neq w_1 \wedge \cdots \wedge x \neq w_m) \Big] \wedge \phi$$

$$\exists x \Big[(x \neq w_1 \wedge \cdots \wedge x \neq w_m) \wedge R(x) \Big] \wedge \phi$$

$$\exists x \Big[(x \neq w_1 \wedge \cdots \wedge x \neq w_m) \wedge \neg R(x) \Big] \wedge \phi$$

In all three of these cases, the given form is equivalent to the formula ϕ (in which x does not occur): because the universe A of the structure is infinite and R is both infinite and co-infinite, and so the conjunction within "$\Big[$" and "$\Big]$" is always true of some x, no matter what values are assigned to $w_1, \ldots, w_m$.

Problem 3. Let $f(n) = n! = 1 \cdot 2 \cdot 3 \cdots n$ be the factorial function on $\mathbb{N}$, so $f(0) = 1$ (by convention), $f(1) = 1, f(2) = 2, f(3) = 6, \cdots$ Prove (in outline) that $f(n)$ is arithmetical.

SOLUTION. The function $f(n)$ is defined by the following primitive recursion:

$$f(0) = 0, \quad f(S(n)) = f(n) \cdot S(n).$$

Problem 4. To do this problem you will need to use the fact that

(∗) *Every set can be ordered,*

i.e., for every set A there exists a binary relation $x \leq y$ which is a linear ordering of A. This is a non-trivial (but basic) set-theoretic fact about sets which we will just assume here. (It is proved in every standard course in set theory).

Recall also the first order definition of what it means for $\leq$ to be a linear ordering,

$$\mathsf{LO} \equiv \forall x[x \leq x] \wedge \forall x \forall y \forall z[(x \leq y \wedge y \leq z) \rightarrow x \leq z] \\ \wedge \forall x \forall y[(x \leq y \wedge y \leq x) \rightarrow x = y] \wedge \forall x \forall y[x \leq y \vee y \leq x].$$

Prove that if τ is any vocabulary without the relational symbol $\leq$, T is a τ-theory and χ is a τ-sentence, then

$$\text{if } T, \mathsf{LO} \vdash \chi, \text{ then } T \vdash \chi.$$

Hint: You need to use the Soundness and Completeness Theorems to do this problem, it can't be done by manipulating proofs.

Solution. By the Completeness Theorem, it is enough to show that every model of T satisfies χ, so assume that $\mathbf{A} \models T$. Choose by (∗) an ordering $\leq$ of the universe A and let $(\mathbf{A}, \leq)$ is the expansion of $\mathbf{A}$ with the ordering $\leq$. Notice that

$$(\mathbf{A}, \leq) \models T,$$

since by the Compositionality Theorem, for any τ-sentence ϕ,

$$(\mathbf{A}, \leq) \models \phi \iff \mathbf{A} \models \phi$$

and the proposition on the right is true of every $\phi \in T$. It follows that

$$(\mathbf{A}, \leq) \models T \cup \{\mathsf{LO}\},$$

and so, by the hypothesis (and the Soundness Theorem),

$$(\mathbf{A}, \leq) \models \chi.$$

The relational symbol $\leq$ does not occur in χ, and so

$$(\mathbf{A}, \leq) \models \chi \iff \mathbf{A} \models \chi$$

by the Compositionality Theorem again; and this gives the required $\mathbf{A} \models \chi$.