Mathematics 114L, Spring 2016
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Final Examination, Wednesday, June 8, 2016

Name (last name first): ______________________________

Signature: ______________________________

Two of the problems may look familiar, as they were also in the Take Home Part of the final.

Each part of a problem is worth 10 points, for a total of 130 points.

You may use all the results we have covered, including the Soundness, Completeness and Compactness Theorems.

Page 8 has a copy of Theorem 3J.1 in the Notes which you need for one problem and page 9 is blank.

Try to be concise and clear, making sure the grader understands how you are going to prove something—the "architecture" of your argument.

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Problem 6. __________

Total: ______________

Problem 1. Suppose x, y, z are distinct variables and consider the following (misspelled) formula in the language of arithmetic:

$$\phi \equiv (\forall x)(\exists y)\Big(x + y = z \wedge (\exists z)(z \cdot z = y)\Big)$$

(1a) Write out ϕ correctly, i.e., with prefix notation ($+(x, y)$ for $x + y$) and all the parens it needs in the proper places.

(1b) In the correctly spelled out ϕ, circle all the free occurrences of variables.

Problem 2. Prove that if $Q(x, y)$ is elementary in a structure $\mathbf{A}$, $f : A \to A$ is an $\mathbf{A}$-elementary function and

$$R(x) \iff (\exists y)Q(x, f(y)),$$

then $R(x)$ is also $\mathbf{A}$-elementary. (You may use only the definitions and Theorem 3J.1 on page 8.)

Problem 3. Let S be the smallest set of natural numbers which contains 1 and is closed under the operations $x \mapsto 3x, x \mapsto 5^x$, so that for example,

$$1, 3 = 3 \cdot 1, 5 = 5^1, 125 = 5^3, 375 = 3 \cdot 125, 5^{375}, \dots \text{ are all in } S.$$

(3a) Give a precise definition of S (as we gave precise definitions of the sets of terms and formulas in PL and $\mathsf{LPCI}(\tau)$ which were defined by similar inductions).

(3b) Prove that no member of S is an even number.

(3c) Prove that S is arithmetical.

Problem 4.

(4a) Prove that if a theory T in any (finite) vocabulary τ has arbitrarily large, finite models, then it has an infinite model.

(4b) Prove that for any (finite) vocabulary τ and every natural number $n \geq 1$, there is a τ-structure $\mathbf{A}$ with exactly n elements in its universe.

(4c) Define what it means for a class of τ-structures Φ to be *elementary*.

(4d) Let τ be a (finite) vocabulary and let Fin_τ be the class of all finite τ-structures; is Fin_τ elementary? (You must prove your answer.)

Problem 5. For a given binary relation $Q(x, y)$ on a set A and $x \in A$, let

$$P(x) \iff (\text{for infinitely many } y)Q(x, y).$$

(5a) TRUE or FALSE: if $A = \mathbb{N}$ and $Q(x, y)$ is arithmetical, then $P(x)$ is also arithmetical. (Prove this or give a counterexample.)

(5b) TRUE or FALSE: if $A = \mathbb{N}^*$ is the universe of a non-standard model $\mathbf{N}^*$ of (true) arithmetic and $Q(x, y)$ is elementary in $\mathbf{N}^*$, then $P(x)$ is also elementary in $\mathbf{N}^*$. (Prove this or give a counterexample.)

Problem 6. This problem gives a simple generalization of the First Incompleteness Theorem of Gödel which shows that it does not depend on any special properties of the axioms and rules of inference of first order logic.

A **verification procedure** for arithmetic is a binary relation $V(\chi, y)$ between sentences of the languages of Peano arithmetic and numbers; we read this

$$V(\chi, y) \iff y \text{ is a verification (or justification) that } \chi \text{ is true in } \mathbf{N}.$$

A verification procedure is **sound** if it only verifies true sentences, i.e., for all sentences χ and numbers y,

$$(*) \qquad \text{if } V(\chi, y), \text{ then } \mathbf{N} \models \chi.$$

A verification procedure is **arithmetical** if the number theoretic relation

$$V^{\#}(c, y) \iff c \text{ is the code of a sentence } \chi \text{ such that } V(\chi, y)$$

is arithmetical. For example, by Lemma 2.6 of Part 3 of the Notes, the relation

$$V_T(\chi, y) \iff y \text{ is the code of a proof of } \chi \text{ from } T$$

is sound and arithmetical for every sound and arithmetical theory T.

Prove that there is no sound and arithmetical verification procedure which verifies all true sentences of the language of arithmetic, i.e.,

$$(**) \qquad \mathbf{N} \models \chi \Longrightarrow (\exists y)V(\chi, y).$$

Theorem 3J.1. The collection $\mathcal{E}(\mathbf{A})$ of $\mathbf{A}$-elementary functions and relations on the universe of a structure

$$\mathbf{A} = (A, \{c^{\mathbf{A}}\}_{c \in \text{Const}}, \{R^{\mathbf{A}}\}_{R \in \text{Rel}}, \{f^{\mathbf{A}}\}_{f \in \text{Funct}}).$$

has the following properties:

(1) Each primitive relation $R^{\mathbf{A}}$ and the (binary) identity relation $x = y$ on A are $\mathbf{A}$-elementary.

(2) For each constant symbol c and each n, the n-ary constant function

$$g(\vec{x}) = c^{\mathbf{A}}$$

is $\mathbf{A}$-elementary; each primitive function $f^{\mathbf{A}}$ is $\mathbf{A}$-elementary; and every projection function

$$P_i^n(x_1, \dots, x_n) = x_i \quad (1 \leq i \leq n)$$

is $\mathbf{A}$-elementary.

(3) $\mathcal{E}(\mathbf{A})$ is closed under substitutions of $\mathbf{A}$-elementary functions: i.e., if $h(u_1, \dots, u_m)$ is an m-ary $\mathbf{A}$-elementary function and $g_1(\vec{x}), \dots, g_m(\vec{x})$ are n-ary, $\mathbf{A}$-elementary, then the function

$$f(\vec{x}) = h(g_1(\vec{x}), \dots, g_m(\vec{x}))$$

is $\mathbf{A}$-elementary; and if $P(u_1, \dots, u_m)$ is an m-ary $\mathbf{A}$-elementary relation, then the n-ary relation

$$Q(\vec{x}) \iff P(g_1(\vec{x}), \dots, g_m(\vec{x}))$$

is $\mathbf{A}$-elementary.

(4) $\mathcal{E}(\mathbf{A})$ is closed under the propositional operations: i.e., if $P_1(\vec{x})$ and $P_2(\vec{x})$ are $\mathbf{A}$-elementary, n-ary relations, then so are the following relations:

$$\begin{aligned} Q_1(\vec{x}) &\iff \neg P_1(\vec{x}), \\ Q_2(\vec{x}) &\iff P_1(\vec{x}) \wedge P_2(\vec{x}), \\ Q_3(\vec{x}) &\iff P_1(\vec{x}) \vee P_2(\vec{x}), \\ Q_4(\vec{x}) &\iff P_1(\vec{x}) \to P_2(\vec{x}). \end{aligned}$$

(5) $\mathcal{E}(\mathbf{A})$ is closed under quantification on A, i.e., if $P(\vec{y})$ is $\mathbf{A}$-elementary, then so are the relations

$$\begin{aligned} Q_1(\vec{x}) &\iff (\exists y) P(\vec{x}, y), \\ Q_2(\vec{x}) &\iff (\forall y) P(\vec{x}, y). \end{aligned}$$

Moreover: $\mathcal{E}(\mathbf{A})$ is the smallest collection of functions and relations on A which satisfies (1) – (5).