Mathematics 114L, Spring 2017
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Name (last name first): ______________________

Signature: ______________________

There are 12 questions (organized in 5 problems) and each is worth 10 points for a total of 120; 100 points count for a "perfect" score, so you can skip two questions or afford to make some errors.

Some of the questions are completely trivial while others require some work; so do first those that you know immediately how to do.

You may use all the results we have covered, including the Deduction and Completeness Theorems for PL.

Page 6 has the Hilbert and Gentzen systems and page 7 is blank.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

Solved!

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Total: __________

Problem 1. This problem is about the **Propositional Calculus** PL, and you may assume the definition and basic properties of the Satisfaction Relation

$$v \models \chi \iff v \textit{ satisfies } \chi$$

between an assignment v to the propositional variables and a formula of PL.

(1a) Define the relation of *logical consequence* between a set T if PL-formulas and a formula χ,

$$T \models \chi \iff \chi \textit{ is a logical consequence of } T.$$

Solution. Put first

$$v \models T \iff \text{for every } \phi \in T, v \models \phi,$$

and then put

$$T \models \chi \iff \text{for every assignment } v, \text{ if } v \models T, \text{ then } v \models \chi.$$

(1b) Prove or give a counterexample: for all formulas ϕ, ψ,

$$\neg\phi \models (\phi \to \psi).$$

Solution. This is true; because for any assignment v, if $v \models \neg\phi$, then $v \models (\phi \to \psi)$ whether $v \models \psi$ or not.

(1c) Prove or give a counterexample: for all formulas ϕ, ψ,

$$(\phi \to \psi) \to \psi \models \psi.$$

Solution. This is not always true; it fails when $v \models \phi$ but $v \models \neg\psi$.

(1d) Prove or give a counterexample: for all formulas ϕ, ψ and all assignments v, if $v \models (\phi \wedge \psi) \vee \neg\psi$, then $v \models (\psi \to \phi)$

Solution. This is true. To check it, consider cases on a given assignment v and the truth values it assigns to the parts ϕ, ψ of the hypothesis.

If $v \models \neg\psi$, then $v \models (\psi \to \phi)$, whether $v \models \phi$ or not; while if $v \models (\phi \wedge \psi)$, then clearly $v \models (\psi \to \phi)$.

(1e) Prove or give a counterexample: for all formulas ϕ, ψ,

$$\neg\phi \vdash (\phi \to \psi).$$

Solution. This is the same as part (1b), except for the "proves" sign $\vdash$ in place of the "validates" sign $\models$. The Completeness Theorem, however, guarantees that

$$\text{if } \neg\phi \models (\phi \to \psi) \text{ then } \neg\phi \vdash (\phi \to \psi),$$

and so the claim is true.

Problem 2. This problem is about which connectives are really needed in propositional logic. We set

$$\phi \text{ is equivalent to } \psi \iff \models (\phi \to \psi) \wedge (\psi \to \phi).$$

(2a) Prove or give a counterexample:

Every formula is equivalent to one in which the negation symbol $\neg$ *does not occur.*

Solution. This is false, for the formula $\neg p$ among many others. To see this, we prove by structural induction the following proposition about formulas:

$(*)$ *If there is no negation in* ϕ *and all its variables are in the list* $p_1, \dots, p_n$*, then*

$$p_1, \dots, p_n \models \phi.$$

We skip the steps of the induction which are easy.

(2b) Prove or give a counterexample:

Every formula is equivalent to one in which the only connective that occurs is the negation symbol $\neg$.

Solution. This is false, for example for the formula $p \wedge q$; because the only formulas that can be constructed using only the unary connective $\neg$ have only one variable in them, and so cannot restrict both p and q.

(2c) Let $\bullet$ be any one of the binary connectives $\wedge, \vee, \to$. Prove or give a counterexample:

Every formula is equivalent to one in which the only connectives that occur are $\neg$ *and* $\bullet$.

Solution. This is true for every binary connective $\bullet$. The relevant definitions are

$$\begin{array}{lll} \vee & (\phi \wedge \psi) :\equiv \neg((\neg\phi) \vee (\neg\psi)), & (\phi \to \psi) :\equiv \neg\phi \vee \psi \\ \wedge & (\phi \vee \psi) :\equiv \neg((\neg\phi) \wedge (\neg\psi)), & (\phi \to \psi) :\equiv \neg\phi \vee \psi \\ \to & (\phi \vee \psi) :\equiv (\neg\phi) \to \psi, & (\phi \wedge \psi) :\equiv \neg((\neg\phi) \vee (\neg\psi)). \end{array}$$

Problem 3. This problem is also about the **Propositional Calculus** PL, and you may assume the definition and basic properties of the Satisfaction Relation $v \models \chi$ for PL.

(3a) (The Compositionality principle for PL). Prove that for any two assignments v, u and any formula χ,

if $v(p) = u(p)$ for every variable p which occurs in χ,

$$\text{then } \Big(v \models \chi \iff u \models \chi\Big).$$

Solution. We use structural induction.

If χ is a propositional variable p, then using the hypothesis,

$$v \models p \iff v(p) = 1 \iff u(p) = 1 \iff u \models p.$$

If $\chi \equiv \neg\phi$, then using the induction hypothesis this time and the (obvious) fact that the same variables occur in ϕ and $\neg\phi$,

$$v \models \neg\phi \iff v \not\models \phi \iff u \not\models \phi \iff u \models \neg\phi.$$

The arguments about the binary connectives is similar.

(3b) Let T be a set of formulas, let p be a propositional variable which does not occur in any formula of T, and let χ be any formula such that p also does not occur in χ. Prove that

$$\textit{if } \ T, p \vdash \chi, \ \textit{then } T \vdash \chi.$$

Solution. The simplest proof of this is to use the Completeness Theorem, by which it is enough to prove the "semantic version" of the claim, i.e.,

$$T, p \models \chi \Longrightarrow T \models \chi.$$

Written out in full, this means that

$$(\text{for every } v)\Big(v \models T \cup \{p\} \Longrightarrow v \models \chi\Big) \Longrightarrow (\text{for every } v)\Big(v \models T \Longrightarrow v \models \chi\Big);$$

and this is true, because by the first part of the problem (and the hypothesis that p does not occur in for any formula of T or in χ), for any v, if we let

$$v\{p := 1\}(q) = \begin{cases} 1, & \text{if } q \equiv p, \\ v(q), & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} v \models T \Longrightarrow (\forall \phi \in T)[v \models \phi] \Longrightarrow (\forall \phi \in T \cup \{p\})[v\{p := 1\} \models \phi] \\ \Longrightarrow v\{p := 1\} \models \chi \Longrightarrow v \models \chi. \end{aligned}$$

Problem 4. Prove in PL that $\neg p, p \vdash q$; you may give a deduction in the Hilbert system or use the rules of $\mathbf{G}_s$.

SOLUTION. Here is a Hilbert deduction of q from the hypotheses $\neg p, p$ without showing the justifications:

$$p \to (\neg q \to p), \quad p, \quad \neg q \to p, \quad \neg p \to (\neg q \to \neg p), \quad \neg p, \quad \neg q \to \neg p,$$
$$(\neg q \to p) \to \Big((\neg q \to \neg p) \to \neg\neg q\Big), \quad (\neg q \to \neg p) \to \neg\neg q, \quad \neg\neg q,$$
$$\neg\neg q \to q, \quad q.$$

The proof in $\mathbf{G}_s$ is completely trivial: we start with the axiom $p \vdash p$ and apply the $\neg$-elim rule with $\chi \equiv q$ to get $p, \neg p \vdash q$.

Problem 5. Consider the following "misspelled" formula in the language of number theory:

$$\chi \equiv (\exists x)[x + S(y) = S(x+y)] \wedge x + y = S(x)$$

Write out the official, correct version of χ and circle all the free occurrences of the variables x and y.

SOLUTION.

$$(\exists x + (x, S(\boxed{y})) = S(+(x, \boxed{y})) \wedge +(\boxed{x}, \boxed{y}) = S(\boxed{x}))$$

The Hilbert system for PL

(1) $\phi \to (\psi \to \phi)$
(2) $(\phi \to \psi) \to ((\phi \to (\psi \to \chi)) \to (\phi \to \chi))$
(3) $(\phi \to \psi) \to ((\phi \to \neg\psi) \to \neg\phi)$
(4) $\neg\neg\phi \to \phi$
(5) $\phi \to (\psi \to (\phi \wedge \psi))$
(6a) $(\phi \wedge \psi) \to \phi$
(6b) $(\phi \wedge \psi) \to \psi$
(7a) $\phi \to (\phi \vee \psi)$
(7b) $\psi \to (\phi \vee \psi)$
(8) $(\phi \to \chi) \to ((\psi \to \chi) \to ((\phi \vee \psi) \to \chi))$

$$\frac{T \vdash \phi \qquad T \vdash \phi \to \psi}{T \vdash \psi} \text{ (Modus Ponens)}$$

The Gentzen system $\mathbf{G}_s$ for PL

$$\chi \vdash \chi \text{ (hypothesis)}$$

$$\frac{T \vdash \chi}{T, \phi \vdash \chi} \text{ (thinning)} \qquad \frac{T, \neg\phi \vdash \phi}{T \vdash \phi} \text{ (simp}_1\text{)} \qquad \frac{T, \phi \vdash \neg\phi}{T \vdash \neg\phi} \text{ (simp}_2\text{)}$$

$$\frac{T, \phi \vdash \neg\chi}{T, \chi \vdash \neg\phi} \text{ ($\neg$-intro)} \qquad \frac{T \vdash \phi}{T, \neg\phi \vdash \chi} \text{ ($\neg$-elim)}$$

$$\frac{T \vdash \phi \qquad T \vdash \psi}{T \vdash \phi \wedge \psi} \text{ ($\wedge$-intro)} \qquad \frac{T, \phi \vdash \chi}{T, \phi \wedge \psi \vdash \chi} \quad \frac{T, \psi \vdash \chi}{T, \phi \wedge \psi \vdash \chi} \text{ ($\wedge$-elim)}$$

$$\frac{T \vdash \phi}{T \vdash \phi \vee \psi} \quad \frac{T \vdash \psi}{T \vdash \phi \vee \psi} \text{ ($\vee$-intro)} \qquad \frac{T, \phi \vdash \chi \qquad T, \psi \vdash \chi}{T, \phi \vee \psi \vdash \chi} \text{ ($\vee$-elim)}$$

$$\frac{T, \chi \vdash \phi}{T \vdash \chi \to \phi} \text{ ($\to$-intro)} \qquad \frac{T \vdash \phi \qquad T, \psi \vdash \chi}{T, \phi \to \psi \vdash \chi} \text{ ($\to$-elim)}$$

$$\frac{T \vdash \phi \qquad T, \phi \vdash \psi}{T \vdash \psi} \text{ (Cut)}$$